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BLOCK INTERSECTION NUMBERS OF BLOCK DESIGNS II

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1. Introduction

Let t, v, k and λ be positive integers with $v \geq k \geq t$. A t-(v, k, λ) design is a pair consisting of a v-set Ω and a family $\boldsymbol{B}$ of k-subsets of Ω, such that each t-subset of Ω is contained in just λ elements of $\boldsymbol{B}$. Elements of Ω and $\boldsymbol{B}$ are called points and blocks, respectively. A t-$(v, k, 1)$ design is often called a Steiner system $S(t, k, v)$. A t-(v, k, λ) design is called nontrivial provided $\boldsymbol{B}$ is a proper subfamily of the family of all k-subsets of Ω, then $t<k<v$. In this paper we assume that all designs are nontrivial. For a t-(v, k, λ) design $\boldsymbol{D}$ we use $\lambda_i (0 \leq i \leq t)$ to represent the number of blocks which contain a given set of i points of $\boldsymbol{D}$. Then we have

$$\lambda_i = \frac{\binom{v-i}{t-i}}{\binom{k-i}{t-i}}\lambda = \frac{(v-i)(v-i-1)\cdots(v-t-1)}{(k-i)(k-i-1)\cdots(k-t-1)}\lambda \quad (0 \leq i \leq t).$$

A t-(v, k, λ) design $\boldsymbol{D}$ is called block-schematic if the blocks of $\boldsymbol{D}$ form an association scheme with the relations determined by size of intersection (cf. [3]). Any Steiner system $S(2, k, v)$ $(t=2)$ is block-schematic (cf. [2]). For a block B of a t-(v, k, λ) design $\boldsymbol{D}$ we use $x_i(B)$ $(0 \leq i \leq k)$ to denote the number of blocks each of which has exactly i points in common with B. If, for each i $(i=0, \cdots, k)$, $x_i(B)$ is the same for every block B, we say that $\boldsymbol{D}$ is block-regular and we write x_i instead of $x_i(B)$. Any Steiner system $S(t, k, v)$ is block-regular (cf. [6]), and any block-schematic t-(v, k, λ) design is also block-regular.

Atsumi [1] proved

Result 1. If a Steiner system $S(t, k, v)$ is block-schematic with $t \geq 3$, then $v \leq k^4 \binom{k}{\left[\frac{k}{2}\right]}$ holds.

Yoshizawa [7] extended Result 1 and prove

Result 2. (a) For each $n \geq 1$ and $\lambda \geq 1$, there exist at most finitely many block-schematic t-(v, k, λ) designs with $k-t=n$ and $t \geq 3$.

(b) For each $n \geq 1$ and $\lambda \geq 2$, there exist at most finitely many block-schematic t-(v, k, λ) designs with $k-t=n$ and $t \geq 2$.

In §2 we first prove the following proposition, and we prove the following theorem related to the above results.

Proposition. *$x_{t-1}^2 \geq x_0$ holds for any block-schematic Steiner system $S(t, k, v)$ with $k \geq 2(t-1)$.*

Theorem 1. *Let ε be a positive real number. Then for each $t \geq 3$ there exist at most finitely many block-schematic Steiner systems $S(t, k, v)$ with $v < k^{2-\varepsilon}$, and for each $t > \dfrac{2}{\varepsilon}+2$ there exist at most finitely many block-schematic Steiner systems $S(t, k, v)$ with $v > k^{3+\varepsilon}$.*

Yoshizawa [7] proved the following result about block-regular designs.

Result 3. Let c be a real number with $c > 2$. Then for each $n \geq 1$ and $l \geq 0$, there exist at most finitely many block-regular t-(v, k, λ) designs with $k-t=n$, $v \geq ct$ and $x_i \leq l$ for some i $(0 \leq i \leq t-1)$.

