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HOMEOMORPHISMS OF 3-MANIFOLDS AND TOPOLOGICAL ENTROPY

Dedicated to Professor Itiro Tamura on his 60th birthday

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1. Introduicton

The *topological entropy* $h(f)$ of a self-map f of a metric space is a measure of its dynamical complexity (for the definition of topological entropy see section 2 below). In [T_1] Thurston has shown that any homeomorphism f of a compact hyperbolic surface is isotopic to φ which is either periodic, pseudo-Anosov, or reducible (see also [F-L-P], [H-T], [M]). We call φ *Thurston's cannonical form* of f. In section 2 we show that $h(\varphi)\leq h(f)$ i.e. φ attains the minimal entropy in its isotopy class. Hence from the dynamical viewpoint Thurston's cannonical form plays an important role ([H], [K], [Smi]).

In this paper, we find a similar cannonical form of a homeomorphism of a class of geometric 3-manifolds (for the definition and fundamental properties of geometric 3-manifolds see [Sc]). We note that every self-homeomorphism of a hyperbolic 3-amnifold is homotopic to a periodic one ([Mo]). In the following, we consider homeomorphisms of an $H^2\times R$, $\widetilde{SL_2(R)}$, or Nil 3-manifold M.

Then our main result is:

Theorem 2. *Let f be a homeomorphism of an $H^2\times R$, $\widetilde{SL_2(R)}$, or Nil 3-manifold M. Then f is homotopic to φ such that either*:

(i) *φ is of type periodic,*

(ii) *φ is of type pseudo-Anosov, or*

(iii) *there is a system $\mathcal{T}$ of tori in M such that φ is reducible by $\mathcal{T}$. There is a φ-invariant regular neighborhood $\eta(\mathcal{T})$ of $\mathcal{T}$ such that each φ-component of $M-Int\ \eta(\mathcal{T})$ satisfies* (i) *or* (ii). *Each component $\eta(T_j)$ of $\eta(\mathcal{T})$ is mapped to itself by some positive iterate φ^{m_j} of φ and $\varphi^{m_j}|_{\eta(T_j)}$ is a twist homeomorphism.*

For the definitions of the terms which appear in Theorem 2, see section 4 below. We note that if M is sufficiently large, then φ is isotopic to f ([Wa]).

In section 5 we show that the above φ attains the minimal entropy in its homotopy class, and $h(\varphi)$ is positive if and only if φ contains a component of type pseudo-Anosov.

2. Preliminaries

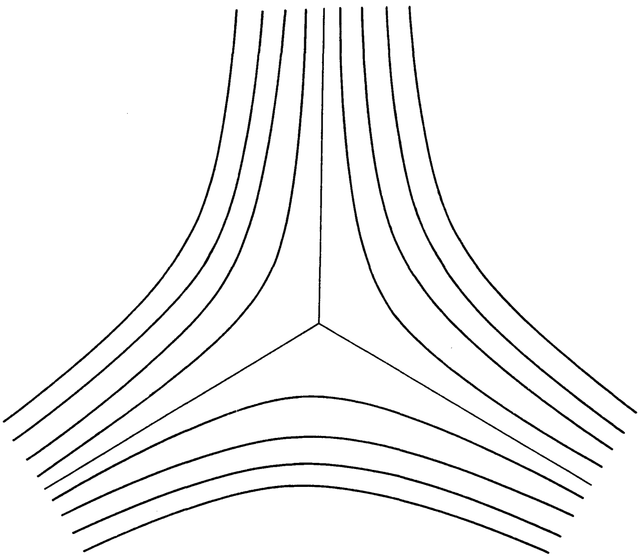
Let f: $X \to X$ be a continuous map of a metric space (X, d). In this section we recall the definition of the topological entropy $h(f)$ of f in [Bo], and show that Thurston's cannonical form of a surface homeomorphism attains the minimal entropy in its homotopy class.

