Naitoh, H.
Osaka J. Math.
21 (1984), 733-764

PSEUDO-RIEMANNIAN SYMMETRIC R-SPACES

HIROO NAITOH(*)

(Received April 18, 1983)

Introduction. Symmetric R-spaces are specific riemannian symmetric spaces. These spaces are canonically realized as complete connected full parallel submanifolds of euclidean spaces and as minimal submanifolds of certain hyperspheres in the euclidean spaces. Conversely, a complete connected full parallel submanifold is congruent to the product image of imbeddings homothetic to the canonical imbeddings of symmetric R-spaces (Ferus [3], Takeuchi [18]). Roughly speaking, symmetric R-spaces are constructed as follows. Take a non-degenerate Jordan triple system which is compact and let $(\mathfrak{g}, \rho)$ be the positive definite symmetric graded Lie algebra constructed from its Jordan triple system in the Koecher's fashion, i.e., $\mathfrak{g}=\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$ is a graded Lie algebra of non compact type and ρ is a Cartan involution of $\mathfrak{g}$. Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be the Cartan decomposition. Then there exists a unique element $\nu\in\mathfrak{p}$ such that $\mathfrak{g}_\mu$, $\mu=0, \pm 1$, are eigen spaces of $\mathrm{ad}(\nu)$ with eigen values ν respectively. Then a symmetric R-space is defined as the orbit of ν by the connected Lie subgroup of $GL(\mathfrak{g})$ with the Lie algebra $\mathrm{ad}(\mathfrak{k})|\mathfrak{p}$. Here the euclidean metric on $\mathfrak{p}$ is given by the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{p}$.

Naitoh [11] has defined notions "orthogonal Jordan triple system", "orthogonal symmetric graded Lie algebra". An orthogonal Jordan triple system $(V, \{\ \}, \langle\ \rangle)$ is a Jordan triple system $(V, \{\ \})$ with a non-degenerate symmetric bilinear form $\langle\ \rangle$ on V. Non-degenerate Jordan triple systems are orthogonal Jordan triple systems with their trace forms. An orthogonal symmetric graded Lie algebra $(\mathfrak{g}, \rho, \langle\ \rangle_{\mathfrak{p}})$ is a symmetric graded Lie algebra $(\mathfrak{g}, \rho)$ with a non-degenerate symmetric bilinear form $\langle\ \rangle_{\mathfrak{p}}$ on $\mathfrak{p}$. Semi-simple symmetric graded Lie algebras are orthogonal symmetric graded Lie algebras with the restrictions of the Killing forms of $\mathfrak{g}$ to $\mathfrak{p}$. Between these objects there exists a natural one-to-one correspondence, which is the extension of the Koecher's way. (See § 1 for these precise definitions and the correspondence.) Moreover, in the above paper, we have constructed pseudo-riemannian

(*) Partially supported by the Sakkokai Foundation and the Grant-in-aid for Scientific Research, No. 57740045.

symmetric R-spaces from orthogonal Jordan triple systems satisfying some condition (S) in the same way as riemannian symmetric R-spaces. The condition (S) provides the existence of the point $\nu\in\mathfrak{p}$. (See § 2 in detail.) Pseudo-riemannian symmetric R-spaces are specific pseudo-riemannian symmetric spaces.

In this paper, firstly, we will show the following. Let M be a complete connected parallel submanifold of a pseudo-euclidean space $\boldsymbol{E}$ satisfying that, for any point $p\in M$, (1) the normal space at p is linearly spanned by the image of the second fundamental form at p and (2) there exists a normal vector at p such that the shape operator for it is the identity map. Then,

(A) All such spaces M are exhausted by pseudo-riemannian symmetric R-spaces (Theorem 2.6).

Moreover assume that $\boldsymbol{E}$ is a pseudo-hermitian space and that M is a totally real submanifold of $\boldsymbol{E}$ such that $\dim_{\boldsymbol{R}} M=\dim_{\boldsymbol{C}} \boldsymbol{E}$. Then;

(B) All such spaces M are exhausted by pseudo-riemannian symmetric R-spaces constructed from orthogonal Jordan triple systems such that their Jordan triple systems are associated with Jordan algebras with unity (Theorem 2.11).

Next we will show the following.

(C) Pseudo-riemannian symmetric R-spaces are imbedded as minimal submanifolds of certain pseudo-riemannian hyperspheres in pseudo-euclidean spaces if and only if the associated orthogonal Jordan triple systems are non-degenerate Jordan triple systems (Theorem 3.1).

Moreover we will list up pseudo-riemannian symmetric R-spaces associated with simple (non-degenerate) Jordan triple systems by using the classification given by Neher [13, 14].

The author wishes to express his hearty thanks to Professor M. Takeuchi for useful comments.

1. Preliminaries

Let $\boldsymbol{K}$ be either the field $\boldsymbol{R}$ of real numbers or the field $\boldsymbol{C}$ of complex numbers. A *Jordan triple system* over $\boldsymbol{K}$ (abbreviated to *JTS*), by definition, is a finite dimensional vector space V over $\boldsymbol{K}$ with a $\boldsymbol{K}$-trilinear map $\{\ \}: V\times V\times V\rightarrow V$ satisfying the following two conditions:

(JTS 1) $\{x, y, z\} = \{z, y, x\}$,

(JTS 2) $[L(x, y), L(u, v)] = L(L(x, y)u, v)-L(u, L(y, x)v)$

for $x, y, z, u, v\in V$, where $L(x, y)$, $x, y\in V$, denote $\boldsymbol{K}$-endomorphisms of V defined by $L(x, y)z=\{x, y, z\}$ for $z\in V$. Two JTS's $(V, \{\ \})$, $(V', \{\ \}')$ are *equivalent* to each other if there exists a $\boldsymbol{K}$-linear isomorphism δ of V onto V' such that

$$\delta(\{x, y, z\}) = \{\delta(x), \delta(y), \delta(z)\}'$$

for $x, y, z \in V$. The *trace form* β of a JTS $(V, \{\ \})$ is defined by

$$\beta(x, y) = \text{Trace of } L(x, y)$$

for $x, y \in V$. A JTS is called *non-degenerate* if the trace form is non-degenerate. The trace form β of a non-degenerate JTS $(V, \{\ \})$ is a non-degenerate symmetric bilinear form satisfying

$$(1.1) \qquad \beta(L(x, y)z, w) = \beta(z, L(y, x)w)$$

for $x, y, z, w \in V$.

A *symmetric graded Lie algebra* over $\boldsymbol{K}$ (abbreviated to SGLA), by definition, is a Lie algebra $\mathfrak{g}$ over $\boldsymbol{K}$ with an involutive automorphism ρ of $\mathfrak{g}$ satisfying the following four conditions:

(SGLA 1) $\mathfrak{g}=\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$ is a graded Lie algebra, i.e., $[\mathfrak{g}_\mu, \mathfrak{g}_\nu]\subset\mathfrak{g}_{\mu+\nu}$ where $\mathfrak{g}_\lambda=\{0\}$ for $\lambda\neq 0, \pm 1$,

(SGLA 2) $\rho(\mathfrak{g}_\mu)=\mathfrak{g}_{-\mu}$ for $\mu=0, \pm 1$,

(SGLA 3) $\mathfrak{g}_0$ acts faithfully on $\mathfrak{g}_{-1}\neq\{0\}$ by the adjoint representation ad,

(SGLA 4) $\mathfrak{g}_0=[\mathfrak{g}_{-1}, \mathfrak{g}_1]$.

Two SGLA's $(\mathfrak{g}, \rho)$, $(\mathfrak{g}', \rho')$ are *equivalent* to each other if there exists a Lie algebra isomorphism λ of $\mathfrak{g}$ onto $\mathfrak{g}'$ such that

$$\lambda(\mathfrak{g}_\mu) = \mathfrak{g}'_\mu \qquad \text{for } \mu = 0, \pm 1, \text{ and } \lambda\circ\rho = \rho'\circ\lambda .$$

An SGLA $(\mathfrak{g}, \rho)$ is called *semi-simple* if $\mathfrak{g}$ is semi-simple, i.e., the Killing form of $\mathfrak{g}$ is non-degenerate. Returning to an SGLA $(\mathfrak{g}, \rho)$, put

$$\mathfrak{k} = \{X\in\mathfrak{g};\, \rho(X) = X\}\,, \quad \mathfrak{p} = \{X\in\mathfrak{g};\, \rho(X) = -X\}\,.$$

Then we have the canonical decomposition: $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ and the Lie subalgebra $\mathfrak{k}$ acts on the vector space $\mathfrak{p}$ by the adjoint representation. Denote by B the Killing form of $\mathfrak{g}$. Assume that the SGLA is semi-simple. Then the restriction $B|\mathfrak{p}$ of B to $\mathfrak{p}$ is a non-degenerate symmetric bilinear form satisfying

$$(1.2) \qquad B|\mathfrak{p}(\mathrm{ad}(T)X, Y)+B|\mathfrak{p}(X, \mathrm{ad}(T)Y) = 0$$

for $T\in\mathfrak{k}$, $X, Y\in\mathfrak{p}$.

