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HOMOLOGY LOCALIZATIONS AFTER APPLYING SOME RIGHT ADJOINT FUNCTORS

Dedicated to Professor Nobuo Shimada on his sixtieth birthday

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0. Introduction

Each homology theory E_* determines a natural E_*-localization $\eta: X \to L_E X$ in the homotopy category $h\mathcal{CW}$ of CW-complexes or $h\mathcal{CWS}$ of CW-spectra. It is full of interest to study the behavior of E_*-localizations after application of various functors T to the category $h\mathcal{CW}$ or $h\mathcal{CWS}$. Consider as T the 0-th space functor $\Omega^\infty: h\mathcal{CWS} \to h\mathcal{CW}$ which is right adjoint to the suspension spectrum functor Σ^∞. Bousfield [4] showed that the E_*-localization of an infinite loop space $\Omega^\infty X$ is still an infinite loop space. More precisely, he proved

Theorem 0.1 ([4, Theorem 1.1]). *There exists an idempotent monad L: $h\mathcal{CWS}_0 \to h\mathcal{CWS}_0$ and $\eta: 1 \to L$ such that the map $\Omega^\infty\eta: \Omega^\infty X \to \Omega^\infty LX$ is an E_*-localization in $h\mathcal{CW}$. Here $h\mathcal{CWS}_0$ denotes the full subcategory of $h\mathcal{CWS}$ consisting of (-1)-connected CW-spectra.*

As remarked by Bousfield [4], this implies

Proposition 0.2. *If f: $A \to B$ is an E_*-equivalence in $h\mathcal{CW}$, then so is $\Omega^\infty\Sigma^\infty f$: $\Omega^\infty\Sigma^\infty A \to \Omega^\infty\Sigma^\infty B$.*

On the other hand, Kuhn [7, Proposition 2.4] gave recently a simple proof of Proposition 0.2 using the stable decompositions of $\Omega^\infty\Sigma^\infty A$ and $\Omega^\infty\Sigma^\infty B$ (see [9]).

In this note we will show that Proposition 0.2 is essential to the existence theorem 0.1. Thus, by use of only Proposition 0.2 we give a direct proof of the existence theorem 0.1 along the primary line of Bousfield [1, 2 and 3]. In our proof we don't need the knowledge of very special Γ-spaces although Bousfield did in [4].

Let T: $\mathcal{C} \to \mathcal{B}$ be a functor with a left adjoint S and $\mathcal{W}$ be a morphism class in $\mathcal{B}$. In §1 we introduce $T^*\mathcal{W}$- and $(\mathcal{W}, T)$-localizations in $\mathcal{C}$ and discuss a relation between them. Following our notation Theorem 0.1 says that there exists an (E_*, Ω^∞)-localization in $h\mathcal{CWS}_0$ where E_* stands for the morphism class of E_*-equivalences in $h\mathcal{CW}$. Don't confuse our notation with Bousfield's [4]. We next give three conditions (C.1)–(C.3) under which we can construct a $(\mathcal{W}, T)$-localization $\eta: X \to LX$ for each $X \in \mathcal{C}$ where $\mathcal{C}=h\mathcal{CW}$ or $h\mathcal{CWS}$, by the same method as Bousfield used in constructing E_*-localizations in [1, 3].

It might be indistinctly known that the 0-th space functor Ω^∞ converts generally a cofiber sequence in $h\mathcal{CWS}$ to a fiber sequence in $h\mathcal{CW}$. Nevertheless we prove this fact in §2 by making use of secondary operations on mappings [10]. This result yields a key lemma for proving the existence theorem of (E_*, Ω^∞)-localization.

In §3 we first check that the conditions (C.1)–(C.3) are satisfied for the triple $(\mathcal{W}, T, S)=(E_*, \Omega^\infty, \Sigma^\infty)$. As a result we can give a new proof of the existence theorem of (E_*, Ω^∞)-localization in $h\mathcal{CWS}$. Since the equivariant version of Proposition 0.2 is valid when G is a finite group (use [8, V]), we obtain the equivariant version of Theorem 0.1. Of course we may prove it by using very special G-Γ spaces following Bousfield's approach. Let G be a compact Lie group and ϕ_K be the K-fixed point functors. Applying our method to $T=\prod \phi_K$ we also obtain the existence theorem of $(\prod E_{K*}, \prod \phi_K)$-localization which was studied in [11, Theorem 2.1].

1. $(\mathcal{W}, T)$- and $T^*\mathcal{W}$-localizations

1.1. Let $\mathcal{B}$ be a category. We call a functor and transformation $L: \mathcal{B} \to \mathcal{B}$, $\eta: 1 \to L$ *idempotent* if $\eta_{LA}=L\eta_A: LA \to L^2A$ and it is an equivalence for each $A \in \mathcal{B}$. It is easy to show

(1.1) *A functor $L: \mathcal{B} \to \mathcal{B}$ and transformation η: $1 \to L$ is idempotent if and only if $\eta_A: A \to LA$ induces a bijection $\eta_A^*: \mathcal{B}(LA, LB) \to \mathcal{B}(A, LB)$ for any $A, B \in \mathcal{B}$.*

Given a morphism class $\mathcal{W}$ in a category $\mathcal{B}$, an object $D \in \mathcal{B}$ is called *$\mathcal{W}$-local* if each $f: A \to B$ in $\mathcal{W}$ induces a bijection $f^*: \mathcal{B}(B, D) \to \mathcal{B}(A, D)$. For each $A \in \mathcal{B}$ a morphism $g: A \to D$ is called a *$\mathcal{W}$-localization of A* if g belongs to $\mathcal{W}$ and D is $\mathcal{W}$-local. If all objects of $\mathcal{B}$ admit $\mathcal{W}$-localizations, then there exists a functor $L: \mathcal{B} \to \mathcal{B}$ and transformation η: $1 \to L$ such that $\eta_A: A \to LA$ is a $\mathcal{W}$-localization for each $A \in \mathcal{B}$. Such an (L, η) is unique up to natural equivalence, so it is called the $\mathcal{W}$-localization in $\mathcal{B}$. It follows from (1.1) that the $\mathcal{W}$-localization is idempotent [1].

