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ON FREE BOUNDARY PLATEAU PROBLEM FOR GENERAL DIMENTIONAL SURFACES

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The aim of this paper is to deal with a free boundary Plateau problem following Reifenberg's method.

The so-called Plateau problem of seeking surfaces of least area bounded by a prescribed contour has been attacked from various aspects. Classical approaches due to Radó, Douglas and Courant have limited the problem to the case of dimension two, where the admissible surface is a mapping image of some fixed parameter domain. Reifenberg is one of the first who considered minimal surfaces, with a given boundary, of various topological types simultaneously. In his pioneering paper [11], [12] and [13] he fixed a compact subset A of R^p and regarded any compact set X containing A as a surface with boundary A so far as X spans A homologically.

In the present paper we are concerned with a free boundary problem which is not discussed in Reifenberg's studies. It will be natural to define the free boundary of a surface homologically. We formulate a free boundary problem approximately and prove existence and regularity results with some simple examples illustrating our situation. The result of this paper remains valid in case the ambient space is a Riemannian manifold, which is close to the Euclidean space in the sense of Morrey [10], especially, a compact Riemannian manifold.

1. Formulation of free boundary problem

In this section we formulate the free boundary problem mentioned above. Suppose that we are given a compact subset E of R^p. It is just the set on which all our free boundaries should lie. For any compact subset X of R^p, we define $FB(X):=X\cap E$ to be free boundary of X on E. Then the inclusion maps

$$(1) \qquad i\colon FB(X)\hookrightarrow X\,,\quad j\colon FB(X)\hookrightarrow E$$

induce the following diagram

$$
\begin{array}{ccccc}
H_m(X, FB(X); G) & \xrightarrow{\partial} & H_{m-1}(FB(X); G) & \xrightarrow{i_*} & H_{m-1}(X; G) \\
 & & \downarrow j_* & & \\
 & & H_{m-1}(E; G), & &
\end{array}
$$

where G is a fixed compact abelian coefficient group.
Though we want to regard Ker i_* as its "algebraic boundary" following Reifenberg, unfortunately $H_{m-1}(FB(X); G)$ changes as X varies. Then we measure its boundary in $H_{m-1}(E; G)$: We call $j_*(\text{Ker } i_*)=\text{Im}(j_* \circ \partial)$ to be *algebraic free boundary* of X on E.

Definition 1. Take and fix a subgroup Γ of $H_{m-1}(E; G)$. We say that X is a *surface with free boundary including* Γ if and only if the algebraic free boundary of X contains Γ. $\mathfrak{g}_{\text{free}}(\Gamma)$ denotes the totality of such X.

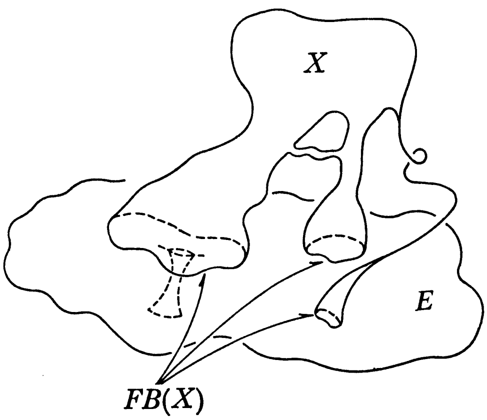

Fig. 1

The m-dimensional volume of X shall be now

$$\text{Vol}^m(X) = \mathcal{H}^m(X \setminus FB(X)),$$

where $\mathcal{H}^m$ denotes m-dimensional Hausdorff measure, and accordingly

$$V := \inf \{\text{Vol}^m(X) ; X \in \mathfrak{g}_{\text{free}}(\Gamma)\} .$$

We are in a position to raise a question.

Problem FB. Is there any X^* of $\mathfrak{g}_{\text{free}}(\Gamma)$ satisfying $\text{Vol}^m(X^*)=V$?