Let K ($\subset X$) be compact, $\varepsilon > 0$ a positive number, and n a positive integer. We say that $E(\subset K)$ is (n, ε)-*separated*, if $x, y \in E$, $x \neq y$, then there is $0 \leq i < n$ such that $d(f^i(x), f^i(y)) \geq \varepsilon$. Let $s_K(n, \varepsilon)$ be the maximal cardinality of an (n, ε)-separated set in K. We say that $E(\subset K)$ is (n, ε)-*spanning* for K, if $x \in K$, then there is a $y \in E$ such that $d(f^i(x), f^i(y)) < \varepsilon$ for all i with $0 \leq i < n$. Let $r_K(n, \varepsilon)$ be the minimal cardinality of an (n, ε)-spanning set in K. Let $\bar{s}_K(\varepsilon) = \lim_{x \to \infty} \sup 1/n \cdot \log(s_K(n, \varepsilon))$, and $\bar{r}_K(\varepsilon) = \lim_{x \to \infty} \sup 1/n \cdot \log(r_K(n, \varepsilon))$. Then it can be shown that $\lim_{\varepsilon \to 0} \bar{s}_K(\varepsilon) = \lim_{\varepsilon \to 0} \bar{r}_K(\varepsilon)$, and we denote this value by $h_K(f)$. Finally, we put $h(f) = \sup \{(h_K(f) \mid K: \text{compact} \subset X\}$.

For the proof of the following theorem see [Bo].

Theorem A. *Let X, Y be compact metric spaces, p: $X \to Y$, f: $X \to X$, g: $Y \to Y$ continuous maps such that $f \circ p = p \circ g$. Then $h(g) \leq h(f) \leq h(g) + \sup \{h_{p^{-1}(y)}(f); y \in Y\}$.*

Let F be a compact hyperbolic surface. A *measured foliation* $(\mathcal{F}, \mu)$ on F is a pair of a singular foliation on F and a transverse invariant measure μ of $\mathcal{F}$



3-pronged saddle

Fig. 1

([F-L-P], [T_1]). $\mathcal{F}$ may have a finite number of singularities $a_1, \cdots, a_l$, where a_i is an r_i-pronged saddle with $r_i \geq 3$. If M has boundary, then each boundary component is a leaf of the foliation and has at least one singularity. A self-homeomorphism $f: F \to F$ is *pseudo-Anosov* if there is a pair of mutually transverse measured foliations $(\mathcal{F}^s, \mu^s)$, $(\mathcal{F}^u, \mu^u)$ and a number $\lambda > 1$ such that f preserves two foliations $\mathcal{F}^s$, $\mathcal{F}^u$ and $f_*(\mu^s) = 1/\lambda \cdot \mu^s$, $f_*(\mu^u) = \lambda \cdot \mu^u$. Then λ is called the *expanding factor* of f. f is *reducible* by Γ if Γ is a system of mutually disjoint and non-parallel loops, each of which is non-contractible, non-peripheral and $f(\Gamma) = \Gamma$. f is *periodic* if there is a positive integer n such that $f^n = id_F$. Let A be an annulus, $g: A \to A$ a homeomorphism, $\tilde{A} = [-1, 1] \times R$ the universal cover of A where the covering translations are generated by $(x, y) \to (x, y+1)$. g is a *twist homeomorphism* if there is a lift $\tilde{g}: \tilde{A} \to \tilde{A}$ of g such that $\tilde{g}(x, y) = (\pm x, h(x, y))$ for some map h.

Then the precise statement of Thurston's result is:

Theorem B (Thurston [T_1]). *If f is a self-homeomorphism of a compact hyperbolic surface F then f is isotopic to φ such that either:*

(i) *φ is periodic,*

(ii) *φ is pseudo-Anosov, or*

(iii) *there is a system of simple loops Γ on F such that φ is reducible by Γ. There is a φ-invariant regular neighborhood $\eta(\Gamma)$. Each φ-component of $F-$ Int $\eta(\Gamma)$ satisfies* (i) *or* (ii). *Each component, A_i, of $\eta(\Gamma)$ is mapped to itself by some positive iterate φ^{m_i} of φ, and $\varphi^{m_i}|_{A_i}$ is a twist homeomorphism.*

Then we can show:

Proposition 2.1. *Thurston's cannonical form φ attains the minimal entropy in its homotopy class. Moreover $h(\varphi) > 0$ if and only if φ contains a pseudo-Anosov component.*