Let T: $\mathcal{C} \to \mathcal{B}$ be a functor and $\mathcal{W}$ be a morphism class in $\mathcal{B}$. An idempotent monad $L: \mathcal{C} \to \mathcal{C}$ and $\eta: 1 \to L$ is called the *$(\mathcal{W}, T)$-localization in $\mathcal{C}$* if $T\eta_X$: $TX \to TLX$ is a $\mathcal{W}$-localization for each $X \in \mathcal{C}$.

We here restrict to a morphism class $\mathcal{W}$ in $\mathcal{B}$ satisfying the condition:

(C.0) i) Each equivalence f: $A \to B$ is contained in $\mathcal{W}$.
ii) If two of f: $A \to B$, g: $B \to C$ and gf: $A \to C$ are in $\mathcal{W}$, so is the third.

Lemma 1.1. *Let $T: \mathcal{C} \to \mathcal{B}$ be a functor with a left adjoint S, and $\mathcal{W}$ be* *a morphism class in* $\mathcal{B}$ *satisfying the condition* (C.0). *Assume that there exists a* $(\mathcal{W}, T)$*-localization* (L, η) *in* $\mathcal{C}$. *If* $f: A \to B$ *is contained in* $\mathcal{W}$, *then so is* TSf: $TSA \to TSB$. (Cf., [4, Remark following Proposition 1.2]).

Proof. Each $f: A \to B$ in $\mathcal{W}$ induces a bijection $f^*: \mathcal{B}(B, TLX) \to \mathcal{B}(A, TLX)$ for any $X \in \mathcal{C}$ since TLX is $\mathcal{W}$-local. By adjointness Sf^*: $\mathcal{C}(SB, LX) \to \mathcal{C}(SA, LX)$ is bijective, too. Making use of (1.1) we easily verify that $LSf: LSA \to LSB$ is an equivalence. It is now immediate that TSf: $TSA \to TSB$ is in $\mathcal{W}$ because $\mathcal{W}$ satisfies the condition (C.0).

Given a functor $T: \mathcal{C} \to \mathcal{B}$ and a morphism class $\mathcal{W}$ in $\mathcal{B}$ we denote by $T^*\mathcal{W}$ the morphism class in $\mathcal{C}$ which consists of all $u: X \to Y$ with $Tu \in \mathcal{W}$. We here study a relation between the $T^*\mathcal{W}$-localization and the $(\mathcal{W}, T)$-localization.

Proposition 1.2. *Let* $T: \mathcal{C} \to \mathcal{B}$ *be a functor with a left adjoint* S, *and* $\mathcal{W}$ *be a morphism class in* $\mathcal{B}$ *satisfying the condition* (C.0). *Assume that* $u: X \to Y \in \mathcal{C}$ *is an equivalence whenever so is* $Tu: TX \to TY$. *Then an idempotent monad* (L, η) *is the* $(\mathcal{W}, T)$*-localization in* $\mathcal{C}$ *if and only if it is the* $T^*\mathcal{W}$*-localization in* $\mathcal{C}$ *and moreover* $TSf: TSA \to TSB$ *is in* $\mathcal{W}$ *when so is* $f: A \to B$.

Proof. The "if" part: It is sufficient to show that TLZ is $\mathcal{W}$-local for each $Z \in \mathcal{C}$. Given any $f: A \to B$ in $\mathcal{W}$, $Sf^*: \mathcal{C}(SB, LZ) \to \mathcal{C}(SA, LZ)$ is bijective since LZ is $T^*\mathcal{W}$-local. By adjointness this means that TLZ is $\mathcal{W}$-local.

The "only if" part: The latter part follows from Lemma 1.1. So we only have to show that LZ is $T^*\mathcal{W}$-local for each $Z \in \mathcal{C}$. Taking any $u: X \to Y$ in $T^*\mathcal{W}$, $TLu: TLX \to TLY$ is an equivalence since it is in $\mathcal{W}$ and TLX, TLY are both $\mathcal{W}$-local. Under our assumption $Lu: LX \to LY$ is also an equivalence. It is immediate from (1.1) that $u^*: \mathcal{C}(Y, LZ) \to \mathcal{C}(X, LZ)$ is bijective, thus LZ is $T^*\mathcal{W}$-local.

1.2. Let G be a compact Lie group. Let $G\mathcal{T}$ denote the category of based G-spaces with G-fixed basepoint, and $G\mathcal{S}\mathcal{A}$ the category of G-spectra indexed on an indexing set $\mathcal{A}$ in a G-universe U. Let us write $G\mathcal{S}U$ for $G\mathcal{S}\mathcal{A}$ when $\mathcal{A}$ is the standard indexing set in U. The category $G\mathcal{S}\mathcal{A}$ is equivalent to $G\mathcal{S}U$ for any indexing set $\mathcal{A}$ in U. The suspension spectrum functor $\Sigma^\infty: G\mathcal{T} \to G\mathcal{S}\mathcal{A}$ has a right adjoint functor $\Omega^\infty: G\mathcal{S}\mathcal{A} \to G\mathcal{T}$ called the 0-th space functor [8, Proposition II. 2.3].

Let $\bar{h}G\mathcal{T}$ or $\bar{h}G\mathcal{S}\mathcal{A}$ be the category obtained from the homotopy category $hG\mathcal{T}$ or $hG\mathcal{S}\mathcal{A}$ by formally inverting the weak equivalences respectively. The category $\bar{h}G\mathcal{T}$ is equivalent to the homotopy category $hG\mathcal{C}\mathcal{W}$ of G-CW complexes and cellular maps. Similarly the stable category $\bar{h}G\mathcal{S}\mathcal{A}$ is equivalent to the homotopy category $hG\mathcal{C}\mathcal{W}\mathcal{S}\mathcal{A}$ of G-CW spectra and cellular maps indexed on $\mathcal{A}$ [8, Theorem II. 5.12].