The answer is affirmative and we have

Theorem A. *There exists at least one solution to Problem FB.*

Moreover we can easily show the following result, using the regularity of Reifenberg's fixed boundary solution:

Theorem B. *There exists a solution of Problem FB which is real analytic* $\mathcal{H}^m$*-almost everywhere in the interior.*

2. Some examples

In this section we observe some simple examples helpful to understand our formulation in §1. In dealing with free boundary problems in general, one usually requires some classical linking condition (cf. Courant [5], pp. 213-218) or a homotopical non-triviality (cf. Meeks-Yau [9], §1). In more general cases in which the set of free boundary does not lie on a manifold, careful treatment is needed. Indeed the mere non-retract condition is insufficient to guarantee the non-degeneracy as the following simple example illustrates:

EXAMPLE 1. Given a infinite ladder

$$\begin{aligned} E: = & \{(x, y, 0);\ x-2y = 0,\ 0 \leqq x \leqq 1\} \\ & \cup \{(x, y, 0);\ x+2y = 0,\ 0 \leqq x \leqq 1\} \\ & \cup \bigcup_{n=1}^{\infty} \{(1/n, y, 0);\ -1/2n \leqq y \leqq 1/2n\} \end{aligned}$$

lying on the (x, y)-plane in R^3, we consider a free boundary problem with a non-degenerate condition such as non-retract or homological (homotopical) non-triviality. Obviously such a condition is not adequate in this situation, while E is compact. Indeed we may take the plane domains

$$X_n: = \{(x, y, 0);\ x-2y \geqq 0,\ x+2y \geqq 0,\ 1/(n+1) \leqq x \leqq 1/n\}\ , \quad (n=1, 2, \cdots)$$

as a minimizing sequence for this free boundary problem: X_n is not retract to ∂X_n, and ∂X_n is non-trivial as an element of both $H_1(E; G)$ and $\pi_1(E)$. But $\{X_n\}_{n=1,2,\dots}$ degenerates to the origin.

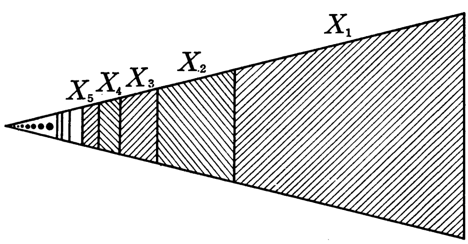

Fig. 2

EXAMPLE 2 (a disk with two wiry handles). Consider the configuration in R^3

$$E = C_1 \cup C_2 \cup D$$

with

$$C_n := \{((-1)^n r/2, \cos\theta, \sin\theta);\ 0 \leqq \theta \leqq \pi\}\ (n = 1, 2),$$
$$D := \{(x, y, 0);\ x^2 + y^2 \leqq R\} \quad (R \gg 1).$$

We look for the surface of least area spanning partly the two semi-circles C_1, C_2 and partly the free boundary on D. The solution will be either simply connected or of degenerate type according as which of them provides smaller area. This alternative depends on r, or equivalently the proportion of the distance to the size of handles. Courant showed the existence of connected type under a sufficient condition that the infimum of areas in all simply connected surfaces is smaller than that in all surfaces of degenerate type (cf. Courant [5], pp. 208–209).