Now there exists a one-to-one correspondence between non-degenerate JTS's and semi-simple SGLA's (cf. Koecher [8]). The correspondence keeps the equivalence of each object. Assume that $\boldsymbol{K}=\boldsymbol{R}$. Then Naitoh [11] has extended notions "non-degenerate JTS", "semi-simple SGLA" to notions "orthogonal JTS", "orthogonal SGLA" by taking note of properties (1.1), (1.2) respectively, and moreover has extended the above correspondence to a correspondence between new notions.

Orthogonal Jordan triple system (abbreviated to OJTS): It is a JTS $(V, \{\ \})$ with a non-degenerate symmetric bilinear form $\langle\ \rangle$ on V satisfying

$$(1.1') \qquad \langle L(x, y)z, w\rangle = \langle z, L(y, x)w\rangle$$

for $x, y, z, w \in V$. Obviously a non-degenerate JTS, together with the trace form, is an OJTS. Two OJTS's $(V, \{\ \}, \langle\ \rangle)$, $(V', \{\ \}', \langle\ \rangle')$ are *equivalent* to each other if there exists an isomorphism δ of $(V, \{\ \})$ onto $(V', \{\ \}')$ such that

$$\langle \delta(x), \delta(y)\rangle' = \langle x, y\rangle$$

for $x, y, z \in V$.

Orthogonal symmetric graded Lie algebra (abbreviated to OSGLA): It is an SGLA $(\mathfrak{g}, \rho)$ with a non-degenerate symmetric bilinear form $\langle\rangle_{\mathfrak{p}}$ on $\mathfrak{p}$ satisfying

$$(1.2') \qquad \langle \mathrm{ad}(T)X, Y\rangle_{\mathfrak{p}} + \langle X, \mathrm{ad}(T)Y\rangle_{\mathfrak{p}} = 0$$

for $T \in \mathfrak{k}$, $X, Y \in \mathfrak{p}$. Obviously a semi-simple SGLA, together with the restriction $B|\mathfrak{p}$, is an OSGLA. Two OSGLA's $(\mathfrak{g}, \rho, \langle\ \rangle_{\mathfrak{p}})$, $(\mathfrak{g}', \rho', \langle\ \rangle_{\mathfrak{p}'})$ are *equivalent* to each other if there exists an isomorphism λ of $(\mathfrak{g}, \rho)$ onto $(\mathfrak{g}', \rho')$ such that

$$\langle \lambda(X), \lambda(Y)\rangle_{\mathfrak{p}'} = \langle X, Y\rangle_{\mathfrak{p}}$$

for $X, Y \in \mathfrak{p}$.

The correspondence between OJTS's and OSGLA's: (I) OJTS→OSGLA. Let $(V, \{\ \}, \langle\ \rangle)$ be an OJTS. Put

$$\mathfrak{g} = V \oplus L \oplus V,$$

where $L = \{L(x, y);\ x, y \in V\}_{\boldsymbol{R}}$. A bracket product $[\ ]$ is defined in the following way:

$$\begin{aligned}&[(x, f, y), (z, g, w)] \\ &= (f(z)-g(x), [f, g]-(1/2)L(x, w)+(1/2)L(z, y), g^*(y)-f^*(w))\end{aligned}$$

for $(x, f, y), (z, g, w) \in \mathfrak{g}$, where f^* denotes the transposed map of f with respect to $\langle\ \rangle$. Then $\mathfrak{g}$, together with the bracket product $[\ ]$, is a graded Lie algebra over $\boldsymbol{R}$ with the grading:

$$\mathfrak{g}_{-1} = V+\{0\}+\{0\}, \quad \mathfrak{g}_0 = \{0\}+L+\{0\}, \quad \mathfrak{g}_1 = \{0\}+\{0\}+V.$$

An involutive automorphism ρ of $\mathfrak{g}$ is defined by

$$\rho((x, f, y)) = (y, -f^*, x)$$

for $(x, f, y) \in \mathfrak{g}$. Note that an element in $\mathfrak{p}$ can be written in the following form:

$$(x, f, -x);\ 2f = \Sigma_i \{L(z_i, w_i) + L(w_i, z_i)\}$$

for some $x, z_i, w_i \in V$ by (1.1′). Suppose

$$2f = \Sigma_i \{L(z_i, w_i) + L(w_i, z_i)\}\ ,\quad 2g = \Sigma_j \{L(u_j, v_j) + L(v_j, u_j)\}\ .$$

Then a non-degenerate symmetric bilinear form $\langle\ \rangle_{\mathfrak{p}}$ is defined by

$$\begin{aligned}\langle (x, f, -x), (y, g, -y)\rangle_{\mathfrak{p}} &= \langle x, y\rangle + \Sigma_j \langle f(u_j), v_j\rangle \\ &= \langle x, y\rangle + \Sigma_i \langle g(z_i), w_i\rangle\end{aligned}$$

for $(x, f, -x), (y, g, -y) \in \mathfrak{p}$. (This is well-defined.) The triple $(\mathfrak{g}, \rho, \langle\ \rangle)$ constructed in the above way is an OSGLA.

(II) OSGLA→OJTS. Let $(\mathfrak{g}, \rho, \langle\ \rangle)$ be an OSGLA. Put

$$V = \mathfrak{g}_{-1}$$

and

$$\begin{aligned}\{X, Y, Z\} &= -2[[X, \rho(Y)], Z] \\ \langle X, Y\rangle &= \langle X - \rho(X),\ Y - \rho(Y)\rangle_{\mathfrak{p}}\end{aligned}$$

for $X, Y, Z \in V$. Then the triple $(V, \{\ \}, \langle\ \rangle)$ is an OJTS.

Results in Naitoh [11]. (A) *Two constructions (I), (II) keep the equivalence of each object and are the inverses of each other.*

(B) *Let* $(V, \{\ \})$ *be a non-degenerate JTS and* $(\mathfrak{g}, \rho)$ *a semi-simple SGLA corresponding to* $(V, \{\ \})$ *in the way in Koecher* [8]. *Then the OJTS* $(V, \{\ \}, 2\beta)$ *corresponds to the OSGLA* $(\mathfrak{g}, \rho, B|\mathfrak{p})$.

REMARK. In the result (B), β is positive definite if and only if ρ is a Cartan involution. In this case the non-degenerate JTS $(V, \{\ \})$ is called *compact.*

Now we define specific OJTS's, OSGLA's, called "orthogonal Jordan algebra", "hermitian symmetric graded Lie algebra" respectively, and give a construction (III) of the former into the latter.

Orthogonal Jordan algebra (abbreviated to OJA): A finite dimensional non-associative algebra A over $\boldsymbol{K}$ is called *Jordan algebra* if its product satisfies

$$x\cdot y = y\cdot x\ ,\quad x\cdot(x^2\cdot y) = x^2\cdot(x\cdot y)$$

for $x, y \in A$. Denote by T_x, $x \in A$, $\boldsymbol{K}$-endomorphisms of A defined by $T_x(y) = x\cdot y$ for $y \in A$. An *orthogonal Jordan algebra* is a Jordan algebra A over $\boldsymbol{R}$ with a non-degenerate symmetric bilinear form $\langle\ \rangle_A$ on A satisfying

$$(1.3) \qquad \langle T_x(y), z\rangle_A = \langle y, T_x(z)\rangle_A$$

for $x, y, z \in A$. We can associate an OJTS $(V, \{\ \}, \langle\ \rangle)$ with an OJA $(A, \langle\ \rangle_A)$ by putting

$$V = A\,,$$

and

$$\{x, y, z\} = (T_{x\cdot y} + [T_x, T_y])(z),\ \langle x, y\rangle = \langle x, y\rangle_A$$

for $x, y, z \in V$. If A has the unity e, the OJA is reproduced from the associated OJTS by the relation: $x\cdot y = \{x, y, e\}$.

Two OJA's $(A, \langle\ \rangle_A)$, $(A', \langle\ \rangle_{A'})$ are *equivalent* to each other if there exists an algebra isomorphism δ of A onto A' such that

$$\langle \delta(x), \delta(y)\rangle_{A'} = \langle x, y\rangle_A$$

for $x, y \in A$. If two OJA's are equivalent to each other, the associated OJTS's are also equivalent to each other.