Proof. If φ is periodic then $h(\varphi) = 0$ for $h(\varphi^n) = n \cdot h(\varphi)$. If φ is pseudo-Anosov then by [F-L-P] Exposé 10, $h(\varphi) > 0$ and it attains the minimal entropy in its homotopy class. Suppose that φ is reducible. Let $A_1, \cdots, A_m$ $(m \geq 1)$ be the components of $\eta(\Gamma)$ and $F_1, \cdots, F_n$ $(n \geq 1)$ be the closures of the components of $F - \cup A_i$. Let l be a positive integer such that $\varphi^l(A_i) = A_i$ $(1 \leq i \leq m)$ and $\varphi^l(F_j) = F_j$ $(1 \leq j \leq n)$. By the definition of topological entropy we see that $h(\varphi^l) = \max_{i,j} \{h(\varphi^l|_{A_i}), h(\varphi^l|_{F_j})\}$. It is easy to show that for each homeomorphism g of a circle we have $h(g) = 0$. By Theorem A we see that $h(\varphi^l|_{A_i}) = 0$ for each i. Hence $h(\varphi^l) = \max_j \{h(\varphi^l|_{F_j})\}$.

If each $\varphi^l|_{F_i}$ is periodic, then $h(\varphi^l) = 0$. Hence $h(\varphi) = 0$.

If φ^l contains a pseudo-Anosov component, then $h(\varphi^l) = \log \lambda_i$, where λ_i is the expanding factor of some pseudo-Anosov component $\varphi^l|_{F_i}$. Since $\chi(F_i)$

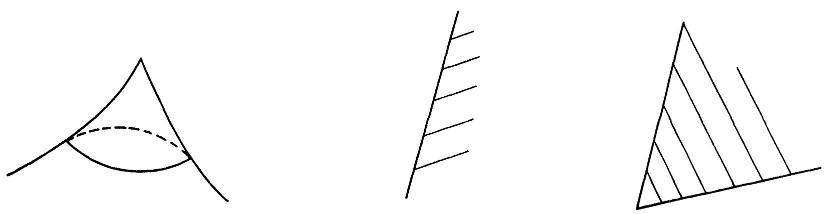
<0, there is an essential, non-peripheral (not necessarily simple) loop α on F_i. Then by [F-L-P] $\lim_{x\to\infty} 1/n\cdot\log(L(\varphi^{nl}(\alpha)))=\lambda_i$, where $L(\alpha)$ denotes the infimum of the length of loops which are homotopic to α. By [F-L-P] if g is homotopic to φ^l then $h(g)\geq\log\lambda_i$i.e. φ^l attains the minimal entropy in its homotopy class. Since $h(\varphi^l)=l\cdot h(\varphi)$ we see that φ attains the minimal entropy in its homotopy class.

This completes the proof of Proposition 2.1.

3. Homeomorphisms of 2-dimensional orbifolds

In this section, we give a classification theorem for homeomorphisms of 2-dimensional orbifolds. We assume that the reader is familiar with [T_2, §13] or [Sc, §2].

By [Sc], [T_2] every singularity on a 2-dimensional orbifold is either cone, reflector line, or corner reflector. Throughout this paper we consider orbifolds



Singuralities of 2-orbifolds
Fig. 2

whose singularities are cones. Let O(O' resp.) be a 2-dimensional orbifold with singularities $x_1, \cdots, x_n$ $(n\geq 1)$ $(x'_1, \cdots, x'_{n'}$ $(n'\geq 1)$ resp.), where the cone angle of x_i (x'_i resp.) is $2\pi/p_i$ ($2\pi/p'_i$ resp.).

Let f be a map from O to O'. f is called *O-homeomorphism* if f satisfies:

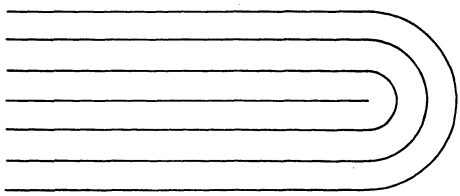
(i) f is a topological homeomorphism,

(ii) $f(\mathrm{sing}(O))=\mathrm{sing}(O')$, where $\mathrm{sing}(O)$ denotes the set of the singularities of O,

(iii) if $f(x_i)=x'_j$ then $p_i=p'_j$.