Let us abbreviate by GC the category $GC\mathcal{W}$ of G-CW complexes or the category $GC\mathcal{W}S\mathcal{A}$ of G-CW spectra indexed on $\mathcal{A}$, and by hGC its homotopy category. Let $S: \mathcal{B}\to hGC$ be a functor and $\mathcal{W}$ be a morphism class in $\mathcal{B}$. For a fixed infinite cardinal number σ we consider the subclass $\mathcal{W}_\sigma=\{f_\alpha;\ A_\alpha\to B_\alpha\}_{\alpha\in I}$ consisting of morphisms in $\mathcal{W}$ with $\# SA_\alpha\leqq\sigma$ and $\# SB_\alpha\leqq\sigma$, where $\# X$ denotes the number of G-cells in $X\in GC$. Note that $Sf_\alpha: SA_\alpha\to SB_\alpha$ may be represented by an inclusion i_α, when replacing SB_α by the mapping cylinder of Sf_α if necessary.

We say an inclusion map $u: X\to Y\in GC$ admits an *$(S, \mathcal{W}_\sigma)$-decomposition* if there exists a transfinite sequence

$$X=X_0\subset X_1\subset\cdots\subset X_s\subset X_{s+1}\subset\cdots\subset X_\gamma=Y$$

in GC such that $X_\lambda=\bigcup_{s<\lambda}X_s$ when λ is a limit ordinal and $X_s\subset X_{s+1}$ is obtained from a pushout square

$$\text{(1.2)}\qquad \begin{array}{ccc} \vee SA_\alpha & \to & X_s \\ {\scriptstyle \vee i_\alpha}\downarrow & & \downarrow \\ \vee SB_\alpha & \to & X_{s+1} \end{array}$$

in GC where the inclusion i_α is a representative of Sf_α for $f_\alpha: A_\alpha\to B_\alpha$ in $\mathcal{W}_\sigma$.

Let γ be the first infinite ordinal of cardinality greater than σ. For each $X\in GC$ we inductively construct a transfinite sequence

$$X=X_0\subset X_1\subset\cdots\subset X_s\subset X_{s+1}\subset\cdots$$

in GC where $X_\lambda=\bigcup_{s<\lambda}X_s$ for each limit ordinal λ and $X_s\subset X_{s+1}$ is given by the pushout square

$$\begin{array}{ccc} \vee_{\alpha\in I}\vee_g SA_\alpha & \to & X_s \\ \downarrow & & \downarrow \\ \vee_{\alpha\in I}\vee_g SB_\alpha & \to & X_{s+1} \end{array}$$

in which g ranges over all representative cellular maps $SA_\alpha\to X_s$ (cf., [2]). Putting $LX=X_\gamma$, we see immediately

(1.3) *The inclusion map $\eta_X: X\to LX$ admits an $(S, \mathcal{W}_\sigma)$-decomposition.*

Each cellular map $k: SA_\alpha\to LX$ passes through SB_α because the image of k is contained in X_s for some $s<\gamma$. Therefore any $f_\alpha: A_\alpha\to B_\alpha$ in $\mathcal{W}_\sigma$ induces a surjection $Sf_\alpha^*: hGC(SB_\alpha, LX)\to hGC(SA_\alpha, LX)$ This implies

(1.4) *If an inclusion map $v: Y\to Z$ admits an $(S, \mathcal{W}_\sigma)$-decomposition, then $v^*: hGC(Z, LX)\to hGC(Y, LX)$ is surjective.*

Let $S_\sharp\mathcal{W}_\sigma$ denote the morphism class consisting of morphisms in hGC, each of which is represented by some inclusion having an $(S, \mathcal{W}_\sigma)$-decomposition. We now assume that $S_\sharp\mathcal{W}_\sigma$ satisfies the condition:

(C.1) Given $u: X\to Y$ in $S_\sharp\mathcal{W}_\sigma$ and $f, g: Y\to Z$ such that $fu=gu$ in hGC, there exists $w: Z\to W$ in $S_\sharp\mathcal{W}_\sigma$ such that $wf=wg$ in hGC.

Under the condition (C.1) it is easy to show

(1.5) *Each $v: Y\to Z$ in $S_\sharp\mathcal{W}_\sigma$ induces a bijection $v^*: hGC(Z, LX)\to hGC(Y, LX)$* (see [1, Lemma 2.5]).

By use of (1.1), (1.3) and (1.5) we obtain

Lemma 1.3. *Let $S: \mathcal{B}\to hGC$ be a functor and $\mathcal{W}$ be a morphism class in $\mathcal{B}$. Fix an infinite cardinal number σ and assume that the morphism class $S_\sharp\mathcal{W}_\sigma$ satisfies the condition* (C.1). *Then the inclusion map $\eta_X: X\to LX$ give rise to an idempotent monad (L, η) in hGC.*

Let $S: \mathcal{B}\to hGC$ be a functor with a right adjoint T and $\mathcal{W}$ be a morphism class in $\mathcal{B}$. We moreover assume that the following conditions are satisfied:

(C.2) For each $f: A\to B$ in $\mathcal{W}$ the morphism $Sf: SA\to SB$ is in $S_\sharp\mathcal{W}_\sigma$.

(C.3) If $u: X\to Y$ is in $S_\sharp\mathcal{W}_\sigma$, then the morphism $Tu: TX\to TY$ is in $\mathcal{W}$.

Note that both (C.2) and (C.3) imply

(C.4) If $f: A\to B$ is in $\mathcal{W}$, then so is $TSf: TSA\to TSB$.