In §3 we notice that the block-regularity of Result 3 is essentially unnecessary, and we prove

Theorem 2. *Let c be a real number with $c > 2$, and n, l be integers with $n \geq 1$, $l \geq 0$. Then there exist at most finitely many t-(v, k, λ) designs each of which satisfies the following conditions:* (i) $k-t=n$, (ii) $v \geq ct$, (iii) *there exist a block B and an integer $i (0 \leq i \leq t-1)$ with $x_i(B) \leq l$.*

2. Proof of Theorem 1

Let $\boldsymbol{D}$ be a t-(v, k, λ) design. Let $B_1, \cdots, B_{\lambda_0}$ be the blocks of $\boldsymbol{D}$, and $A_h (0 \leq h \leq k)$ be the h-adjacency matrix of $\boldsymbol{D}$ of degree λ_0 defined by

$$A_h(i,j) = \begin{cases} 1 & \text{if } |B_i \cap B_j| = h\,, \\ 0 & \text{otherwise.} \end{cases}$$

If $\boldsymbol{D}$ is block-schematic, then

$A_i A_j = \sum_{h=0}^{k} \mu(i, j, h)\, A_h (0 \leq i, j \leq k)$ where $\mu(i, j, h)$ is a non-negative integer defined by the following: When there exist blocks B_p and B_q with $|B_p \cap B_q| = h$,

$$\mu(i, j, h) = |\{B_r\colon\ |B_p \cap B_r| = i,\ |B_q \cap B_r| = j,\ 1 \leq r \leq \lambda_0\}|\,,$$

and when there exist no blocks B_p and B_q with $|B_p \cap B_q| = h$, $\mu(i, j, h) = 0$. Let $\boldsymbol{a}$ be the all -1 column vector of degree λ_0. Then

$$A_i A_j\, \boldsymbol{a} = \sum_{h=0}^{k} \mu(i, j, h)\, A_h\, \boldsymbol{a}\,.$$

Hence we have

Lemma 1. *For a block-schematic t-(v, k, λ) design,* $x_i x_j = \sum_{h=0}^{k} \mu(i, j, h) x_h$ *holds* $(0 \leq i, j \leq k)$.

REMARK. Lemma 1 is essentially well-known (cf. [1], [7]).

Lemma 2. *Let $\boldsymbol{D}$ be a Steiner system $S(t, k, v)$ with $t \geq 2$ and $k \geq 2(t-1)$. If $(t, k, v) \neq (4, 7, 23)$, $(2, n+1, n^2+n+1)$ $(n \geq 2)$, then there exist three blocks B_1, B_2 and B_3 of $\boldsymbol{D}$ such that $|B_1 \cap B_2| = |B_1 \cap B_3| = t-1$ and $|B_2 \cap B_3| = 0$.*

Proof. By [6] we have

$$x_{t-1} = (\lambda_{t-1} - 1) \binom{k}{t-1} = \frac{v-k}{k-t+1} \binom{k}{t-1} > 0 .$$

Hence we may assume that there exist two blocks B_1 and B_2 with $|B_1 \cap B_2| = t-1$. Since $k \geq 2(t-1)$, $B_1 - B_2$ has (distinct) $t-1$ points $\alpha_1, \cdots, \alpha_{t-1}$. Let M_1 $(=B_1)$, $M_2, \cdots, M_{\lambda_{t-1}}$ be the blocks which contain $\alpha_1, \cdots, \alpha_{t-1}$. If $M_i \cap B_2 = \phi$ for some i $(2 \leq i \leq \lambda_{t-1})$, then $|B_1 \cap M_i| = t-1$ and $|B_2 \cap M_i| = 0$ hold. Let us suppose $M_i \cap B_2 \neq \phi$ for $i = 2, \cdots, \lambda_{t-1}$. Then we have