Let f, f': $O\to O'$ be O-homeomorphisms. f and f' are *O-isotopic* if there is a topological isotopy F_t: $O\to O'$ $(0\leq t\leq 1)$ such that each F_t is an O-homeomorphism and $F_0=f$, $F_1=f'$.

The definition of a measured foliation $(\mathcal{F}, \mu)$ on O is the same as the definition for surfaces in section 2 except the fact that if x is a singularity of O then $\mathcal{F}$ may have 1-pronged saddle at x. A self-O-homeomorphism f: $O\to O$ is *pseudo-Anosov* if there are a pair of mutually transverse measured foliations $(\mathcal{F}^s, \mu^s)$, $(\mathcal{F}^u, \mu^u)$ and a number $\lambda>1$ such that f preserves the two foliations, and $f_*(\mu^s)=1/\lambda\cdot\mu^s$, $f_*(\mu^u)=\lambda\cdot\mu^u$.



1-pronged saddle
Fig. 3

Let a_1, $a_2 \subset O$ be simple loops which do not meet singularities of O. a_1 and a_2 are *parallel* if $a_1 \cup a_2$ bounds an annulus which does not contain singularities. a_1 is *peripheral* if a_1 is parallel to a component of ∂O. a_1 is *essential* if a_1 is not peripheral and a_1 does not bound a disk on O which contains at most one singular point.

A self-O-homeomorphism f: $O \to O$ is *reducible* by Γ if Γ is a system of simple loops on O each of which does not meet a singular point and is essential, which are mutually disjoint and non-parallel, and $f(\Gamma)=\Gamma$.

Then we have:

Theorem 1. *Let O be a compact 2-dimensional orbifold whose (possibly empty) singular points are all cone type and f a self-O-homeomorphism of O. Then f is O-isotopic to φ such that either:*

(i) *φ is periodic,*

(ii) *φ is pseudo-Anosov, or*

(iii) *there is a system Γ of simple loops on O such that φ is reducible by Γ. There is a φ-invariant regular neighborhood $\eta(\Gamma)$ of Γ such that each φ-component of $O - Int\, \eta(\Gamma)$ satisfies* (i) *or* (ii). *Each component, A_j, of $\eta(\Gamma)$ is mapped to itself by some positive iterate φ^{m_j} of φ and $\varphi^{m_j}|_{A_j}$ is a twist homeomorphism.*

Proof. First, suppose that O contains no singularities i.e. O is a surface. In case of $\chi(O)<0$ Theorem 1 is just Theorem B in section 2. There are four distinct compact surfaces with Euler characteristic zero, say annulus, Möbius band, Klein bottle, and torus [Sc]. It is easy to see that every homeomorphism of an annulus or a Möbius band is homotopic to a periodic one, and then is isotopic to a periodic one [E]. By Lickorish [Li] every homeomorphism of a Klein bottle is isotopic to a periodic one. Let O be a torus, A a 2×2 matrix representing f_*: $(O)\pi_1 \to \pi_1(O)$ for a fixed basis of $\pi_1(O)$. Then f is isotopic to a reducible, periodic, or (pseudo-)Anosov map according to whether A is conjugate to $\begin{pmatrix} \varepsilon & n \\ 0 & \varepsilon \end{pmatrix}$ ($\varepsilon=\pm 1$, $n \neq 0$), to $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ with $|\lambda_1|=|\lambda_2|=1$, or to $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ with $|\lambda_1| \neq |\lambda_2|$ [F-L-P, Expose 1]. There are three distinct compact surfaces with positive Euler characteristic say sphere, disk, and projective plane. By Smale [Sma], [F-L-P, Exposé 2] every homeomorphism of them is isotopic to a periodic one.

We suppose that O has singularities $x_1, \cdots, x_n$ ($n \geq 1$) which are cone type. Let S be a surface obtained from $O-\{x_1, \cdots, x_n\}$ by adding a circle to each non-compact end. By moving f by an O-isotopy we may suppose that $f|_{O-\{x_1,\cdots,x_n\}}$ extends to $\bar{f}$: $S \to S$.

If $\chi(S) \geq 0$, then by the above $\bar{f}$ is isotopic to a periodic map $\bar{\varphi}$. Let φ: $O \to O$ be the projection of $\bar{\varphi}$. Then φ is O-isotopic to f, and periodic.