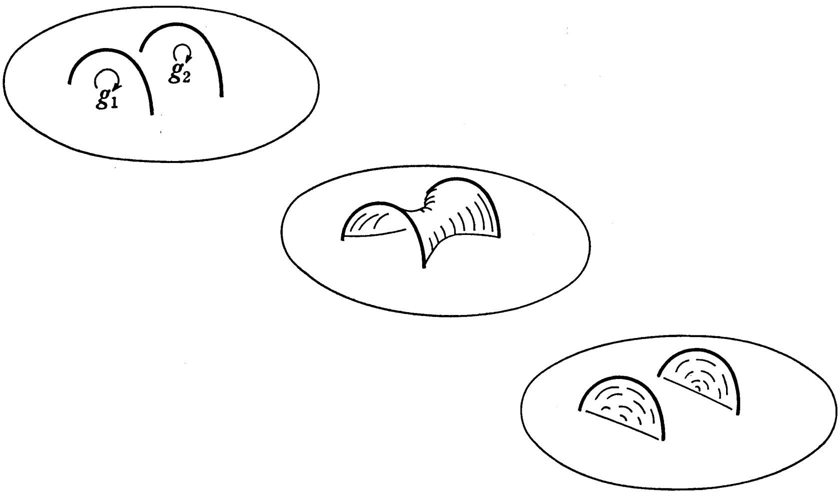

Fig. 3

Let us consider the same problem in our formulation. We take the 1-chains g_1, g_2 as homological basis, $\check{H}_1(E;\ G) = \langle g_1 \rangle \oplus \langle g_2 \rangle$, and $\Gamma = \langle g_1 - g_2 \rangle$. (See Fig. 3. In this case we can identify the Čech homology group with the simplicial homology group because E is regarded as a singular simplicial complex.) Note that X necessarily spans C_1 and C_2 owing to the definition requiring that free boundary includes Γ in the present case. Both surfaces of the above two types are the ones with free boundary including Γ.

EXAMPLE 3 (cf. Reifenberg [11], p. 80, Almgren [1], p. 6).

$$T = ((2+\cos\theta)\cos\varphi,\ (2+\cos\theta)\sin\varphi,\ \sin\theta);\ 0 \leqq \theta,\ \varphi \leqq 2\pi\}$$

is the 2-torus obtained by rotating about the z-axis the unit meridian circle

$(x-2)^2+z^2=1$ lying on the (x, z)-plane. On the other hand

$$X_\varphi := \{((2+r\cos\theta)\cos 3(\theta+\varphi), (2+r\cos\theta)\sin 3(\theta+\varphi), \sin\theta); \\ 0\leqq\theta\leqq 2\pi, 0\leqq r\leqq 1\}$$

is a triple Möbius strip whose boundary lies on T. If we take $E=T\setminus\bigcup_{|\varphi|<\varepsilon} X_\varphi$ (ε: sufficiently small positive) for the manifold on which the free boundary should lie, we have the isomorphisms

$$H_1(E; G) \cong H_1(S^1; G) \cong G.$$

Then X_φ is admissible and attains the minimum of least area for some subgroups of $H_1(E; G)$. If some X_{φ_0} $(\varepsilon\leqq\varphi_0\leqq 2\pi/3-\varepsilon)$ is a solution, then any X_φ $(\varepsilon\leqq\varphi\leqq 2\pi/3-\varepsilon)$ is also a solution because the latter is obtained by rotating X_{φ_0} through the angle $\varphi-\varphi_0$. In case $G=Z_3$, all X_φ $(\varepsilon\leqq\varphi\leqq 2\pi/3-\varepsilon)$ are solution surfaces with free boundary including $\Gamma=H_1(E; Z_3)$.