$$\frac{v-t+1}{k-t+1} - 1 \leq k-t+1 . \tag{1}$$

On the other hand by Theorems 3A. 3 and 4 in [4], we have $v-t+1 \geq (k-t+2)(k-t+1)$, with equality only when $(t, k, v) = (2, n+1, n^2+n+1)$ $(n \geq 2)$, $(3, 4, 8)$, $(3, 6, 22)$, $(3, 12, 112)$, $(4, 7, 23)$ or $(5, 8, 24)$. Hence by (1) and the assumption of Lemma 2, we have $(t, k, v) = (3, 4, 8)$, $(3, 6, 22)$, $(3, 12, 112)$ or $(5, 8, 24)$. But we can easily check that $S(3, 4, 8)$, $S(3, 6, 22)$, $S(3, 12, 112)$ and $S(5, 8, 24)$ satisfy the conclusion of Lemma 2 if $S(3, 12, 112)$ exists (cf. [5, Corollary 1]).

Proof of Proposition. Let us suppose that $\boldsymbol{D}$ is a block-schematic Steiner system $S(t, k, v)$ with $k \geq 2(t-1)$. Then by Lemma 1, we have

$$x_{t-1}^2 = \sum_{h=0}^{k} \mu(t-1, t-1, h)\, x_h .$$

Now by Lemma 2, $\mu(t-1, t-1, 0) > 0$ or $x_0 = 0$ holds when $k \geq 2(t-1)$ and $t \geq 2$ hold. Hence we have $x_{t-1}^2 \geq x_0$.

Proof of Theorem 1. First let us suppose that $\boldsymbol{D}$ is a block-schematic Steiner system $S(t, k, v)$ with $t \geq 3$ and $v < k^{2-\varepsilon}$. By Theorems 3A. 3 and 4 in [4], we have $v \geq (k-t+2)(k-t+1)+t-1$, where the right hand of this inequality is a polynomial in k of degree two. Hence there exists a positive number $N(\varepsilon, t)$ with $k<N(\varepsilon, t)$, where $N(\varepsilon, t)$ depends only on ε and t. Hence by Result 1, v is bounded above by a function of ε and t.

Next suppose that $\boldsymbol{D}$ is a block-schematic Steiner system $S(t, k, v)$ with $t>\dfrac{2}{\varepsilon}+2$ and $v>k^{3+\varepsilon}$.

By [6] we have

$$x_{t-1}^2=(\lambda_{t-1}-1)^2\binom{k}{t-1}^2=\frac{(v-k)^2}{(k-t+1)^2}\binom{k}{t-1}^2 \tag{2}$$

By [5, Lemma 6] (or [7, Lemma 5]) we have

$$x_0=\left\{\binom{v-k}{k}+(-1)^{t+1}\sum_{q=0}^{k-t-1}\binom{t-1+q}{q}\binom{v-k+q}{k-t}\right\}\Big/\binom{v-t}{k-t},$$

$$x_0\geq\frac{\binom{v-k}{k}}{\binom{v-t}{k-t}}-\frac{(k-t)\binom{k-2}{k-t-1}\binom{v-t-1}{k-t}}{\binom{v-t}{k-t}}. \tag{3}$$

Hence by (2) and (3) we have

$$x_0-x_{t-1}^2\geq\frac{(v-k)\cdots(v-2k+1)}{(v-t)\cdots(v-k+1)k\cdots(k-t+1)}-(k-t)(k-2)^{t-1}-\frac{(v-k)^2}{(k-t)^2}\binom{k}{t-1}^2,$$

$$x_0-x_{t-1}^2\geq\frac{(v-2k)^k}{v^{k-t}\,k^t}-k^t-\frac{v^2k^{2t-2}}{(k-t)^2}.$$

Since we may assume $k\geq 2t$ by Result 1, we have

$$x_0-x_{t-1}^2\geq\frac{(v-2k)^k}{v^{k-t}\,k^t}-k^t-4v^2\,k^{2t-4}\geq\frac{(v-2k)^k}{v^{k-t}\,k^t}-5v^2\,k^{2t-4}.$$