If $\chi(S) < 0$, then by Theorem B $\bar{f}$ is isotopic to Thurston's cannonical form $\bar{\varphi}$. Let φ: $O \to O$ be the projection of $\bar{\varphi}$. Then φ is O-isotopic to f, and we easily see that φ satisfies the conclusion (i), (ii), or (iii) of Theorem 1 according to $\bar{\varphi}$ is periodic, pseudo-Anosov, or reducible.

This completes the proof of Theorem 1.

4. Homeomorphisms of $H^2 \times R$, $\widetilde{SL_2(R)}$ and Nil-manifolds

Throughout this section let M be a compact, orientable 3-manifold with an $H^2 \times R$, $\widetilde{SL_2(R)}$, or Nil structure. In this section we prove Theorem 2 and investigate some properties of homeomorphisms of type periodic and pseudo-Anosov. By [Sc] M admits a Seifert fibration p: $M \to O$ where O is a good 2-dimensional orbifold whose (possibly empty) singularities are all cones and by moving f by a homotopy we may suppose that f is fiber preserving. We note that this deformation can be realized by an isotopy if M is sufficiently large [Wa]. Then we have an O-homeomorphism ψ: $O \to O$ which satisfies:

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ O & \xrightarrow[\psi]{} & O \end{array}$$

A fiber preserving self-homeomorphism f is of *type periodic* if ψ is periodic. f is of *type pseudo-Anosov* if ψ is pseudo-Anosov. Let F be a 2-sided surface properly embedded in a 3-manifold M'. F is *incompressible* if i_*: $\pi_1(F) \to \pi_1(M')$ is injective. Let E be a subset of a Seifert fibered manifold S. E is *saturated* if E is a union of fibers of S. f: $M \to M$ is *reducible* by $\mathcal{T}$ if $\mathcal{T}$ is a system of mutually disjoint, non-parallel incompressible tori, and $f(\mathcal{T}) = \mathcal{T}$. Let q: $N \to A$ be a circle bundle over an annulus A, g: $N \to N$ a fiber preserving homeomorphism, φ: $A \to A$ a map induced from g. g is a *twist homeomorphism* if φ is a twist homeomorphism.

Lemma 4.1. *Let M be an $H^2 \times R$, $\widetilde{SL_2(R)}$, or Nil-manifold with a Seifert fibration, T an incompressible torus in M. Then T is isotopic to a saturated torus.*

Proof. Asumme that T is not isotopic to a saturated torus. By [Sc],

[Wa] Seifert fibrations on M are unique up to isotopy. Then by [J] Theorem VI. 34 we have either:

(i) M is a torus bundle over a circle and T is a fiber of the bundle, or

(ii) $M=M_1\cup M_2$ where $M_1\cap M_2=\partial M_1=\partial M_2=T$, and M_i $(i=1, 2)$ is a twisted I-bundle over the Klein bottle.

Assume that (i) holds. The monodromy of M is represented by a 2×2 matrix $A\in SL_2(Z)$. By [Sc] a torus bundle over the circle admits either an E^3, Nil, or Sol structure, and M admits a Nil structure if and only if A is conjugate to $\begin{pmatrix}\varepsilon & n\\ 0 & \varepsilon\end{pmatrix}$ $(\varepsilon=\pm 1,\ n\neq 0)$. Then we easily see that there is a Seifert fibration on M such that T is saturated with respect to the fibration. This contradicts the assumption. Assume that (ii) holds. By [Sc] M admits an E^3 or Sol structure and does not admit an $H^2\times R$, $\widetilde{SL_2(R)}$, or Nil structure, a contradiction. Hence T is isotopic to a saturated torus.

Proof of Theorem 2. If T is a non-peripheral, saturated, incompressible torus in M, then $p(T)$ is an essential loop on O. Conversely, if a is an essential loop on O, then $p^{-1}(a)$ is a non-peripheral, saturated, incompressible torus in M. Moreover, if a_1, a_2 are mutually non-parallel, essential loops on O, then $p^{-1}(a_1)$, $p^{-1}(a_2)$ are mutually non-parallel incompressible tori. From these facts the proof of Theorem 2 follows immediately from Theorem 1.