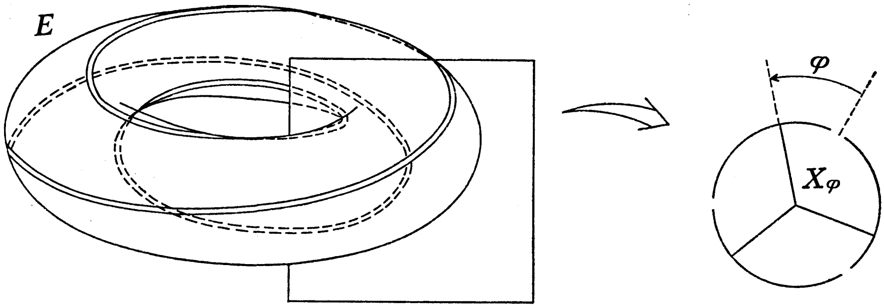

Fig. 4-1

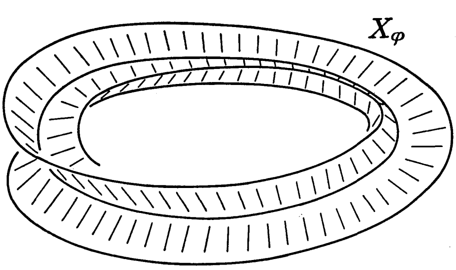

Fig. 4-2

3. Proof of Theorem *A*, *B*

For a compact subset A of R^p and a compact subgroup Δ of $H_{m-1}(A; G)$, we use the notation $\mathfrak{g}(\Delta)$ for the class of all the *surface with boundary including* Δ, where $X\,(\supset A)$ is a surface with boundary including Δ if and only if Δ is contained in the kernel of the homomorphism

$$(\mathrm{inc})_*: H_{m-1}(A; G)\to H_{m-1}(X; G)$$

induced by the inclusion map

$$\mathrm{inc}: A\to X$$

(Reifenberg [11], p. 79, Definition).

We note here the following fact asserting that $\mathfrak{g}_{\mathrm{free}}(\Gamma)$ is closed with respect to an appropriate patching:

Lemma 1 (Patching lemma). *Let X be a surface with free boundary including Γ, i.e. $X\in\mathfrak{g}_{\mathrm{free}}(\Gamma)$.*

1° *If we set $\Delta:=\mathrm{Ker}\ \{i_*: H_{m-1}(FB(X); G)\to H_{m-1}(X; G)\}$, then $Y\in\mathfrak{g}(\Delta)$ implies $Y\in\mathfrak{g}_{\mathrm{free}}(\Gamma)$.*

2° *Further let U be an open set in R^p such that $U\cap X\neq\phi$, $\bar{U}\cap E=\phi$, and let $\Delta^\dagger:=\mathrm{Ker}\ \{(i^\dagger)_*: H_{m-1}(X\cap\partial U; G)\to H_{m-1}(X\cap\bar{U}; G)$, where $i^\dagger: X\cap\partial U \hookrightarrow X\cap U$. Then $Y\in\mathfrak{g}(\Delta^\dagger)$ implies $\hat{X}:=(X\setminus U)\cup Y\in\mathfrak{g}_{\mathrm{free}}(\Gamma)$.*

Proof. Consider the inclusion maps i': $FB(X)\hookrightarrow Y$ in addition to (1). We have then $j_*(\mathrm{Ker}\ (i')_*)\supset j_*(\Delta)=j_*(\mathrm{Ker}\ i_*)\supset\Gamma$, which proves 1°. Next introducing the three more inclusion maps $i^{\dagger\dagger}$: $\{(X\cap\partial U)\cup FB(X)\}\hookrightarrow X\setminus U$, k: $X\cap\partial U\hookrightarrow\{(X\cap\partial U)\cup FB(X)\}$ and l: $FB(X)\hookrightarrow\{(X\cap\partial U)\cup FB(X)\}$, we observe the kernels $\Delta^{\dagger\dagger}:=\mathrm{Ker}\ (i^{\dagger\dagger})_*$, Δ and $\Delta^\dagger$. Then we see $l_*\Delta=k_*\Delta^\dagger+\Delta^{\dagger\dagger}$ (Lemma 12A in Reifenberg [11]), and hence $\hat{X}\in\mathfrak{g}(\Delta)$ (Lemma 11A in Reifenberg [11]). The just proved result 1° enable us to conclude that $\hat{X}\in\mathfrak{g}_{\mathrm{free}}(\Gamma)$. q.e.d.