Hermitian symmetric graded Lie algebra (abbreviated to HSGLA): It is an OSGLA $(\mathfrak{g}, \rho, \langle\ \rangle_\mathfrak{p})$ with a complex structure $J_\mathfrak{p}$ on $\mathfrak{p}$ satisfying

$$(1.4) \quad \mathrm{ad}(T)|\mathfrak{p}\circ J_\mathfrak{p} = J_\mathfrak{p}\circ \mathrm{ad}(T)|\mathfrak{p}\,, \quad \langle J_\mathfrak{p}(X), J_\mathfrak{p}(Y)\rangle_\mathfrak{p} = \langle X, Y\rangle_\mathfrak{p}$$

for $T \in \mathfrak{k}$, $X, Y \in \mathfrak{p}$. Two HSGLA's $(\mathfrak{g}, \rho, J_\mathfrak{p}, \langle\ \rangle_\mathfrak{p})$, $(\mathfrak{g}', \rho', J_{\mathfrak{p}'}, \langle\ \rangle_{\mathfrak{p}'})$ are *equivalent* to each other if there exists an isomorphism λ of $(\mathfrak{g}, \rho, \langle\ \rangle_\mathfrak{p})$ onto $(\mathfrak{g}', \rho', \langle\ \rangle_{\mathfrak{p}'})$ such that

$$J_{\mathfrak{p}'}\circ\lambda = \lambda\circ J_\mathfrak{p}\,.$$

***The construction* (III) *of HSGLA's from OJA's*:** Let $(A, \langle\ \rangle_A)$ be an OJA with the unity e and $(V, \{\ \}, \langle\ \rangle)$ the associated OJTS. By the construction (I) this OJTS induces an OSGLA $(\mathfrak{g}, \rho, \langle\ \rangle_\mathfrak{p})$. Put

$$J_\mathfrak{p} = \mathrm{ad}((e, 0, e))|\mathfrak{p}\,.$$

Then the quadruple $(\mathfrak{g}, \rho, J_\mathfrak{p}, \langle\ \rangle_\mathfrak{p})$ is an HSGLA.

Results in Naitoh [11]. (C) *The construction (III) keeps the equivalence of each object and is injective.*

2. Pseudo-riemannian symmetric $\boldsymbol{R}$-spaces

Let $\boldsymbol{E}$ be a pseudo-euclidean space, i.e., a finite dimensional vector space over $\boldsymbol{R}$ with a non-degenerate symmetric bilinear form $\langle\ \rangle_E$ on it. Let f be an isometric immersion of a pseudo-riemannian manifold M into $\boldsymbol{E}$. Denote by σ the second fundamental form on M, and by ∇, D the Levi-Civita connection of the tangent bundle TM, the normal connection of the normal bundle NM respectively. The isometric immersion f is called *parallel* if $\nabla^*\sigma=0$, i.e.,

$$(2.1) \qquad (\nabla_X^*\sigma)(Y, Z) = D_X(\sigma(Y, Z))-\sigma(\nabla_X Y, Z)-\sigma(Y, \nabla_X Z) = 0$$

for vector fields X, Y, Z tangent to M. If f is an imbedding, the image $f(M)$ is called a *parallel submanifold* of $\boldsymbol{E}$. Returning to an isometric immersion f, we consider the following two conditions for each point $p\in M$:

$$C_1(p):\quad N_pM = \{\sigma_p(X, Y);\, X,\, Y\in T_pM\}_{\boldsymbol{R}}\,,$$
$$C_2(p):\quad \text{There exists } \xi_0\in N_pM \text{ such that } A_{\xi_0}=-1_{T_pM}\,,$$

where A_{ξ_0}, 1_{T_pM} denote the shape operator for ξ_0, the identity map of T_pM respectively. Let $(M, \boldsymbol{E})$ be a pair of a pseudo-euclidean space $\boldsymbol{E}$ and a complete connected parallel submanifold M of $\boldsymbol{E}$ satisfying the conditions $C_1(p)$, $C_2(p)$ for every point $p\in M$. Two such pairs $(M, \boldsymbol{E})$, $(M', \boldsymbol{E}')$ are *equivalent* to each other if there exists an affine isometry ϕ of $\boldsymbol{E}$ onto $\boldsymbol{E}'$ such that

$$\phi(M) = M'\,.$$

Now we consider specific OJTS's $(V, \{\ \}, \langle\ \rangle)$ such that

$$(S) \qquad\qquad 1_V\in L\,,$$

and construct pairs $(M, \boldsymbol{E})$ from them. Conversely, we also construct such specific OJTS's from pairs $(M, \boldsymbol{E})$. Hereafter we call these pairs $(M, \boldsymbol{E})$ *r-pairs*.

***The construction* (IV) *of r-pairs from OJTS's*:** Let $(V, \{\ \}, \langle\ \rangle)$ be an OJTS with the condition (S) and $(\mathfrak{g}, \rho, \langle\ \rangle_{\mathfrak{p}})$ an OSGLA constructed from this OJTS. Let K be the connected Lie subgroup of $GL(\mathfrak{p})$ such that the Lie algebra is $\mathrm{ad}(\mathfrak{k})\,|\,\mathfrak{p}\subset\mathfrak{gl}(\mathfrak{p})$, and put

$$v = (0,\, 1_V,\, 0)\in\mathfrak{p}\,.$$

Then the orbit $M=K(v)$ is a pseudo-riemannian submanifold of the pseudo-euclidean space $\boldsymbol{E}=(\mathfrak{p}, \langle\ \rangle_{\mathfrak{p}})$. Moreover, it, together with the non-degenerate metric $\langle\ \rangle_{\mathfrak{p}}\,|\,M$ induced from $\langle\ \rangle_{\mathfrak{p}}$, is a pseudo-riemannian symmetric space, which is associated with the symmetric pair (K, K_0), where $K_0=\{k\in K;\, k(v)=v\}$ (cf. Naitoh [11]). We call this space $M=K(v)$ the *pseudo-riemannian symmetric R-space* associated with $(V, \{\ \}, \langle\ \rangle)$.

Proposition 2.1. *The pseudo-riemannian symmetric R-space M is a complete connected parallel submanifold of $\boldsymbol{E}$ satisfying the conditions $C_1(p)$, $C_2(p)$ for every point $p\in M$. Moreover, the construction*: $(V, \{\ \}, \langle\ \rangle)\to(M, \boldsymbol{E})$ *keeps the equivalence of each object.*

Proof. It has been proved in Naitoh ([11], Theorem 5.7, (1)) that M is a complete connected parallel submanifold. We show that M satisfies $C_1(p)$, $C_2(p)$ for $p\in M$. Denote by f the map of K/K_0 into E defined by $f(kK_0)=k(\nu)$. Since f is K-equivariant, it is sufficient to see that f satisfies $C_1(o)$, $C_2(o)$ at $o=eK_0$. Put

$$\mathfrak{k}_0=\mathfrak{k}\cap\mathfrak{g}_0\,,\quad \mathfrak{m}=\mathfrak{k}\cap(\mathfrak{g}_{-1}\oplus\mathfrak{g}_1)\,.$$

Then the Lie algebras of K, K_0 are isomorphic to $\mathfrak{k}$, $\mathfrak{k}_0$ respectively and the canonical decomposition of $\mathfrak{k}$ associated with the symmetric pair (K, K_0) is given by $\mathfrak{k}=\mathfrak{k}_0\oplus\mathfrak{m}$ (Naitoh [11]). Identifying $T_o(K/K_0)$ with $\mathfrak{m}$, in the same way as in Ferus ([3], Lemma 1), we have

$$(2.2)\qquad f_{*o}(X)=[X,\nu]\,,\quad \sigma_o(X,Y)=\mathrm{ad}(X)\mathrm{ad}(Y)\nu$$

for X, $Y\in\mathfrak{m}$. It follows that

$$T_\nu M=[\mathfrak{m},\nu]=(\mathfrak{g}_{-1}\oplus\mathfrak{g}_1)\cap\mathfrak{p}\,,\quad N_\nu M=\mathfrak{g}_0\cap\mathfrak{p}=\{\sigma_0(X,Y);\,X,Y\in\mathfrak{m}\}_R\,.$$

This implies the condition $C_1(o)$. The condition $C_2(o)$ also follows from (2.2).
The second claim is obvious. Q.E.D.

REMARK 2.2. A non-degenerate JTS always satisfies the condition (S) (cf. Theorem 3.1). Particularly in the case when the JTS $(V, \{\ \})$ is compact, the pseudo-riemannian symmetric R-space associated with the OJTS $(V, \{\ \}, 2\beta)$ is, the so called, "*riemannian symmetric R-space*".

***The construction* (V) *of OJTS's from r-pairs*:** Let $(M, \boldsymbol{E})$ be an r-pair and fix a point $p\in M$. Denote by R the curvature tensor of M and by σ, A the second fundamental form, the shape operator on M respectively. Put

$$V=T_pM\,,$$

and

$$\{X,Y,Z\}=R_p(X,Y)Z+A_{\sigma_p(X,Y)}Z,\ \langle X,Y\rangle=\langle X,Y\rangle_E$$

for X, Y, $Z\in V$.