Proposition 1.4. *Let $T: hGC\to\mathcal{B}$ be a functor with a left adjoint S and $\mathcal{W}$ be a morphism class in $\mathcal{B}$. Fix an infinite cardinal number σ and assume that the three conditions* (C.1), (C.2) *and* (C.3) *are all satisfied. Then there exists a $(\mathcal{W}, T)$-localization (L, η) in hGC.*

Proof. Under our assumptions it follows from (1.3) and (1.5) that the morphism $T\eta_X$: $TX\to TLX$ is a $\mathcal{W}$-localization. The result is now immediate from Lemma 1.3.

2. Homotopy theoric fiber sequences

Given maps $d_1, d_2: K\wedge I^+\to N$ in $G\mathcal{T}$ such that $d_1|K\times\{1\}=d_2|K\times\{0\}$ we define a G-map $d_1\perp d_2: K\wedge I^+\to N$ as $d_1\perp d_2(x, t)$ is equal to $d_1(x, 2t)$ if $0\leqq t\leqq 1/2$ and to $d_2(x, 2-2t)$ if $1/2\leqq t\leqq 1$. Consider a sequence $K\xrightarrow{f}L\xrightarrow{g}M\xrightarrow{h}N$ in $G\mathcal{T}$ such that the two composite gf, hg are both G-null homotopic. Then there are G-maps $F: CK\to M$ and $H: CL\to N$ such that $F|K\times\{1\}=gf$ and $H|L\times\{1\}=hg$ where C denotes the reduced cone functor. Two maps hF, $H(Cf)$ give rise to a G-map $d(hF, H(Cf))\colon \Sigma K\to N$ obtained as $d(hF, H(Cf))=hF\perp H(f\wedge\tau)$ where Σ denotes the reduced suspension functor and $\tau\colon I^+\to I^+$ is the twisting map. The bracket $\langle f, g, h\rangle$ is defined to be the double coset of $h_*[\Sigma K, M]_G$ and $\Sigma f^*[\Sigma L, N]_G$ in $[\Sigma K, N]_G$ determined by $[d(hF, H(Cf))]$.

Consider the mapping cocylinder

$$E_h=\{(z,\omega)\in M\times F(I,N);\ h(z)=\omega(0)\}$$

of $h\colon M\to N$. The G-map $p\colon E_h\to N$ defined to be $p(z,\omega)=\omega(1)$ is a G-fibration. Let us denote by F_h the fiber of p over the basepoint of N, which is called the mapping fiber of h. The G-map $q\colon F_h\to M$ defined to be $q(z,\omega)=z$ is a G-fibration, too. Notice that the fiber of q is just the loop space ΩN.

Assume that there exist G-maps $b\colon C_f\to M$, $a\colon \Sigma K\to N$ making the diagram below G-homotopy commutative

$$(2.1)\qquad \begin{array}{ccccc} L & \to & C_f & \to & \Sigma K \\ \| & & \downarrow b & & \downarrow a \\ L & \underset{g}{\to} & M & \underset{h}{\to} & N \end{array}$$

where we write C_f for the mapping cone of $f\colon K\to L$. According to [10, Theorem 3.3] the bracket $\langle f, g, h\rangle$ is represented by the map a. So we may choose G-maps $F\colon CK\to M$ and $H\colon CL\to N$ such as $F|K\times\{1\}=gf$, $H|L\times\{1\}=hg$ and $[d(hF, H(Cf))]=[a]\in[\Sigma K, N]_G$.

Using such a map H we define a G-map $\beta\colon L\to F_h$ to be

$$(2.2)\qquad \beta(y)=(g(y),\ \ H(1\wedge\tau)|\{y\}\times I)\in M\times F(I,N)\,.$$

As is easily seen, the following diagram

$$(2.3)\qquad \begin{array}{ccccc} K & \xrightarrow{f} & L & \xrightarrow{g} & M \\ \bar{a}\downarrow & & \beta\downarrow & & \| \\ \Omega N & \to & F_h & \underset{q}{\to} & M \end{array}$$

is G-homotopy commutative where $\bar{a}$ is the adjoint of a.

A sequence $K\xrightarrow{f}L\xrightarrow{g}M\xrightarrow{h}N$ in $G\mathcal{T}$ is said to be a *fiber sequence in* $\bar{h}G\mathcal{T}$ if there exist weak equivalences $\beta\colon L\to F_h$, $\alpha\colon K\to\Omega N$ such that the diagram below is G-homotopy commutative:

$$(2.4)\qquad \begin{array}{ccccc} K & \to & L & \to & M \\ \alpha\downarrow & & \beta\downarrow & & \| \\ \Omega N & \to & F_h & \to & M\,. \end{array}$$

Proposition 2.1. *Let $X\to Y\to Z\to\Sigma X$ be a cofiber sequence in $hGS\mathcal{A}$.*

Then the sequence $\Omega^\infty X \to \Omega^\infty Y \to \Omega^\infty Z \to \Omega^\infty \Sigma X$ *is a fiber sequence in* $\bar{h}G\mathscr{T}$.

Proof. Consider the following diagram

$$\begin{array}{ccccccc}
\Sigma^\infty\Omega^\infty X & \to & \Sigma^\infty\Omega^\infty Y & \to & \Sigma^\infty C_{\Omega^\infty u} & \to & \Sigma\Sigma^\infty\Omega^\infty X \\
\varepsilon\downarrow & & \varepsilon\downarrow & & \downarrow & & \downarrow\Sigma\varepsilon \\
X & \xrightarrow[u]{} & Y & \xrightarrow[v]{} & Z & \xrightarrow[w]{} & \Sigma X
\end{array}$$

in $G\mathcal{S}\mathcal{A}$ where ε's are the adjunction maps. Both of horizontal rows are cofiber sequences in $hG\mathcal{S}\mathcal{A}$ and the left square is commutative. So there exists a G-map $\tilde{b}\colon \Sigma^\infty C_{\Omega^\infty u} \to Z$ such that the remaining squares become G-homotopy commutative. Taking the adjoint situation the maps $b\colon C_{\Omega^\infty u} \to \Omega^\infty Z$ and $a\colon \Sigma\Omega^\infty X \to \Omega^\infty\Sigma X$ give a G-homotopy commutative diagram such as (2.1). From (2.2) and (2.3) we obtain a G-map $\beta\colon \Omega^\infty Y \to F_{\Omega^\infty w}$ such that the following diagram is G-homotopy commutative:

$$\begin{array}{ccccc}
\Omega^\infty X & \longrightarrow & \Omega^\infty Y & \to & \Omega^\infty Z \\
\bar{a}\downarrow & & \beta\downarrow & & \| \\
\Omega\Omega^\infty\Sigma X & \longrightarrow & F_{\Omega^\infty w} & \to & \Omega^\infty Z\,.
\end{array}$$

By use of the desuspension theorem [8, Theorem II. 6.1] we observe that the adjoint $\bar{a}$ of a is a weak equivalence. Applying Five lemma we moreover verify that β is also a weak equivalence.