As mentioned in §1, Theorem B follows from Theorem A and the regularity of Reifenberg's fixed boundary solution. Assuming Theorem A, we first prove Theorem B:

Let X^* denote a solution of Problem FB, of which the existence is guaranteed by Theorem A. We consider the Reifenberg's fixed boundary problem for the compact boundary $FB(X^*)$ and the subgroup $\Delta^*:=\mathrm{Ker}\ \{(i^*)_*: H_{m-1}(FB(X^*); G)\to H_{m-1}(X^*; G)\}$, where i: $FB(X^*)\hookrightarrow X^*$. Then his solution Y for this problem obviously satisfies $\mathrm{Vol}^m(Y)\leq\mathrm{Vol}^m(X^*)$. By 1° of Lemma 1, Y belongs to $\mathfrak{g}_{\mathrm{free}}(\Gamma)$ since $Y\in\mathfrak{g}(\Delta^*)$. Hence Y is also a solution of Problem FB, while it is real analytic $\mathcal{H}^m$-almost everywhere in the interior ([12], [13]). Thus we obtain Theorem B.

We will prove Theorem A. Note first that the Hausdorff distance is introduced in $\mathfrak{g}_{\mathrm{free}}(\Gamma)$ such as in $\mathfrak{g}(\Delta)$. In this paper a limit with respect to the Hausdorff distance is abbreviated, for simplicity, to a Hausdorff limit. Further the $\bar{F}$-limit of a sequence $\{X_n\}_{n=1}^{\infty}$ of a point sets is given by $\bar{F}\text{-}\lim_{n\to\infty} X_n$:

$=\bigcap_{n=1}^{\infty}(\overline{\bigcup_{i=n}^{\infty}X_i})$ (see [8], p. 146). The proof of Theorem A, as in the fixed boundary case, consists of a triple of steps, which are stated as assertions below.

Assertion 1. $\mathfrak{g}_{\text{free}}(\Gamma)$ *is locally sequentially compact with respect to the Hausdorff limit and* $\bar{F}$*-limit.*

The proof is a direct consequence of the following three lemmas. Though these has a counterpart in the fixed boundary case, the Čech homology as well as the category of compact groups play determinant roles in the proof of Lemma 4 which is the main portion, since in general homology theories the image of morphism between groups is not necessarily continuous with respect to the projective limit (see [6], p. 227 Remark).

Lemma 2 (Hausdorff [8], pp. 148–150). *In a compact metric space an arbitrary sequence of closed (therefore compact) sets* X_n $(n=1, 2, \cdots)$ *has the Hausdorff limit and* $\bar{F}$*-limit, and both coincide.*

Lemma 3. *Let* X_1 *belongs to* $\mathfrak{g}_{\text{free}}(\Gamma)$. *Then any surface* X_2 *containing* X_1 *as a subset also belongs to* $\mathfrak{g}_{\text{free}}(\Gamma)$.

Proof. We use the following notations for the running index $n=1, 2$: i_n: $FB(X_n)\hookrightarrow X_n$, j_n: $FB(X_n)\hookrightarrow E$, k: $FB(X_1)\hookrightarrow FB(X_2)$, 1: $(X_1, FB(X_1))\hookrightarrow(X_2, FB(X_2))$ and ∂_n: $H_m(X_n, FB(X_n); G)\to H_{m-1}(FB(X_n); G)$. Then $\text{Im}((j_2)_*\circ\partial_2)\supset \text{Im}((j_2)_*\circ\partial_2\circ 1_*)=\text{Im}((j_2)_*\circ k_*\circ\partial_1)=\text{Im}((j_1)_*\circ\partial_1)\supset\Gamma$. q.e.d.

Lemma 4. *Let a sequence* $\{X_n\}_{n=1}^{\infty}$ *of* $\mathfrak{g}_{\text{free}}(\Gamma)$ *satisfying* $X_1\supset X_2\supset\cdots$ *Then* $X:=\bigcap_{n=1}^{\infty}X_n$ *belongs to* $\mathfrak{g}_{\text{free}}(\Gamma)$.