On the other hand by Proposition, $x_{t-1}^2\geq x_0$ holds because of $k\geq 2t$. Thus we get

$$(v-2k)^k-5v^{k-t+2}\,k^{3t-4}\leq 0. \tag{4}$$

Since $v>k^3$, we have

$$\frac{(v-2k)^k}{5v^{k-t+2}\,k^{3t-4}}=\frac{\left(1-\frac{2k}{v}\right)^{k-t+2}(v-2k)^{t-2}}{5k^{3t-4}}\geq\frac{\left(1-\frac{1}{k}\right)^k(v-2k)^{t-2}}{5\left(1-\frac{1}{k}\right)^{t-2}k^{3t-4}}. \tag{5}$$

Since $\lim_{n\to\infty}\left(1-\dfrac{1}{n}\right)^n=\dfrac{1}{e}$, where e is the Napier number, there is a positive number $M_1(t)$ which depends only on t, such that $\left(1-\frac{1}{n}\right)^n \Big/ \left\{5\left(1-\frac{1}{n}\right)^{t-2}\right\} >$ $M_1(t)$ holds for all integer $n \geq 2$. On the other hand, $(v-2k)^{t-2}/k^{3t-4} \geq v^{t-2}/\{2^{t-2}k^{3t-4}\}$ holds. Hence by (5), we have

$$\frac{(v-2k)^k}{5v^{k-t+2}k^{3t-4}} \geq M_2(t)\,\frac{v^{t-2}}{k^{3t-4}}\,,$$

where $M_2(t)$ is a positive number which depends only on t. Since $v > k^{3+\varepsilon}$ and $t > \frac{2}{\varepsilon}+2$, there exists a positive number $M_3(\varepsilon, t)$ which depends only on ε and t, such that

$$\frac{(v-2k)^k}{5v^{k-t+2}k^{3t-4}} > 1 \text{ holds for any } k \geq M_3(\varepsilon, t)\,.$$

Hence by (4), we must have $k < M_3(\varepsilon, t)$. Hence by Result 1, v is bounded above by a function of ε and t.

3. Proof of Theorem 2

The proof of Theorem 2 is essentially similar to that of Theorem 2 in [7]. So we give its outline.

Let $\boldsymbol{D}$ be a t-(v, k, λ) design, and B be a block of $\boldsymbol{D}$. Counting in two ways the number of the following set

$\{(B', \{\alpha_1, \cdots \alpha_i\})$: B' a block $(\neq B)$, $B' \cap B \ni \alpha_1, \cdots \alpha_i$, $\alpha_j \neq \alpha_{j'}$ if $j \neq j'\}$
gives

$$x_i(B)+\binom{i+1}{i}x_{i+1}(B)+\cdots+\binom{t}{i}x_t(B)+\cdots+\binom{k-1}{i}x_{k-1}(B) = (\lambda_i-1)\binom{k}{i} \tag{6}$$

for $i=0, \cdots, t-1$, and

$$x_i(B)+\binom{i+1}{i}x_{i+1}(B)+\cdots+\binom{k-1}{i}x_{k-1}(B) \leq (\lambda-1)\binom{k}{i} \tag{7}$$

for $i=t, \cdots, k-1$. Let $w_i(B)$ $(t \leq i \leq k-1)$ be the left hand of the above inequality, where $w_t(B)=(\lambda-1)\binom{k}{t}$.

By (6) and (7) we have

$$x_i(B) = \sum_{j=0}^{t-1}\binom{j}{i}(\lambda_j-1)\binom{k}{j}(-1)^{i+j}+\sum_{j=t}^{k-1}\binom{j}{i}w_j(B)(-1)^{1+j}\,, \tag{8}$$

for $i=0, \cdots, t-1$ (cf. [7, Proof of Lemma 1]). By (8) we have that there exists a positive number $C(k, l, t, i)$ which depends only on k, l, t, i, such that $x_i(B)-l>0$ holds if $v \geq C(k, l, t, i)$ (cf. [7, Proof of Lemma 6]). Namely, $v<C(k, l, t, i)$ holds if $x_i(B) \leq l$. Hence we get