2.2. Given two sequences $\Phi\colon K \xrightarrow{f} L \xrightarrow{q} M \xrightarrow{h} N$, $\Phi'\colon K' \xrightarrow{f'} L' \xrightarrow{q'} M' \xrightarrow{h'} N'$ in $G\mathscr{T}$ we consider a morphism $\xi=(k, l, m, n)\colon \Phi \to \Phi'$ such that the induced diagram is G-homotopy commutative. Choose a G-homotopy $P\colon K\wedge I^+ \to L'$ from $f'k$ to lf and define a G-map $\mu\colon C_f \to C_{f'}$ by $\mu|CK = Ck \perp P$ and $\mu|L=l$. We here assume that there are four G-maps b, b', a and a' making the diagram below G-homotopy commutative:

(2.5)
$$\begin{array}{ccccccc}
 & & M & & \xrightarrow{h} & & N \\
 & g\nearrow & \uparrow b & & & & \uparrow a \\
L & \xrightarrow[i]{} & C_f & & \to & & \Sigma K \\
l\downarrow & & \downarrow\mu & m & & & \downarrow\Sigma k \quad n \\
L' & \xrightarrow[i']{} & C_{f'} & & \to & & \Sigma K' \\
 & g'\searrow & \downarrow b' & & & & \downarrow a' \\
 & & M' & & \xrightarrow[h']{} & & N'
\end{array}$$

Choose G-homotopies $U\colon L\wedge I^+ \to M$ from bi to g, $U'\colon L'\wedge I^+ \to M'$ from $b'i'$ to g' and $V\colon C_f\wedge I^+ \to M'$ from mb to $b'\mu$, and then define a G-map $b_1\colon C_f \to M$ by $b_1|CK = b|CK \perp U(f\wedge 1)$ and $b_1|L=g$, and similarly a G-map b_1':

$C_{f'}\to M'$ using the homotopy U'. Combine U, U' and V to obtain a G-homotopy $Q\colon L\wedge I^+\to M'$ from mg to $g'l$ defined to be $Q=mU(1\wedge\tau)\perp V(i\wedge 1)\perp U'(l\wedge 1)$. Putting $F=b_1|CK$ and $F'=b_1'|CK'$ we have

Claim 2.2. *$mF\perp Q(f\wedge 1)$ is G-homotopic rel $K\wedge\partial I^+$ to $F'(CK)\perp g'P$.*

Proof. $b'\mu|CK$ is G-homotopic rel $K\wedge\partial I^+$ to $mb|CK\perp V(if\wedge 1)$ and also $b'i'P\perp U'(lf\wedge 1)$ is so to $U'(f'k\wedge 1)\perp g'P$. Hence the result is easily shown.

Since $[b]=[b_1]\in[C_f, M]_G$ we get a G-map $H\colon CL\to N$ such that $[d(hF, H(Cf)]=[a]\in[\Sigma K, N]_G$ (see [10, Lemma 3.2 and Theorem 3.3]), and similarly a G-map $H'\colon CL'\to N'$ such that $[d(h'F', H'(Cf'))]=[a']\in[\Sigma K', N']_G$. Choose a G-homotopy $R\colon M\wedge I^+\to N'$ from $h'm$ to nh. Then we have

Claim 2.3. *There exists a G-map $W\colon\Sigma M\to N'$ such that $R(g\wedge 1)\perp nH(1\wedge\tau)\perp W(\Sigma g)$ is G-homotopic rel $L\wedge\partial I^+$ to $h'Q\perp H'(l\wedge\tau)$.*

Proof. nhF is G-homotopic rel $K\wedge\partial I^+$ to $h'mF\perp R(gf\wedge 1)$ and similarly $H'(f'k\wedge\tau)$ is so to $h'g'P\perp H'(lf\wedge\tau)$. By means of Claim 2.2 the equality $[d(nhF, nH(Cf)]=[d(h'F'(Ck), H'(Cf'k)]\in[\Sigma K, N']_G$ implies that $R(gf\wedge 1)\perp nH(f\wedge\tau)$ is G-homotopic rel $K\wedge\partial I^+$ to $h'Q(f\wedge 1)\perp H'(lf\wedge\tau)$. The result is now immediate.

Using the maps R and W we define a G-map $\lambda\colon F_h\to F_{h'}$ to be

$$\lambda(z, \omega)=(mz, R|\{z\}\times I\perp n\omega\perp W|\{z\}\times I). \tag{2.6}$$

By means of Claim 2.3 we see easily that the following diagrams are G-homotopy commutative:

$$\begin{array}{ccccc} \Omega N & \to & F_h & \xrightarrow{q} & M \\ \Omega n\downarrow & & \downarrow\lambda & & \downarrow m \\ \Omega N' & \to & F_{h'} & \xrightarrow[q']{} & M' \end{array} \qquad \begin{array}{ccc} L & \xrightarrow{\beta} & F_h \\ l\downarrow & & \downarrow\lambda \\ L' & \xrightarrow[\beta']{} & F_{h'} \end{array} \tag{2.7}$$

where β and β' are defined as (2.2).