Proof. We recall fundamental properties of Čech homology with a compact abelian coefficient group (see Eilenberg–Steenrod [6], Chap. X):

(a) Čech homology is the homology theory which take values in the category of compact abelian groups.

(b) Čech homology is natural and continuous with respect to the projective limit.

Since $\varprojlim_n FB(X_n)=FB(X)$, we have in view of (b)

$$\varprojlim_n H_{m-1}(FB(X_n); G) = H_{m-1}(\varprojlim_n FB(X_n); G) = H_{m-1}(FB(X); G)\,.$$

Let the inclusion i, j, i_n, j_n be earlier, except that n runs 1 to infinity. Then it follows from (b) that

$$\begin{aligned} j_*(\text{Ker } i_*) &= (\varprojlim_n j_n)_* \, (\text{Ker } (\varprojlim_n i_n)_*) \\ &= (\varprojlim_n (j_n)_*) \, (\varprojlim_n \text{Ker } (i_n)_*)\,. \end{aligned} \tag{2}$$

We consider the exact sequence

$$0 \to L_n \to H_{m-1}(FB(X_n); G),$$

where $L_n := \operatorname{Ker}(i_n)_*$. Passing to the limit, we still have

$$0 \to \varprojlim_n L_n \to \varprojlim_n H_{m-1}(FB(X_n); G) = H_{m-1}(FB(X); G)$$

because the projective limit preserves the exactness in the category of compact groups (Eilenberg-Steenrod [6], Chap. VIII Theorem 5.6). Then we have

$$\varprojlim_n (j_n)_* \Big|_{\varprojlim_n L_n} = \varprojlim_n \left(j_n \Big|_{L_n}\right).$$

Hence

$$(3) \qquad \begin{aligned} (\varprojlim_n (j_n)_*)(\varprojlim_n L_n) &= \operatorname{Im}\left(\varprojlim_n (j_n)_* \Big|_{\varprojlim_n L_n}\right) \\ &= \operatorname{Im} \varprojlim_n \left((j_n)_* \Big|_{L_n}\right) = \varprojlim_n \operatorname{Im}\left((j_n)_* \Big|_{L_n}\right) \end{aligned}$$

(see Eilenberg–Steenrod [6], p. 227 Remark). On the other hand

$$(4) \qquad \begin{aligned} \operatorname{Im}\left((j_n)_* \Big|_{L_n}\right) &= (j_n)_*(L_n) \\ &= (j_n)_*(\operatorname{Ker}(i_n)^*) \supset \Gamma, \end{aligned}$$

since $X_n \in \mathfrak{g}_{\text{free}}(\Gamma)$. Therefore it follows from (2), (3) and (4) that

$$j_*(\operatorname{Ker} i_*) = \varprojlim_n \operatorname{Im}\left((j_n)_* \Big|_{L_n}\right) \supset \Gamma,$$

i.e. $X \in \mathfrak{g}_{\text{free}}(\Gamma)$. q.e.d.

Since the Hausdorff measure is not semi-continuous over $\mathfrak{g}_{\text{free}}(\Gamma)$ (cf. Almgren [2], p. 57 (1)), it is necessary to find a nice minimizing sequence such as in the fixed boundary case. We can reduce its existence to Reifenberg's results, by taking a sequence each of whose terms is his fixed boundary solution:

Assertion 2. *There exists a sequence* $\{X_n\}_{n=1}^{\infty}$ *satisfying the following two conditions*:

(i) $\{X_n\}_{n=1}^{\infty}$ *is a minimizing sequence for Problem FB, i.e.* $X_n \in \mathfrak{g}_{\text{free}}(\Gamma)$ *for all* n, *and* $\operatorname{Vol}^m(X_n) \to V$ *as* $n \to \infty$.