Lemma 3. *For each $k \geq 2$ and $l \geq 0$, there exist at most finitely many t-(v, k, λ) designs each of which satisfies that there exists a block B and an integer i $(0 \leq i \leq t-1)$ with $x_i(B) \leq l$.*

Proof of Theorem 2. By Lemma 3 we may assume that $t \geq \frac{2n+((2n+2)\,!)^2}{c-2}+2n$. Let $\boldsymbol{D}$ be a t-(v, k, λ) design satisfying $v \geq ct$ and $t \geq \frac{2n+((2n+2)\,!)^2}{c-2}+2n$. Set $v=mt$ $(m \geq c)$, where m is not always integral.
By (8) we have

$$x_i(B)=\frac{\lambda\binom{k}{i}}{\binom{v-t}{k-t}}\left\{\binom{v-k}{k-i}+(-1)^{t+i+1}\sum_{q=0}^{k-t-1}\binom{t-i-1+q}{q}\binom{v-k+q}{k-}\right\}$$
$$+(\lambda-1)\sum_{j=i}^{t-1}\binom{j}{i}\binom{k}{j}(-1)^{i+j}+\sum_{j=t}^{k-1}\binom{j}{i}w_j(B)(-1)^{i+j},$$

where $x_j(B) \leq w_j(B) \leq (\lambda-1)\binom{k}{j}=(\lambda-1)\binom{t+n}{j}$ $(t \leq j \leq k-1)$ (cf. [7, Proof of Lemma 5]). Hence we get

$$x_i(B)=\frac{\lambda\binom{t+n}{i}}{\binom{(m-1)\,t}{n}}\left\{\binom{(m-1)\,t-n}{t+n-i}+(-1)^{t+i+1}\sum_{q=0}^{n-1}\binom{t-i-1+q}{q}\binom{(m-1)t-n+q}{n}\right\}$$
$$+(\lambda-1)\sum_{j=i}^{t-1}\binom{j}{i}\binom{t+n}{j}(-1)^{i+j}\sum_{j=t}^{t+n-1}\binom{j}{i}w_j(B)(-1)^{i+j},$$

for $i=0, \cdots, t-1$. By the above equality and the condition on t, we have

$$x_i(B)>\frac{((c-1)\,t-n)}{((n+1)\,!)^2}\left(\frac{c-2}{c-1}\right)^n-5n \qquad \text{(cf. [7, pp. 797, 798])}.$$

We remark that the right hand of the above inequality does not depend on i. Hence there exists a positive number $N(c, n, l)$ $\left(\geq \frac{2n+((2n+2)\,!)^2}{c-2}+2n\right)$ which depends only on c, n, l, such that $x_i(B)-l>0$ holds for $i=0, \cdots, t-1$ if $t \geq N(c, n, l)$. Namely, $t<N(c, n, l)$ holds if $x_i(B) \leq l$ holds for some i $(0 \leq i \leq t-1)$. Hence by Lemma 3, we complete the proof.

References

[1] T. Atsumi: *An extension of Cameron's result on block schematic Steiner systems*, J. Combin. Theory Ser. A **27** (1979), 388–391.

[2] R.C. Bose: *Strongly regular graphs, partial geometries, and partially balanced designs*, Pacific J. Math. **13** (1963), 389–419.

[3] P.J. Cameron: *Two remarks on Steiner systems*, Geom. Dedicata **4** (1975), 403–418.

[4] P.J. Cameron: Parallelisms of complete designs, London Math. Soc. Lecture note series No. 23, Cambridge Univ. Press, Cambridge 1976.

[5] B.H. Gross: *Intersection triangles and block intersection numbers for Steiner systems*, Math. Z. **139** (1974), 87–104.

[6] N.S. Mendelsohn: *A theorem on Steiner systems*, Canad. J. Math. **22** (1970), 1010–1015.

[7] M. Yoshizawa: *Block intersection numbers of block designs*, Osaka J. Math. **18** (1981), 787–799.

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