Let $\Phi\colon K\to L\to M\to N$, $\Phi'\colon K'\to L'\to M'\to N'$ be fiber sequences in $\bar{h}G\mathcal{T}$. A morphism $\xi=(k, l, m, n)\colon\Phi\to\Phi'$ is said to be a *morphism between fiber sequences in $\bar{h}G\mathcal{T}$* if there are four weak equivalences β, β', α and α' and a G-map λ such that the diagram below is G-homotopy commutative:

$$\begin{array}{cccccccc} & K & \to & L & & & & \\ & \downarrow\alpha & & \downarrow\beta & \searrow & & & \\ & \Omega N & \to & F_h & \to & M & \to & N \\ k & \downarrow\Omega n & l & \downarrow\lambda & & \downarrow m & & \downarrow n \\ & \Omega N' & \to & F_{h'} & \to & M' & \to & N' \\ & \uparrow\alpha' & & \uparrow\beta' & \nearrow & & & \\ & K' & \to & L' & & & & \end{array} \tag{2.8}$$

Proposition 2.4. *Let* $\psi: X \overset{u}{\to} Y \overset{v}{\to} Z \overset{w}{\to} \Sigma X$, $\psi': X' \overset{u'}{\to} Y' \overset{v'}{\to} Z' \overset{w'}{\to} \Sigma X'$ *be cofiber sequences in* $hGS\mathcal{A}$ *and* $\zeta = (r, s, t, \Sigma r): \psi \to \psi'$ *be a morphism between cofiber sequences in* $hGS\mathcal{A}$. *Then* $\Omega^\infty\zeta: \Omega^\infty\psi \to \Omega^\infty\psi'$ *is a morphism between fiber sequences in* $\bar{h}G\mathcal{T}$.

Proof. Pick up a G-homotopy $P: X \wedge I^+ \to Y'$ from $u'r$ to su and consider the G-maps $\mu: C_{\Omega^\infty u} \to C_{\Omega^\infty u'}$ given by $\mu | C\Omega^\infty X = C\Omega^\infty r \perp \Omega^\infty P$ and $\mu | \Omega^\infty Y = \Omega^\infty s$. By observing standard cofiber sequences in $GS\mathcal{A}$ we can easily find G-maps $\tilde{b}: \Sigma^\infty C_{\Omega^\infty u} \to Z$ and $\tilde{b}': \Sigma^\infty C_{\Omega^\infty u'} \to Z'$ in the proof of Proposition 2.1 such as $t\tilde{b}$ is G-homotopic to $\tilde{b}'(\Sigma^\infty\mu)$. Hence we get four G-maps $b: C_{\Omega^\infty u} \to \Omega^\infty Z$, $b': C_{\Omega^\infty u'} \to \Omega^\infty Z'$, $a: \Sigma\Omega^\infty X \to \Omega^\infty \Sigma X$ and $a': \Sigma\Omega^\infty X' \to \Omega^\infty\Sigma X'$ such that the diagram (2.5) is G-homotopy commutative. Making use of Proposition 2.1, (2.6) and (2.7) we immediately obtain four weak equivalences $\beta: \Omega^\infty Y \to F_{\Omega^\infty w}$, β': $\Omega^\infty Y' \to F_{\Omega^\infty w'}$, $\alpha = \bar{a}: \Omega^\infty X \to \Omega\Omega^\infty\Sigma X$, $\alpha' = \bar{a}': \Omega^\infty X' \to \Omega\Omega^\infty\Sigma X'$ and a G-map $\lambda: F_{\Omega^\infty w} \to F_{\Omega^\infty w'}$ making the diagram (2.8) G-homotopy commutative.

3. (E_*, Ω^∞)- and $(\{E_{K*}\}, \Pi\phi_K)$-localizations

3.1. Let E_* be an $RO(G; U)$-graded homology theory defined on the stable homotopy category $hGC\mathcal{W}SU$. A map $u: X \to Y$ in $hGC\mathcal{W}SU$ is called an E_*-*equivalence* if $u_*: E_*X \to E_*Y$ is an isomorphism, and also a map $f: A \to B$ in $hGC\mathcal{W}$ is called an E_*-*equivalence* if so is $\Sigma^\infty f: \Sigma^\infty A \to \Sigma^\infty B$. Let us denote by $\mathcal{W}^E$ the morphism class consisting of all E_*-equivalences in $hGC\mathcal{W}SU$. We simply write $\mathcal{W}^E$ for the class $\Sigma^{\infty *}\mathcal{W}^E$ consisting of all E_*-equivalences in $hGC\mathcal{W}$. As usual we adopt the terms of E_*T- and (E_*, T)-localizations in place of those of $T^*\mathcal{W}$- and $(\mathcal{W}, T)$-localizations when $\mathcal{W} = \mathcal{W}^E$. Obviously the morphism class $\mathcal{W}^E$ in hGC satisfies the condition (C.0), where $hGC = hGC\mathcal{W}$ or $hGC\mathcal{W}SU$.

Lemma 3.1. *Let* σ *be an infinite cardinal number which is at least equal to the cardinality of* E_*. *Then*

$$\mathcal{W}^E = Id_\sharp \mathcal{W}^E_\sigma$$

where Id denotes the identity functor.

Proof. Trivially $Id_\sharp\mathcal{W}^E_\sigma \subset \mathcal{W}^E$. Taking an E_*-equivalence $u: X \to Y$ in hGC, it may be regarded as an inclusion $X \subset Y$. Let γ be an infinite cardinal number of cardinality greater than $\sharp Y - \sharp X$. As in the non-equivariant case (see [3, Lemma 1.13]) we can construct a transfinite sequence $X = X_0 \subset X_1 \subset \cdots \subset X_s \subset X_{s+1} \subset \cdots$ in GC such that i) if λ is a limit ordinal then $X_\lambda = \cup_{s<\lambda} X_s$, ii) if $X_s = Y$ then $X_{s+1} = Y$, and iii) if $X_s \neq Y$ then $X_{s+1} = X_s \cup W$ for some $W \subset Y$ where $\sharp W \leqq \sigma$, $W \not\subset X_s$ and the inclusion $W \cap X_s \to W$ is an E_*-equivalence. Clearly $Y = X_\gamma$. Hence we observe that the inclusion $u: X \to Y$ admits an $(Id, \mathcal{W}^E_\sigma)$-decomposition.