Proposition 2.3. *The triple* $(V, \{\ \}, \langle\ \rangle)$ *is an OJTS satisfying the condition* (S). *Moreover the construction*: $(M, \boldsymbol{E})\rightarrow(V, \{\ \}, \langle\ \rangle)$ *keeps the equivalence of each object.*

Proof. It has been proved in Naitoh ([11], § 6) that the triple $(V, \{\ \}, \langle\ \rangle)$ is an OJTS. We show that this OJTS satisfies the condition (S). Take the normal vector ξ_0 in the condition $C_2(p)$. By the condition $C_1(p)$ it can be written in the following form: $\xi_0=\Sigma_j\sigma_p(X_i, Y_i)$ for some X_i, $Y_i\in V$. Hence we have

$$1_V = -A_{\xi_0} = -\Sigma_i A_{\sigma_p(X_i, Y_i)}$$
$$= -(1/2)\Sigma_i (L(X_i, Y_i) + L(Y_i, X_i)) \in L\,.$$

Note that, since σ is parallel, tensors R, $A_{\sigma(\,,\,)}$ are also parallel. Hence all the OJTS's constructed from points $p \in M$ are equivalent to each other. Under this note the second claim is obvious. Q.E.D.

Now we will show that the above two constructions (IV), (V) are the inverse of each other. Firstly we study "Gauss map" defined analogously to riemannian cases.

Let $\boldsymbol{E} = \boldsymbol{E}_{r,s}$ be a pseudo-euclidean space of signature (r, s), i.e., the signature of $\langle\ \rangle_E$ is (r, s). Taking a base point we identify $\boldsymbol{E}$ with the associated vector space. Fix integers k, t such that $0 \leqq k \leqq r$, $0 \leqq t \leqq s$, and denote by $G(k, r; t, s)$ the set of vector subspaces P of $\boldsymbol{E}$ such that the restrictions $\langle\ \rangle_E | P$ of $\langle\ \rangle_E$ to P are non-degenerate and have the signature (k, t). Let H be the identity component of the Lie group of linear isomorphisms of $\boldsymbol{E}$ which leave $\langle\ \rangle_E$ invariant. The Lie group H is semi-simple and acts transitively on the set $G(k, r; t, s)$. Fix $P \in G(k, r; t, s)$ and put $H_0 = \{h \in H;\ h(P) = P\}$. Then the pair (H, H_0) is a symmetric pair. Let $\mathfrak{h}$ be the Lie algebra of linear endomorphisms of $\boldsymbol{E}$ which are skew symmetric for $\langle\ \rangle_E$ and put

$$\mathfrak{h}_0 = \{h \in \mathfrak{h};\ h(P) \subset P\}\,, \quad \mathfrak{u} = \{h \in \mathfrak{h};\ h(P) \subset P^{\perp}\}\,,$$

where $P^{\perp}$ denotes the orthogonal compliment of P. Then $\mathfrak{h}$, $\mathfrak{h}_0$ are Lie algebras of H, H_0 respectively and the canonical decomposition of $\mathfrak{h}$ associated with the symmetric pair (H, H_0) is given by $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{u}$. The restriction $B|\mathfrak{u}$ of the Killing form B to $\mathfrak{u}$ is non-degenerate. Hence the set $G(k, r; t, s)$ has a structure of pseudo-riemannian symmetric space, associated with the symmetric pair (H, H_0) and with the metric $B|\mathfrak{u}$ at the origin P.

Denote by ∇^G the Levi-Civita connection of $G(k, r; t, s)$. A submanifold N of $G(k, r; t, s)$ is called *totally geodesic* if, for any vector fields X, Y tangent to N, the vector field $\nabla^G_X Y$is also tangent to N. This definition is equivalent to the following condition: for every $Q \in N$ and every $X \in T_Q N$, the geodesic in $G(k, r; t, s)$ starting from Q with the initial vector X is contained in N near Q. A totally geodesic submanifold has an affine connection by the restriction of ∇^G. A linear subspaces $\mathfrak{s}$ in $\mathfrak{u}$ is called a *Lie triple system* if it satisfies that $[\mathfrak{s}, [\mathfrak{s}, \mathfrak{s}]] \subset \mathfrak{s}$. Complete connected totally geodesic submanifolds N of $G(k, r; t, s)$ through P correspond one-to-one to Lie triple systems $\mathfrak{s}$ of $\mathfrak{u}$ by the relation: $T_p N = \mathfrak{s}$, and consequently they are also affine symmetric spaces. (See Kobayashi–Nomizu [7] for totally geodesic submanifolds).

Let M be a pseudo-riemannian manifold of signature (k, t) and f an isometric immersion of M into $\boldsymbol{E} = \boldsymbol{E}_{r,s}$. The *Gauss map* γ_f of f is a mapping of M into $G(k, r; t, s)$ defined by

$$\gamma_f(p) = f_*(T_pM)$$

for $p \in M$. Denote by $\sigma_p(X)$, $X \in T_pM$, linear mappings of T_pM into N_pM defined by

$$\sigma_p(X)(Y) = \sigma_p(X, Y)$$

for $Y \in T_pM$. Let $P \in G(k, r; t, s)$. Identify the vector space $\mathfrak{u}$ with the vector space $\mathfrak{u}'$ of linear mappings of P into $P^\perp$ by the correspondence: $\mathfrak{u} \ni h \to h | P \in \mathfrak{u}'$. Then, the tangent space $T_PG(k, r; t, s)$ is also identified with the vector space $\mathfrak{u}'$. Under this identification, linear mappings $\sigma_p(X)$ can be regarded as vectors in $T_{\gamma_f(p)}G(k, r; t, s)$. Similarly to riemannian cases, we have

$$(2.3) \qquad (\gamma_f)_*(X) = \sigma(X),$$
$$(2.4) \qquad (\nabla^*_X\sigma)(Y) = \nabla^G_X((\gamma_f)_* Y) - (\gamma_f)_*(\nabla_X Y)$$

for vector fields X, Y of M. Thus, f is parallel if and only if γ_f is connection-preserving, i.e.,

$$(2.5) \qquad \nabla^G_X((\gamma_f)_* Y) = (\gamma_f)_*(\nabla_X Y).$$

Lemma 2.4. *Let M be a connected pseudo-riemannian manifold of signature (k, t) and f a parallel isometric immersion of M into $E = E_{r,s}$. Fix a point $p \in M$ and put $P = f_*(T_pM)$. Then the subspace $\mathfrak{S} = \{\sigma_p(X) \in \mathfrak{u}';\ X \in T_pM\}$ of $\mathfrak{u}$ is a Lie triple system. Let N be the complete connected totally geodesic submanifold of $G(k, r; t, s)$ corresponding to the Lie triple system $\mathfrak{S}$. Then the image $\gamma_f(M)$ is contained in N. If M is complete, the image $\gamma_f(M)$ coincides with N.*

Proof. Denote by R, $R^\perp$ the curvature tensors for the Levi-Civita connection, the normal connection respectively. For a linear mapping h of T_pM into N_pM, h^t denotes the linear mapping of N_pM into T_pM defined by

$$\langle h^t(\xi), Y\rangle_E = \langle \xi, h(Y)\rangle_E$$

for $Y \in T_pM$. Since f is parallel, by the Gauss-Ricci equations of f, we have

$$(2.6) \qquad \begin{cases} R(X, Y) = \sigma(X)^t \circ \sigma(Y) - \sigma(Y)^t \circ \sigma(X), \\ R^\perp(X, Y) = \sigma(X) \circ \sigma(Y)^t - \sigma(Y) \circ \sigma(X)^t, \\ R^\perp(X, Y) \circ \sigma(Z) = \sigma(R(X, Y)Z) + \sigma(Z) \circ R(X, Y) \end{cases}$$

for $X, Y, Z \in T_pM$. Noting that, under the identification: $\mathfrak{u} \leftrightarrow \mathfrak{u}'$,

$$[A, [B, C]] = (B \circ A^t - A \circ B^t) \circ C - C \circ (B^t \circ A - A^t \circ B)$$

for $A, B, C \in \mathfrak{u}'$, we have

$$[\sigma(X), [\sigma(Y), \sigma(Z)]] = \sigma(R(Y, X)Z)$$

for X, Y, $Z\in T_pM$ by (2.6). It follows that $[\mathfrak{S}, [\mathfrak{S}, \mathfrak{S}]]\subset\mathfrak{S}$.

Now we show that $\gamma_f(M)\subset N$. Let q be a point of M and combine q with the point p by a broken geodesic $g(t)=\Sigma_j g_j(t)$ in M. Then, by (2.5), the curve $\gamma_f\circ g(t)=\Sigma_j \gamma_f\circ g_j(t)$ is a broken geodesic in $G(k, r; t, s)$ and the parallel translations of initial vectors of $\gamma_f\circ g_j$ to tangent vectors at the point $\gamma_f(p)$ are contained in $\mathfrak{S}$. Hence the complete totally geodesic submanifold N contains the broken geodesic $\gamma_f\circ g(t)$, and particularly the point q. This implies that $\gamma_f(M)\subset N$.

Assume that M is complete. Note that $\gamma_f\colon M\to N$ is a submersion by (2.5). Thus the image $\gamma_f(M)$ is an open submanifold of N. The completeness of M also implies the completeness of $\gamma_f(M)$. Hence $\gamma_f(M)$ is a connected complete totally geodesic submanifold corresponding to the same Lie triple system as N. It follows that $\gamma_f(M)=N$. Q.E.D.