(ii) *If* $B(P, r) \cap FB(X_n) = \phi$ *and* $P \in X_n$, *then*

$$\mathcal{H}^m(X_n \cap B(P, r)) / \alpha(m)\, r^m \geqq 1,$$

where $B(P, r)$ denotes the closed ball centered at P with radius r, and $\alpha(m)$ is the volume of m-dimensional unit ball.

Proof. We take an arbitrary minimizing sequence $\{Y_n\}_{n=1}^{\infty}$ for our free boundary problem. Consider, for each n, the fixed boundary problem for a compact set $FB(Y_n)$ and a subgroup $\Delta_n := \text{Ker}\ \{(i_n)_*: H_{m-1}(FB(Y_n); G) \to H_{m-1}(Y_n; G)\}$, where $i_n: FB(Y_n) \hookrightarrow Y_n$. Then we have a solution X_n of Reifenberg. Obviously the new sepuence $\{X_n\}_{n=1}^{\infty}$ satisfies the property (i) (Lemma 1, 1°). Furthermore we can verify that it is also endowed with the property (ii) since it is Reifenberg's solution (Reifenberg [11], pp. 27–38). q.e.d.

The above minimizing sequence $\{X_n\}_{n=1}^{\infty}$, passing to a subsequence if necessary, converges to an $X_\infty \in \mathfrak{g}_{\text{free}}(\Gamma)$ (Assertion 1). Then we have

Assertion 3. $\text{Vol}^m(X_\infty) \leqq V$.

Proof. We first claim

$$\text{(5)}\quad \inf\ \{\liminf_{n\to\infty} \mathcal{H}^m(X_n \cap B(P, r))/\alpha(m)r^m;\quad B(P, r) \cap E = \phi,\ P \in X_\infty\} \geqq 1\,.$$

We will show it below. Take an arbitrary point P of X_∞. There exists a sequence of points $\{P_n\}_{n=1}^{\infty}$ such that P_n belongs to X_n and $d(P_n, P) \to 0$ as $n \to \infty$, since X_∞ is the Hausdorff limit of $\{X_n\}_{n=1}^{\infty}$. Note that, since $B(P, r) \cap E = \phi$, $B(P_n, r) \cap E = \phi$ for any sufficiently large n and that $\lim_{n\to\infty} \mathcal{H}^m(B(P_n, r) \setminus B(P, r)) = 0$. Then we have in view of Assertion 2 (ii)

$$\begin{aligned}
&\liminf_{n\to\infty}\ \mathcal{H}^m(X_n \cap B(P, r)) \\
&= \liminf_{n\to\infty}\ \mathcal{H}^m(X_n \cap B(P, r)) + \lim_{n\to\infty} \mathcal{H}^m(B(P_n, r) \setminus B(P, r)) \\
&\geqq \liminf_{n\to\infty}\ \mathcal{H}^m(X_n \cap (B(P, r) \cup B(P_n, r))) \\
&\geqq \liminf_{n\to\infty}\ \mathcal{H}^m(X_n \cap B(P_n, r)) \\
&\geqq \alpha(m) r^m\,,
\end{aligned}$$

provided $B(P, r) \cap E = \phi$ and $P \in X_\infty$. Thus we obtain the above claim.

Let $\mathcal{B}(\delta)$ be the set of all the balls B such that $B \cap E = \phi$ and rad $B < \delta$, where rad B denotes the radius of B. For an arbitrary $\delta > 0$ there exists a finite union of subfamilies $\mathcal{B}_1, \cdots\cdots, \mathcal{B}_k$ of $\mathcal{B}(\delta)$, each of which is a mutually disjoint cover of $\mathcal{H}^m$-almost all points on the set $X_\infty \setminus FB(X_\infty)$ (Allard [3], p. 426, 2.7 (1)). By the above inequality (5) and by the disjointness of $\mathcal{B}_j$, we have