As is easily shown, we have

Corollary 3.2. *Let σ be an infinite cardinal number which is at least equal to the cardinality of E_*. Then $\Sigma^\infty_\sharp \mathcal{W}^E_\sigma$ satisfies the condition* (C.2).

It is known that $\mathcal{W}^E$ admits a calculus of left fractions in hGC (see [1, Lemma 3.6]). In particular, $\mathcal{W}^E = Id_\sharp \mathcal{W}^E_\sigma$ satisfies the condition (C.1).

Lemma 3.3. *Fix an infinite cardinal number σ. The morphism class $\Sigma^\infty_\sharp \mathcal{W}^E_\sigma$ admits a calculus of left fractions in $hGC\mathcal{W}SU$. In particular, it satisfies the condition* (C.1).

Proof. We only show that $\Sigma^\infty_\sharp \mathcal{W}^E_\sigma$ satisfies the condition (C.1) because the remainders are easy. Represent $u: X \to Y$ in $\Sigma^\infty_\sharp \mathcal{W}^E_\sigma$ by a transfinite sequence $X = X_0 \subset X_1 \subset \cdots \subset X_s \subset X_{s+1} \subset \cdots \subset X_\gamma = Y$ in $GC\mathcal{W}SU$, where $X_s \subset X_{s+1}$ is given by a pushout square as (1.2). Put $V_t = Y \times \{0\} \cup X_t \wedge I^+ \cup Y \times \{1\}$ and consider the square

$$\begin{array}{ccc} \bigvee_\alpha \Sigma^\infty(B_\alpha \times \{0\} \cup A_\alpha \wedge I^+ \cup B_\alpha \times \{1\}) & \rightarrow & V_s \\ \downarrow & & \downarrow \\ \bigvee_\alpha \Sigma^\infty(B_\alpha \wedge I^+) & \longrightarrow & V_{s+1}, \end{array}$$

which is also pushout. The transfinite sequence

$$Y \times \{0\} \cup X \wedge I^+ \cup Y \times \{1\} = V_0 \subset V_1 \subset \cdots \subset V_s \subset V_{s+1} \subset \cdots \subset V_\gamma = Y \wedge I^+$$

gives a $(\Sigma^\infty, \mathcal{W}^E_\sigma)$-decomposition for the inclusion $v: V_0 \to V_\gamma$. Given $f, g: Y \to Z$ such that $fu = gu$ in $hGC\mathcal{W}SU$, there is a map $k: V_0 \to Z$ with $k|Y \times \{0\} = f$ and $k|Y \times \{1\} = g$. Take the double mapping cylinder W of v and k, then it follows immediately that the inclusion $w: Z \to W$ has a $(\Sigma^\infty, \mathcal{W}^E_\sigma)$-decomposition and $wf = wg$ in $hGC\mathcal{W}SU$.

Without use of the existence theorem of (E_*, Ω^∞)-localization Kuhn [7, Proposition 2.4] proved that $(\mathcal{W}^E, \Omega^\infty\Sigma^\infty)$ satisfies the condition (C.4) in the non-equivariant case. By virtue of [8, Theorem V. 5.6] we can apply the method of Kuhn in the finite groups case to show

Proposition 3.4. *Assume that G is a finite group. If a map $f: A \to B$ in $GC\mathcal{W}$ is an E_*-equivalence, then so is $\Omega^\infty\Sigma^\infty f: \Omega^\infty\Sigma^\infty A \to \Omega^\infty\Sigma^\infty B$.* (Cf., [7] and [5]).

Proposition 3.5. *Given a homotopy pushout square*

$$\begin{array}{ccc} Y & \xrightarrow{v} & Z \\ s\downarrow & & \downarrow t \\ Y' & \xrightarrow[v']{} & Z' \end{array}$$

in $GC\mathcal{W}SU$ such that $\Omega^\infty s\colon \Omega^\infty Y \to \Omega^\infty Y'$ is an E_*-equivalence, then $\Omega^\infty t\colon \Omega^\infty Z \to \Omega^\infty Z'$ is an E_*-equivalence, too.

Proof. Let ΣX be the cofiber of $v\colon Y \to Z$. Then there is a G-homotopy commutative diagram

$$\begin{array}{ccccccc} \Omega^\infty X & \to & \Omega^\infty Y & \to & \Omega^\infty Z & \to & \Omega^\infty \Sigma X \\ \| & & \downarrow & & \downarrow & & \| \\ \Omega^\infty X & \to & \Omega^\infty Y' & \to & \Omega^\infty Z' & \to & \Omega^\infty \Sigma X. \end{array}$$

Propositions 2.1 and 2.4 assert that the horizontal rows may be regarded as fiber sequences of G–CW complexes. Compare the Atiyah–Hirzebruch spectral sequences (see [6, Theorem 1]). Since the base space $\Omega^\infty \Sigma X$ is a G-homotopy commutative H-space and $\pi_0^K(\Omega^\infty \Sigma X)$ is an abelian group for each closed subgroup K of G, the result is now easily shown.

Making use of Propositions 3.4 and 3.5 we have

Corollary 3.6. *Assume that G is a finite group and fix an infinite cardinal number σ. The morphism class $\Sigma_\sharp^\infty \mathcal{W}_\sigma^E$ satisfies the condition* (C.3).

Let σ be an infinite cardinal number which is at least equal to the cardinality of E_*. Lemma 3.3 and Corollaries 3.2 and 3.6 say that the morphism class $\Sigma_\sharp^\infty \mathcal{W}_\sigma^E$ satisfies the conditions (C.1), (C.2) and (C.3) when G is finite. So we can apply Proposition 1.4 to show the existence theorem of (E_*, Ω^∞)-localization.