Lemma 2.5. *Let $\hat{f}$ be a parallel isometric immersion of a complete pseudo-riemannian manifold $\hat{M}$ into $\boldsymbol{E}$. Let f be a parallel isometric immersion of a pseudo-riemannian manifold M into $\boldsymbol{E}$. Assume that there exist points $p\in M$, $\hat{p}\in\hat{M}$ such that*

$$f(p)=\hat{f}(\hat{p}), \quad f_*(T_pM)=\hat{f}_*(T_{\hat{p}}\hat{M}), \tag{2.7}$$

and

$$\sigma_p(X, Y)=\hat{\sigma}_{\hat{p}}(\hat{X}, \hat{Y}) \tag{2.8}$$

for X, $Y\in T_pM$, $\hat{X}$, $\hat{Y}\in T_{\hat{p}}\hat{M}$ such that $f_*(X)=\hat{f}_*(\hat{X})$, $f_*(Y)=\hat{f}_*(\hat{Y})$. Then the image $f(M)$ is contained in the image $\hat{f}(\hat{M})$. Moreover, if M is complete, the image $f(M)$ coincides with the image $\hat{f}(\hat{M})$.

Proof. Let $\boldsymbol{E}=\boldsymbol{E}_{r,s}$ and denote by (k, t) the same signature of M and $\hat{M}$. Two Gauss maps γ_f, $\gamma_{\hat{f}}$ satisfy

$$\gamma_f(p)=\gamma_{\hat{f}}(\hat{p})=P, \quad (\gamma_f)_*(T_pM)=(\gamma_{\hat{f}})_*(T_{\hat{p}}\hat{M})=\mathfrak{S}$$

for some $P\in G(k, r; t, s)$ and some $\mathfrak{S}\subset\mathfrak{u}'$ by (2.7), (2.8). Hence, by Lemma 2.4, there exists a unique complete connected totally geodesic submanifold N of $G(k, r; t, s)$ through P corresponding to the Lie triple system $\mathfrak{S}$ such that $\gamma_f(M)\subset N$, $\gamma_{\hat{f}}(\hat{M})\subset N$. Without a loss of generality, we may assume that M is simply connected. Put $\Phi=(\hat{f}_{*\hat{p}})^{-1}\circ f_{*p}$. Then, by (2.8) and Gauss equations for f, $\hat{f}$, Φ is a linear isometry of T_pM onto $T_{\hat{p}}\hat{M}$ which maps the curvature tensor of M into that of $\hat{M}$. Since M is a simply connected pseudo-riemannian locally symmetric space and $\hat{M}$ is a complete pseudo-riemannian locally symmetric space, there exists a unique isometry ψ of M into $\hat{M}$ such that

$$\psi(p)=\hat{p}, \quad \psi_{*p}=\Phi$$

by Kobayashi–Nomizu ([7], Chapter VI, §§6, 7). Then γ_f, $\gamma_{\hat f}\circ\psi$ are connection-preserving mappings of M into the affine symmetric space N by (2.5) and satisfy that $(\gamma_f)_{*p}=(\gamma_{\hat f}\circ\psi)_{*p}$ by (2.8). Again, by the Kobayashi–Nomizu's argument, we have

$$\gamma_f = \gamma_{\hat f}\circ\psi\,. \tag{2.9}$$

From this it follows that $\hat f_{*q}(T_qM)=\hat f_{*\psi(q)}(T_{\psi(q)}\hat M)$ and thus $N_qM=N_{\psi(q)}\hat M$ for $q\in M$. Define a bundle map ψ^N over ψ of NM onto $N\hat M$ by the parallel translation of vectors with respect to the canonical connection of $\boldsymbol{E}$. Then ψ^N preserves the fibre metrics and, by (2.9), preserves the second fundamental forms, i.e.,

$$\psi^N(\sigma_q(X,\,Y)) = \hat\sigma_{\psi(q)}(\psi_*X,\,\psi_*Y)$$

for $q\in M$, X, $Y\in T_qM$. Since the normal connections of NM, $N\hat M$ are induced from the canonical connection of the universal normal bundle on $G(k,\,r;\,t,\,s)$ (cf. Kobayashi–Nomizu [7]), ψ^N also preserves the normal connections. Denote by $f^{-1}\boldsymbol{TE}$, $\hat f^{-1}\boldsymbol{TE}$ the pull-backs of the tangent bundle $\boldsymbol{TE}$ by f, $\hat f$ respectively. Then we have the canonical identifications:

$$TM\oplus NM = f^{-1}\boldsymbol{TE} = M\times\boldsymbol{E}\,,\quad T\hat M\oplus N\hat M=\hat f^{-1}\boldsymbol{TE}=\hat M\times\boldsymbol{E}\,.$$

Along the proof of the rigidity theorem in Szczarba [17] there exists $h\in H$ such that, under these identifications, the mapping $\psi_*\oplus\psi^N$: $TM\oplus NM\to T\hat M\oplus N\hat M$ is identified with the mapping $\psi\times h$: $M\times\boldsymbol{E}\to\hat M\times\boldsymbol{E}$. Then, by the condition that $f_{*p}=\hat f_{*\hat p}\circ\psi_{*p}$, we have $(\psi\times h)(p\times X)=\hat p\times X$ for $X\in\boldsymbol{E}$ and thus $h=1_{\boldsymbol{E}}$. It follows that $f_*=\hat f_*\circ\psi_*$. Since M is connected, there exists $c\in\boldsymbol{E}$ such that $f=\hat f\circ\psi+c$. The condition (2.7) implies that $f=\hat f\circ\psi$. Hence it follows that $f(M)\subset\hat f(\hat M)$.

Assume that M is complete. Then we also have $\hat f(\hat M)\subset f(M)$ by the above result. Hence it follows that $f(\hat M)=\hat f(\hat M)$. Q.E.D.

Remark. Lemma 2.4 was proved in Vilms [19] and Lemma 2.5 in Ferus [2] for the riemannian cases. The proof of Lemma 2.5 is along the Ferus' argument.

Theorem 2.6. *Two constructions* (IV), (V) *are the inverses of each other.*

Proof. (a) Let $(M,\,\boldsymbol{E})$ be an r-pair. Denote by $(V,\,\{\ \},\langle\ \rangle)$ the OJTS constructed from the r-pair $(M,\,\boldsymbol{E})$ and, moreover, by $(\hat M,\,\hat{\boldsymbol{E}})=(K(\nu),\,(\mathfrak{p},\langle\ \rangle_{\mathfrak{p}}))$ the r-pair constructed from the OJTS $(V,\,\{\ \},\langle\ \rangle)$. We show that two r-pairs $(M,\,\boldsymbol{E})$, $(\hat M,\,\hat{\boldsymbol{E}})$ are equivalent to each other. Fix $p\in M$ and identify the associated vector space of $\boldsymbol{E}$ with $T_pM\oplus N_pM$. Let ξ_0 be the normal vector in the condition $C_2(p)$. Define an affine mapping ϕ of $\boldsymbol{E}$ onto $\hat{\boldsymbol{E}}$ by

$$\phi(X+\xi+(p-\xi_0)) = (-X, A_\xi, X)$$

for $X\in T_pM$, $\xi\in N_pM$. Note that $2A_\xi=\Sigma_j\{L(u_j, v_j)+L(v_j, u_j)\}$ for $\xi=\Sigma_j\sigma(u_j, v_j)\in N_pM$. Then, from the definition of $\langle\ \rangle_{\mathfrak{p}}$, it follows that ϕ is an isometry. Moreover the mapping ϕ satisfies

$$\phi(p) = \nu\,,\quad \phi_*(T_pM) = T_\nu\hat{M},\quad \hat{\sigma}_\nu(\phi_*(X), \phi_*(Y)) = \phi_*(\sigma_p(X, Y))$$

for X, $Y\in T_pM$. The first equality is obvious by the condition $C_2(p)$ and the third equality is calculated by (2.2). Hence two submanifolds $\phi(M)$, $\hat{M}$ of $\hat{\boldsymbol{E}}$ satisfy the assumptions of Lemma 2.5. It follows that $\phi(M)=\hat{M}$. This implies that $(M, \boldsymbol{E})$, $(\hat{M}, \hat{\boldsymbol{E}})$ are equivalent to each other.