$$\begin{aligned}
&\sum_{B \in \mathcal{B}_j} \alpha(m)\,(\text{rad}\ B)^m \leqq \sum_{B \in \mathcal{B}_j} \liminf_{n\to\infty} \mathcal{H}^m(X_n \cap B) \\
\text{(6)}\qquad &\leq \liminf_{n\to\infty} \sum_{B \in \mathcal{B}_j} \mathcal{H}^m(X_n \cap B) = \liminf_{n\to\infty} \mathcal{H}^m\Big(\bigcup_{B \in \mathcal{B}_j} (X_n \cap B)\Big) \\
&\leqq \liminf_{n\to\infty} \mathcal{H}^m(X_n \setminus FB(X_n)) = \lim_{n\to\infty} \text{Vol}^m(X_n) = V\,,
\end{aligned}$$

and accordingly

$$\inf \{ \sum_{B \in \mathcal{B}} \alpha(m) (\mathrm{rad}\, B)^m ; \mathcal{B} \subset \mathcal{B}(\delta),\ X_\infty \setminus FB(X_\infty) \subset \bigcup_{B \in \mathcal{B}} B \} \leqq kV .$$

Let $\delta \to 0$ and we obtain

$$\mathcal{H}^m(X_\infty \setminus FB(X_\infty)) \leqq kV .$$

Thus we conclude that $X_\infty \setminus FB(X_\infty)$ is of finite $\mathcal{H}^m$-measure.

Next we take and fix a sequence $\{\delta_n\}_{n=1}^\infty$ of positive number tending to zero as $n \to \infty$. Since $X_\infty \setminus FB(X_\infty)$ has finite $\mathcal{H}^m$-measure, there exists a subfamily $\mathcal{B}'(\delta_n)$ of $\mathcal{B}(\delta_n)$ and a set e_n of $\mathcal{H}^m$-measure zero such that $\mathcal{B}'(\delta_n)$ covers $(X_\infty \setminus FB(X_\infty)) \setminus e_n$ (Besicovitch's covering theorem [4].) Then the inequalities (6) holds for $\mathcal{B}'(\delta_n)$ instead of $\mathcal{B}_j$, since $\mathcal{B}(\delta_n)$ is a disjoint family:

$$\mathrm{ind} \{ \sum_{B \in \mathcal{B}} \alpha(m) (\mathrm{rad}\, B)^m ; \mathcal{B} \subset \mathcal{B}(\delta_n),\ (X_\infty \setminus FB(X_\infty)) \setminus \bigcup_{n=1}^{\infty} e_n \subset \bigcup_{B \in \mathcal{B}} B \}$$
$$\leqq \sum_{B \in \mathcal{B}'(\delta_n)} \alpha(m) (\mathrm{rad}\, B)^m \leqq V .$$

Hence letting $n \to \infty$, we conclude

$$\mathrm{Vol}^m(X_\infty) = \mathcal{H}^m(X_\infty \setminus FB(X_\infty)) \setminus \bigcup_{n=1}^{\infty} e_n) \leqq V . \qquad \text{q.e.d.}$$

Since $X_\infty \in \mathfrak{g}_{\mathrm{free}}(\Gamma)$, Assertion 3 implies that X_∞ is a solution of Problem *FB*. Thus we obtain Theorem A.

REMARK 2. The m-dimensional density of the solution X_∞ is 1 at $\mathcal{H}^m$-almost all points of $X_\infty \setminus FB(X_\infty)$. Indeed we write $X_n = X_\infty$ for all n, and then the new minimizing sequence $\{X_n\}_{n=1}^\infty$ satisfies the properties (i), (ii) in Assertion 2. Therefore the inequality (5) holds for X_∞ in place of X_n. Hence the m-dimensional density of X_∞ is not less than 1. Conversely it is not greater than 1 at $\mathcal{H}^m$-almost all points of $X_\infty \setminus FB(X_\infty)$ (Federer [7], 2.10.9. (5)).

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