Theorem 3.7. *Assume that G is a finite group. Then there exists an (E_*, Ω^∞)-localization (L, η) in $hGC\mathcal{W}SU$.* (Cf., [4, Theorem 1.1]).

Let $hGC\mathcal{W}SU_0$ denote the full subcategory of $hGC\mathcal{W}SU$ consisting of (-1)-connected G-CW spectra. The 0-th space functor $\Omega^\infty\colon hGC\mathcal{W}SU_0 \to hGC\mathcal{W}$ satisfies the assumption in Proposition 1.2. So we get

Corollary 3.8. *Assume that G is a finite group. Then there exists an $E_*\Omega^\infty$-localization (L, η) in $hGC\mathcal{W}SU_0$.* (See [4]).

3.2. Let G be a compact Lie group and $\mathcal{F}$ be a collection of closed subgroups of G which are not conjugate subgroups each other. We partially order a list $\mathcal{F}$ by writing $H \leq K$ if H is subconjugate to K. Let $\mathcal{E}_\mathcal{F} = \{E_{K*}\}_{K \in \mathcal{F}}$ be a family of homology theories defined on $hC\mathcal{W}SU$. A family $\mathcal{E}_\mathcal{F}$ is said to be order preserving if $E_{K*}X=0$ implies $E_{H*}X=0$ for each pair $H\leqq K$ in $\mathcal{F}$. Write $\mathcal{W}^{\mathcal{E}_{\mathcal{F}}}$ for the morphism class $\prod_{K\in\mathcal{F}}\mathcal{W}^{E_K}$ in $\prod_{K\in\mathcal{F}}h\mathcal{CW}$ or in $\prod_{K\in\mathcal{F}}h\mathcal{CWSU}$.

For each closed subgroup K of G the K-fixed point functor $\phi_K\colon G\mathcal{I}\to\mathcal{I}$ or $G\mathcal{SA}\to\mathcal{SA}$ has a left adjoint functor $(G/K)^+\wedge -$ (see [8, Proposition II. 4.6]). Abbreviate by $\mathcal{C}$ the category $\mathcal{CW}$ or $\mathcal{CWSU}$ and similarly by $G\mathcal{C}$. The fixed points functor $\phi_{\mathcal{F}}=\prod_{K\in\mathcal{F}}\phi_K\colon G\mathcal{C}\to\prod_{K\in\mathcal{F}}\mathcal{C}$ has a left adjoint $\psi_{\mathcal{F}}$: $\prod_{K\in\mathcal{F}}\mathcal{C}\to G\mathcal{C}$ defined to be $\psi_{\mathcal{F}}(\{X_K\})=\bigvee_K(G/K)^+\wedge X_K$. We here show that $(\mathcal{W}^{\mathcal{E}_{\mathcal{F}}}, \phi_{\mathcal{F}}\psi_{\mathcal{F}})$ satisfies the condition (C.4).

Lemma 3.9. *Assume that a family $\mathcal{E}_{\mathcal{F}}=\{E_{K*}\}$ is order preserving. Given E_{K*}-equivalences $f_K\colon X_K\to Y_K$ in $h\mathcal{C}$ for all $K\in\mathcal{F}$, then $\phi_H\psi_{\mathcal{F}}(\{f_K\})$: $(\bigvee_K(G/K)^+\wedge X_K)^H\to(\bigvee_K(G/K)^+\wedge Y_K)^H$ is also an E_{H*}-equivalence for each $H\in\mathcal{F}$.* (Cf., [11, Lemma 2.2]).

Proof. Under the hypothesis on $\mathcal{E}_{\mathcal{F}}$ it follows that $1\wedge f_K\colon (G/K)^{H+}\wedge X_K\to (G/K)^{H+}\wedge Y_K$ is an E_{H*}-equivalence since $(G/K)^H=\phi$ unless $H\leqq K$.

Let $\mathcal{E}_{\mathcal{F}}=\{E_{K*}\}$ be an order preserving family and σ be an infinite cardinal number which is at least equal to the cardinality of $\oplus_{K\in\mathcal{F}}E_{K*}$. By similar arguments to Lemma 3.3 and Corollaries 3.2 and 3.6 involving Lemma 3.9 we easily verify that $\psi_{\mathcal{F}\#}\mathcal{W}_{\sigma}^{\mathcal{E}_{\mathcal{F}}}$ in $hG\mathcal{C}$ satisfies the conditions (C.1), (C.2) and (C.3). Applying Proposition 1.4 we obtain

Theorem 3.10. *Let G be a compact Lie group and $\mathcal{E}_{\mathcal{F}}=\{E_{K*}\}$ be a family of homology theories defined on $h\mathcal{CWSU}$. Assume that $\mathcal{E}_{\mathcal{F}}$ is order preserving. Then there exists an $(\mathcal{E}_{\mathcal{F}}, \phi_{\mathcal{F}})$-localization (L, η) in $hG\mathcal{CW}$ or in $hG\mathcal{CWSU}$ where $\phi_{\mathcal{F}}=\prod_{K\in\mathcal{F}}\phi_K$ denotes the fixed points functor.*

If a list $\mathcal{F}$ contains precisely one subgroup from every conjugacy class of closed subgroups of G, then it is said to be *complete*. As is well known, the fixed points functor $\phi_{\mathcal{F}}$ satisfies the assumption in Proposition 1.2 when $\mathcal{F}$ is complete. Hence we have

Corollary 3.11. *Assume that a list $\mathcal{F}$ is complete and a family $\mathcal{E}_{\mathcal{F}}=\{E_{K*}\}$ is order preserving. Then there exists an $\mathcal{E}_{\mathcal{F}}\phi_{\mathcal{F}}$-localization (L, η) in $hG\mathcal{CW}$ or in $hG\mathcal{CWSU}$.* (Cf., [12], Theorem 2.1]).

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