(b) Let $(V, \{\ \}, \langle\ \rangle)$ be an OJTS satisfying the condition (S). Denote by $(\hat{M}, \hat{\boldsymbol{E}})=(K(\nu), (\mathfrak{p}, \langle\ \rangle_{\mathfrak{p}}))$ the r-pair constructed from this OJTS and, moreover, by $(V', \{\ \}', \langle\ \rangle')$ the OJTS constructed from the r-pair $(\hat{M}, \hat{\boldsymbol{E}})$. Define a linear mapping δ of V onto V' by

$$\delta(x) = (x, 0, x)$$

for $x\in V$. Obviously δ is an isometry. Calculating the shape operator $\hat{A}$ of $\hat{M}$ by using (2.2), we have

$$\hat{A}_{\hat{\sigma}_\nu(\hat{A},\hat{B})}\hat{C} = ((1/2)(L(x, y)+L(y, x))z,\ 0,\ -(1/2)\,(L(x, y)+L(y, x))z)$$

for $\hat{A}=(x, 0, -x)$, $\hat{B}=(y, 0, -y)$, $\hat{C}=(z, 0, -z)\in T_\nu\hat{M}$. Hence, by the Gauss equation of $\hat{M}$, it follows that

$$\begin{aligned}\{\delta(x), \delta(y), \delta(z)\}' &= \hat{A}_{\hat{\sigma}_\nu(\hat{A},\hat{B})}\hat{C}+\hat{A}_{\hat{\sigma}_\nu(\hat{B},\hat{C})}\hat{A}-\hat{A}_{\hat{\sigma}_\nu(\hat{A},\hat{C})}\hat{B}\\ &= \delta(\{x, y, z\})\,.\end{aligned}$$

This implies that two OJTS's $(V, \{\ \}, \langle\ \rangle)$, $(V', \{\ \}', \langle\ \rangle')$ are equivalent to each other. Q.E.D.

REMARK 2.7. (1) Consider the conditions $C_1(p)$, $C_2(p)$ for a pair $(M, \boldsymbol{E})$. Note that, to construct the OJTS from $(M, \boldsymbol{E})$, it is sufficient to satisfy the conditions for some point. Moreover, note that the assumption of Lemma 2.5 is a condition for one point. Hence, by Theorem 2.6, (a), the term "for every point" in the conditions for $(M, \boldsymbol{E})$ can be rewritten into the term "for some point".

(2) Let $(M, \boldsymbol{E})$ be an r-pair. Fix a point $p\in M$ and let ξ_0 be the normal vector in the condition $C_2(p)$. Put $o=p-\xi_0\in\boldsymbol{E}$. By Theorem 2.6, (a), the function: $M\ni q\to\langle q-o, q-o\rangle_{\boldsymbol{E}}\in\boldsymbol{R}$ is constant. Denote by c the constant and put $H(c)=\{q\in\boldsymbol{E};\langle q-o, q-o\rangle_{\boldsymbol{E}}=c\}$. If $c=0$, the submanifold M is contained in the null cone $H(0)$ of $\boldsymbol{E}$.

Assume that $c\neq 0$. Then $H(c)$ containing M is a pseudo-riemannian hypersurface of $\boldsymbol{E}$. Conversely let M be a complete connected parallel submanifold of $\boldsymbol{E}$ satisfying the condition $C_1(p)$ for some $p\in M$ and contained in $H(c)$ $(c\neq 0)$ centered at $o\in\boldsymbol{E}$. Since $c\neq 0$, $\xi_0=p-o$ is a normal vector of $H(c)$ and thus of M. Note that the shape operator A^H of $H(c)$ is given by

$$(2.11) \qquad A^H_\xi(X) = -(1/c)\langle\xi_0, \xi\rangle_E X$$

for $X\in T_pH(c)$, $\xi\in N_pH(c)$. Then we have $A_{\xi_0}=A^H_{\xi_0}|T_pM=-1_{T_pM}$. This implies that M satisfies the condition $C_2(p)$. Hence the classification of complete connected parallel submanifolds of $\boldsymbol{E}$ satisfying $C_1(p)$ for some p and contained in $H(c)$, $c\neq 0$, is reduced to that of OJTS's satisfying (S) and $\langle v, v\rangle_{\mathfrak{p}}=c$. Particularly, in riemannian cases, the classification gives that of complete connected strongly full parallel submanifolds of a euclidean sphere (cf. Ferus [3], Takeuchi [18]).

A pseudo-hermitian space, by definition, is a pseudo-euclidean space $\boldsymbol{E}$ with an almost complex structure J satisfying that

$$\langle J(x), J(y)\rangle_E = \langle x, y\rangle_E$$

for x, $y\in\boldsymbol{E}$. A pseudo-riemannian submanifold M of a pseudo-hermitian space $\boldsymbol{H}=(\boldsymbol{E}; J)$ is called *totally real* if $J(TM)\subset NM$. Consider pairs $(M, \boldsymbol{H})$ of pseudo-hermitian spaces $\boldsymbol{H}$ and $(\dim \boldsymbol{H}/2)$-dimensional complete connected totally real parallel submanifolds M of $\boldsymbol{H}$ satisfying the conditions $C_1(p)$, $C_2(p)$ for every $p\in M$. We call such pairs $(M, \boldsymbol{H})$ *h-pairs*. Two h-pairs $(M, \boldsymbol{H})$, $(M', \boldsymbol{H}')$ are *equivalent* to each other if there exists a holomorphic isometry ϕ of $\boldsymbol{H}$ onto $\boldsymbol{H}'$ such that

$$\phi(M) = M'.$$

Now we will give a one-to-one correspondence between h-pairs and OJA's with unity. The correspondence is a special case of the constructions (IV), (V).

***The construction* (VI) *of h-pairs from OJA's*:** Let $(A, \langle\ \rangle_A)$ be an OJA with the unity e. Denote by $(V, \{\ \}, \langle\ \rangle)$ the OJTS associated with $(A, \langle\ \rangle_A)$. Then, since $L(e, e)=T_e=1_V$, this OJTS satisfies the condition (S). Let $(\mathfrak{g}, \rho, J_{\mathfrak{p}}, \langle\ \rangle_{\mathfrak{p}})$ be the HSGLA constructed from the OJA $(A, \langle\ \rangle_A)$ and put

$$M = K(v)\,, \quad \boldsymbol{H} = (\mathfrak{p}, J_{\mathfrak{p}}, \langle\ \rangle_{\mathfrak{p}})\,.$$

Proposition 2.8. *The pseudo-riemannian symmetric R-space M is* $(\dim \boldsymbol{H}/2)$-*dimensional and totally real in $\boldsymbol{H}$. Moreover the construction*: $(A, \langle\ \rangle_A)\rightarrow(M, \boldsymbol{H})$ *keeps the equivalence of each object.*

Proof. Note that $N_vM=\{(0, T_x, 0);\ x\in A\}$. Then the first claim is obvious from the definition of $J_{\mathfrak{p}}$ (cf. Naitoh [11], Theorem 5.7, (2)). The second claim is obvious. Q.E.D.

The construction (VII) ***of OJA's from h-pairs***: Let $(M, \boldsymbol{H})$ be an h-pair and denote $\boldsymbol{H}=(\boldsymbol{E}; J)$. Fix $p \in M$ and define a product $\cdot$ on T_pM by

$$X \cdot Y = J\sigma_p(X, Y)$$

for $X, Y \in T_pM$. Denote by A this algebra and put $\langle\ \rangle_A = \langle\ \rangle_{\boldsymbol{E}}$.

Proposition 2.9. *The pair $(A, \langle\ \rangle_A)$ is an OJA and the vector $E=-J(\xi_0)$ is the unity of A. Moreover the construction: $(M, \boldsymbol{H}) \to (A, \langle\ \rangle_A)$ keeps the equivalence of each object.*

Proof. It has been proved in Naitoh ([11], Lemma 6.1) that the pair $(A, \langle\ \rangle_A)$ is an OJA. We show that the vector E is the unity of A. Note that

$$J\sigma_p(X, Y) = -A_{JX}Y \tag{2.12}$$

for $X, Y \in T_pM$, since M is totally real (cf. Naitoh [9], Lemma 2.4). Thus we have

$$X \cdot E = E \cdot X = -J\sigma_p(J(\xi_0), X) = -A_{\xi_0}(X) = X$$

for $X \in T_pM$ by the condition $C_2(p)$. It follows that E is the unity of A.

Note that, since σ is parallel and M is totally real, the tensor $J\sigma$ on M is parallel with respect to the Levi-Civita connection of M (cf. Naitoh [10], Lemma 1.2). Hence all the OJA's constructed from points $p \in M$ are equivalent to each other. Under this note the second claim is obvious. Q.E.D.

REMARK 2.10. The OJTS associated with this OJA is the same as defined in Proposition 2.3 (cf. Naitoh [11], Lemma 6.1).

Theorem 2.11. *Two constructions* (VI), (VII) *are the inverses of each other.*

Proof. (a) Let $(A, \langle\ \rangle_A)$ be an OJA with unity. Denote by $(A', \langle\ \rangle_{A'})$ the OJA constructed from $(A, \langle\ \rangle_A)$ by carrying out the constructions (VI), (VII) succesitively. It has been proved in Naitoh ([11], Theorem 6.3, (2)) that two OJA's are equivalent to each other.