Now, we investigate homeomorphisms of type periodic on M. Suppose that f is of type periodic. Let G be a subgroup of Out $\pi_1(M)$ generated by f_*. Let ψ: Out $\pi_1(M)\to$ Out $\pi_1^{\mathrm{orb}}(O)$ be a cannonical homomorphism, where $\pi_1^{\mathrm{orb}}(O)$ denotes the fundamental group of O as an orbifold ([T_2 § 13]). Then we have an exact sequence:

$$1\to \operatorname{Ker}\psi|_G\to G\xrightarrow{\psi}\psi(G)\to 1\,.$$

$\psi(G)$ is a finite cyclic group. If O is non-orientable, then by Kojima [Ko] Ker $\psi|_G$ is a finite group and G itself a finite cyclic group. This fact together with [Zi] implies:

Proposition 4.2. *If O is non-orientable and f: $M\to M$ is a homeomorphism of type periodic, then f is homotopic to a periodic one.*

5. Topological entropy of homeomorphisms of $H^2\times R$, $\widetilde{SL_2(R)}$, or Nil-manifolds

In this section we see that the map φ obtained in Theorem 2 attains the minimal entropy in its homotopy class. Throughout this section, M denotes a compact, orientable $H^2\times R$, $\widetilde{SL_2(R)}$, or Nil manifold with Seifert fibration $M\to O$.

Lemma 5.1. *Let g: $M \to M$ be a fiber preserving homeomorphism. Then $h(g)=h(\psi)$, where ψ: $O \to O$ is the homeomorphism induced from g.*

Proof. By the argument in the proof of Lemma 3.1 of [S-S] we see that $h_C(g)=0$ for each fiber C of M. Hence, by Theorem A we have $h(g)=h(\psi)$.

By Lemma 5.1 we have:

Proposition 5.2. *If f: $M \to M$ is a homeomorphism of type periodic, then $h(f)=0$.*

For the homeomorphisms of type pseudo-Anosov, we have:

Proposition 5.3. *If f: $M \to M$ is a homeomorphism of type pseudo-Anosov, then $h(f)>0$. Moreover, it attains the minimal entropy in its homotopy class.*

Proof. By [Sc] M admits a finite covering p: $\bar{M} \to M$ such that the Seifert bundle structure on M lifts to a circle bundle structure q: $\bar{M} \to S$. We may suppose that some power of f, f^n, lifts to a homeomorphism $\bar{f}$: $\bar{M} \to \bar{M}$. Let $\bar{\psi}$: $S \to S$ be a homeomorphism induced from $\bar{f}$. Then we have $h(f)=1/n \cdot h(f^n)$. By Lemma 5.1 $h(f^n)=h(\bar{f})=h(\bar{\psi})$.

Then we note that $\bar{\psi}$ is a pseudo-Anosov homeomorphism. Let λ (>1) be the expanding factor of f. Then λ^n is the expanding factor of $\bar{\psi}$. By [F-L-P] $h(\bar{\psi})=n \cdot \log \lambda > 0$. Hence, we have $h(f)=\log \lambda$. Since $\chi(S)<0$, there is a loop l in $\bar{M}$ such that $q(l)$ ($\subset S$) is an essential loop of S. We note that $q(\bar{f}(l))=\bar{\psi}(q(l))$ ($\subset S$). Then by [Sc] we see that $L(l') \geq L(q(l'))$ for each loop l' on $\bar{M}$, where $L(l')$ denotes the infimum of the length of loops which are homotopic to l'. Since $\lim_{m \to \infty} 1/m \cdot \log L(\bar{\psi}^m(q(l))=\lambda^n$, we see that $\lim_{m \to \infty} 1/m \cdot \log L(\bar{f}^m(l)) \geq \lambda^n$. Hence if f' is homotopic to $\bar{f}$ then $h(f') \geq h(\bar{f})=n \cdot \log \lambda$. From this we see that f attains the minimal entropy in its homotopy class.

By using the argumant as in the proof of Proposition 2.1 and by Lemma 5.1 we easily have:

Proposition 5.4. *Let φ: $M \to M$ be as in Theorem 2. Then φ attains the minimal entropy in its homotopy class and $h(\varphi)$ is positive if and only if φ contains a component of type pseudo-Anosov.*

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