(b) Let $(M, \boldsymbol{H})$ be an h-pair and fix a point $p \in M$. Denote by $(A, \langle\ \rangle_A)$ the OJA constructed from the h-pair and the point p and, moreover, by $(\hat{M}, \hat{\boldsymbol{H}})$ the h-pair constructed from $(A, \langle\ \rangle_A)$. We show that two h-pairs $(M, \boldsymbol{H})$, $(\hat{M}, \hat{\boldsymbol{H}})$ are equivalent to each other. Let ϕ be the affine mapping of $\boldsymbol{H}$ onto $\hat{\boldsymbol{H}}$ defined in the proof of Theorem 2.6. Then ϕ is an isometry such that $\phi(M)=\hat{M}$. Note that

$$\phi_*(X+JY)=(-X,\ -T_Y,\ X)$$

for $X,\ Y\in T_pM$ by (2.12). Then it follows that $\phi_*\circ J=J_{\mathfrak{p}}\circ\phi_*$ from the definitionof $J_{\mathfrak{p}}$. Hence two h-pairs $(M,\ \boldsymbol{H})$, $(\hat{M},\ \hat{\boldsymbol{H}})$ are equivalent to each other.
Q.E.D.

REMARK 2.12. Theorem 6.3 in Naitoh [11] is a special case of Theorem 2.11.

3. Minimal pseudo-riemannian symmetric R-spaces

Let M, $\bar{M}$ be pseudo-riemannian manifolds and f an isometric immersion of M into $\bar{M}$. Denote by $\langle\ \rangle_M$ the metric on M and let (r, s) be the signature of M. Let $\{e_1, \cdots, e_r, e_{r+1}, \cdots, e_{r+s}\}$ be an orthonormal basis of T_pM, i.e., $\langle e_i, e_i\rangle_M=-1$, $\langle e_j, e_j\rangle_M=1$, $\langle e_a, e_b\rangle_M=0$ for $1\leqq i\leqq r$, $1+r\leqq j\leqq r+s$, $1\leqq a\neq b\leqq r+s$. The *mean curvature vector* η_p at $p\in M$ is defined by

$$(r+s)\eta_p=-\sum_{i=1}^r\sigma_p(e_i, e_i)+\sum_{j=r+1}^{r+s}\sigma_p(e_j, e_j)\,,$$

where σ denotes the second fundamental form of f. The mean curvature vector is independent of the choice of an orthonormal basis of T_pM. The isometric immersion f is called *minimal* if $\eta_p=0$ for all $p\in M$. Moreover, if f is an imbedding, the image $f(M)$ is also called a *minimal submanifold* of $\bar{M}$.

Theorem 3.1.(*) *Non-degenerate JTS's satisfy the condition (S). Fix a real number $c\neq 0$ and let $(V,\ \{\ \},\ \langle\ \rangle)$ be an OJTS satisfying the condition (S). Then the pseudo-riemannian symmetric R-space associated with the OJTS is a minimal submanifold of $H(c)$ centered at the origin of the vector space $\mathfrak{p}$ if and only if $(V,\ \{\ \})$ is non-degenerate and $\langle\ \rangle=(c/\dim V)\cdot\beta$, where β denotes the trace form of $(V,\ \{\ \})$.*

Proof. Let $(V,\ \{\ \},\ \langle\ \rangle)$ be an OJTS with trace form β and l a linear endomorphism of V defined by $\beta(x, y)=\langle l(x), y\rangle$ for $x, y\in V$. Then we have

$$l=-\sum_{i=1}^r L(e_i, e_i)+\sum_{j=r+1}^{r+s}L(e_j, e_j)\,, \tag{3.1}$$

where $\{e_1, \cdots, e_r, e_{r+1}, \cdots, e_{r+s}\}$ denotes an orthonormal basis of V for the metric $\langle\ \rangle$ of signature (r, s) (cf. Naitoh [11], (7.1)). Assume that the OJTS $(V,\ \{\ \},\ \langle\ \rangle)$ satisfies the condition (S) and that the pseudo-riemannian symmetric R-space $\hat{M}=K(\nu)$ associated with the OJTS is contained in $H(\hat{c})$ $(\hat{c}\neq 0)$ centered at the origin of the vector space $\mathfrak{p}$. Denote by $\hat{\eta}_\nu$, η_ν the mean curvature vectors at $\nu\in\hat{M}$ of the inclusions: $\hat{M}\to\mathfrak{p}$, $\hat{M}\to H(\hat{c})$ respectively.

(*) C. Blomstrom has proved this theorem using the Lie algebra theory in: Symmetric R-spaces with indefinite metric, Abstracts of Amer. Math. Soc., 4 (1983), No. 4, 83T-53-246.

Then we have

$$(3.2)\qquad \hat{\eta}_\nu = \eta_\nu - (1/\hat{c})\cdot\nu$$

by (2.11) and, on the other hand, we have

$$(3.3)\qquad (\dim V)\cdot\hat{\eta}_\nu = (0,\, l,\, 0)$$

by (2.2), (3.1). Note that the inclusion: $\hat{M}\to\mathfrak{p}$ is K-equivariant. Then, by (3.2), (3.3), $\hat{M}$ is a minimal submanifold of $H(\hat{c})$ if and only if $l=(\dim V/\hat{c})\cdot 1_V$.

Now we show the first claim. Assume that $(V, \{\ \})$ is non-degenerate and consider the OJTS $(V, \{\ \}, \beta)$. Then, since $l=1_V$, we have $1_V\in L$ by (3.1). Hence this JTS satisfies the condition (S).

Next we show the second claim. Assume that $(V, \{\ \})$ is non-degenerate and put $(V, \{\ \}, \langle\ \rangle)=(V, \{\ \}, (c/\dim V)\cdot\beta)$. Then, since $l=(\dim V/c)\cdot 1_V$, it follows that $\langle\nu, \nu\rangle_{\mathfrak{p}}=c$ by (3.1) and thus $\hat{M}=K(\nu)\subset H(c)$. Moreover, by the above note, $\hat{M}$ is a minimal submanifold of $H(c)$. The converse is obvious by the above note. Q.E.D.

REMARK 3.1. Theorem 3.1 implies that the classification of complete connected parallel submanifolds M of $\boldsymbol{E}$ satisfying the condition $C_1(p)$ for every $p\in M$ and being minimal in $H(c)$, $c\neq 0$, is reduced to that of non-degenerate JTS's. Ferus [4] has proved the theorem for riemannian cases.

REMARK 3.2. Two examples of pseudo-riemannian symmetric R-spaces which are not minimal have been given in Naitoh [11]. One is the case when the Lie algebra $\mathfrak{g}$ is semi-simple, and the other is the case when the Lie algebra $\mathfrak{g}$ is not reductive.

Now we study OSGLA's and pseudo-riemannian symmetric R-spaces associated with non-degenerate OJTS's $(V, \{\ \}, (c/\dim V)\cdot\beta)$, $c\neq 0$. A JTS $(V, \{\ \})$ over $\boldsymbol{K}$ is called *simple* if all the subspaces W of V satisfying that $\{W, V, V\}\subset W$, $\{V, W, V\}\subset W$, $\{V, V, W\}\subset W$ are only $\{0\}$ and V. Obviously simple JTS's are non-degenerate and it is known that any non-degenerate JTS is decomposed into the finite sum of simple JTS's uniquely up to an order. Return to the case when $\boldsymbol{K}=\boldsymbol{R}$. Let $(V, \{\ \})$ be a non-degenerate JTS and fix a real number $c\neq 0$. Suppose that the JTS is decomposed into the sum of simple JTS's $(V_j, \{\ \}_j)$, $1\leqq j\leqq k$, and denote by β, β_j the trace forms of $(V, \{\ \})$, $(V_j, \{\ \}_j)$ respectively. Put $c_j=(\dim V_j/\dim V)c$. Then the OJTS $(V, \{\ \}, (c/\dim V)\cdot\beta)$ is decomposed into the sum of the OJTS's $(V_j, \{\ \}_j, (c_j/\dim V_j)\cdot\beta_j)$, $1\leqq j\leqq k$. Hence objects associated with $(V, \{\ \}, (c/\dim V)\cdot\beta)$ is also decomposed into the sum of objects associated with $(V_j, \{\ \}_j, (c_j/\dim V_j)\cdot\beta_j)$, $1\leqq j\leqq k$. Particularly, it follows that

$$(3.4)\qquad (\hat{M}, \hat{\boldsymbol{E}}) = (\hat{M}_1, \hat{\boldsymbol{E}}_1)\times\cdots\times(\hat{M}_k, \hat{\boldsymbol{E}}_k)\,,$$

where $(\hat{M}, \hat{E})$,$(\hat{M}_j, \hat{E}_j)$ denote the pseudo-riemannian symmetric R-spaces associated with OJTS's $(V, \{ \}, (c/\dim V)\cdot\beta)$, $(V_j, \{ \}_j, (c_j/\dim V_j)\cdot\beta_j)$ respectively. Next we note that, in the construction: OJTS→OSGLA, SGLA's $(\mathfrak{g}, \rho)$ are independent of changing the metrics of OJTS's. Hence homogeneous spaces K/K_0 associated with pseudo-riemannian symmetric R-spaces are also independent of the metrics. Under these notes, our aim is to list up objects $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{k}_0$ associated with all simple JTS's.

Now Neher has classified all simple JTS's over $\boldsymbol{R}$ by constructing new JTS's from compact simple JTS's in the following two ways: one is the "complexification", and another is the "modification".

(a) ***Complexification.*** Let $(V, \{ \})$ be a JTS over $\boldsymbol{R}$. Put $V^C = \boldsymbol{C}\otimes V$ and extend the $\boldsymbol{R}$-trilinear mapping $\{ \}$ into a $\boldsymbol{C}$-trilinear mapping $\{ \}^C$: $V^C\times V^C\times V^C\to V^C$. Then the pair $(V^C, \{ \}^C)$ is a JTS over $\boldsymbol{C}$ and thus can be regarded as a JTS over $\boldsymbol{R}$ naturally. We call this JTS the *complexification* of $(V, \{ \})$.

(b) ***Modification.*** Let $(V, \{ \})$ be a JTS over $\boldsymbol{R}$ and α an involutive automorphism of the JTS. Denote by $V_\pm$ the ± 1-eigenspaces of α respectively. Put

$$V^\alpha = V_+\oplus\sqrt{-1}V_-\subset V^C,\ \{ \}^\alpha = \{ \}^C|V^\alpha.$$

Then the pair $(V^\alpha, \{ \}^\alpha)$ is a JTS over $\boldsymbol{R}$. We call this JTS the *modification* of $(V, \{ \})$ by α.

Results in Neher [12]. (1) *All simple JTS's over* $\boldsymbol{R}$ *are realized as all complexifications and modifications of compact simple JTS's.*

(2) *Two complexifications of compact simple JTS's are equivalent to each other if and only if the original JTS's are so.*

(3) *Two modifications of compact simple JTS's are equivalent to each other if and only if there exists an isomorphism between the original JTS's through which the involutive automorphisms are conjugate to each other.*

(4) *Complexifications and modifications of compact simple JTS's are never equivalent to each other.*

Moreover, in Neher [13], [14], all involutive automorphisms on compact simple JTS's have been given explicitly up to conjugacy. To list up our objects we will use his explicit results and the following three lemmas.

Let $(V, \{ \})$ be a non-degenerate JTS over $\boldsymbol{R}$ and $(V^C, \{ \}^C)$ the complexification. Denote by $(\mathfrak{g}, \rho)$, $(\mathfrak{g}^C, \rho^C)$ their semi-simple SGLA's constructed from $(V, \{ \})$, $(V^C, \{ \}^C)$ respectively.

Lemma 3.3. *The Lie algebra* $\mathfrak{g}^C$ *is the complexification of* $\mathfrak{g}$ *and the involute automorphism* ρ^C *is the* $\boldsymbol{C}$*-linear extension of* ρ *to* $\mathfrak{g}^C$. *Moreover* $\nu^C=\nu$.

Hence Lie algebras $(\mathfrak{g}^C)_\mu$, $\mu=0, \pm 1$, $\mathfrak{k}^C$, $(\mathfrak{k}^C)_0$ *are the complexifications of* $\mathfrak{g}_\mu$, $\mu=0, \pm 1$, $\mathfrak{k}$, $\mathfrak{k}_0$ *respectively.*

Let $(V^\alpha, \{\ \}^\alpha)$ be the modification of $(V, \{\ \})$ by an involutive automorphism α. Denote by $(\mathfrak{g}^\alpha, \rho^\alpha)$ the semi-simple SGLA constructed from $(V^\alpha, \{\ \}^\alpha)$. Note that $(V^C, \{\ \}^C)$ is also the complexification of $(V^\alpha, \{\ \}^\alpha)$. For an automorphism δ of $(V, \{\ \})$ denote by $\bar{\delta}$ an automorphism of $(\mathfrak{g}, \rho)$ defined by

$$\bar{\delta}((x, f, y)) = (\delta(x), \delta\circ f\circ\delta^{-1}, \delta(y))$$

for $(x, f, y)\in\mathfrak{g}$,and by $\bar{\delta}^C$ the $\boldsymbol{C}$-linear extension of $\bar{\delta}$ to $\mathfrak{g}^C$. Denote by τ the conjugation of $\mathfrak{g}^C$ with respect to the real form $\mathfrak{g}$.

Lemma 3.4. *The conjugation* τ^α *of* $\mathfrak{g}^C$ *with respect to* $\mathfrak{g}^\alpha$ *is given by* $\tau^\alpha=\bar{\alpha}^C\circ\tau$ *and, moreover,* $\rho^\alpha=\rho^C|\mathfrak{g}^\alpha$, $\nu^\alpha=\nu$. *Hence the restrictions of* τ^α *to* $(\mathfrak{g}^C)_\mu$, $\mu=0, \pm 1$, $\mathfrak{k}^C$, $(\mathfrak{k}^C)_0$ *are conjugations with respect to* $(\mathfrak{g}^\alpha)_\mu$, $\mu=0, \pm 1$, $\mathfrak{k}^\alpha$, $(\mathfrak{k}^\alpha)_0$ *respectively.*

Proof. Note that $[\bar{\alpha}^C, \tau]=0$. Then it follows easily that $\bar{\alpha}^C\circ\tau$ is a conjugation of $\mathfrak{g}^C$. Denote by $L_\pm$ the ± 1-eigenspaces of the involutive automorphism: $L\ni f\to\alpha\circ f\circ\alpha\in L$. Regard L, L^α as subalgebras of L^C naturally. Then we can see easily that $L^\alpha=L_+\oplus\sqrt{-1}L_-$. Under this note it follows that $\mathfrak{g}^\alpha=\{X\in\mathfrak{g}^C;\ \bar{\alpha}^C\circ\tau(X)=X\}$. This implies that $\bar{\alpha}^C\circ\tau$ is a conjugation with respect to $\mathfrak{g}^\alpha$.

The other claims are obvious by Lemma 3.3. Q.E.D.

A JTS $(V, \{\ \})$ over $\boldsymbol{R}$ is called *hermitian* if there exists a complex structure J on V satisfying that

$$J\{x, y, z\} = \{Jx, y, z\} = -\{x, Jy, z\} = \{x, y, Jz\}$$

for $x, y, z\in V$. A *conjugation* of a hermitian JTS $(V, \{\ \})$ is, by definition, an involutive automorphism δ of the JTS such that $\delta\circ J+J\circ\delta=0$. Put

$$V_\delta = \{x\in V;\ \delta(x)=x\}, \quad \{\ \}_\delta = \{\ \}|V_\delta.$$

This JTS $(V_\delta, \{\ \}_\delta)$ is called a *real form* of $(V, \{\ \})$ defined by the conjugation δ. A real form of a hermitian JTS is non-degenerate if and only if so is the original JTS.

Let $(V, \{\ \})$ be a non-degenerate hermitian JTS and $(V_\delta, \{\ \}_\delta)$ a real form of the JTS defined by a conjugation δ. Note that

$$(L_\delta)^C(x, y) = L(x, \delta(y)) \tag{3.5}$$

for $x, y\in V$ and thus $(L_\delta)^C=L$. For $f\in L$, denote by f^*, $f^{*(\delta)}$ the transposed maps of f with respect to the trace forms of $(V, \{\ \})$, $(V_\delta, \{\ \}_\delta)$ respectively.

Then, by (3.5), we have

$$f^{*(\delta)} = \delta \circ f^* \circ \delta . \tag{3.6}$$

Let $(\mathfrak{g}, \rho)$, $(\mathfrak{g}_\delta, \rho_\delta)$ be the semi-simple SGLA's constructed from $(V, \{ \})$, $(V_\delta, \{ \}_\delta)$ respectively. Note that $(\mathfrak{g}_\delta)^C = \mathfrak{g}$ as linear spaces. Define an $\boldsymbol{R}$-linear mapping Λ of $(\mathfrak{g}_\delta)^C$ onto $\mathfrak{g}$ by

$$\Lambda((x, f, y)) = (x, f, \delta(y))$$

for $(x, f, y) \in (\mathfrak{g}_\delta)^C$. Then Λ is a Lie algebra isomorphism by (3.5), (3.6). Let $((V_\delta)^\alpha, \{ \}_\delta^\alpha)$ be the modification of $(V_\delta, \{ \}_\delta)$ by an involutive automorphism α of $(V_\delta, \{ \}_\delta)$.

Lemma 3.5. *The image* $\Lambda((\mathfrak{g}_\delta)^\alpha)$ *is given by*

$$\Lambda((\mathfrak{g}_\delta)^\alpha) = \{X \in \mathfrak{g}; \bar{\alpha}^C \circ \bar{\delta}(X) = X\}$$

and, moreover, $\Lambda \circ (\rho_\delta)^\alpha \circ \Lambda^{-1} = \bar{\delta} \circ \rho \,|\, \Lambda((\mathfrak{g}_\delta)^\alpha)$, $\Lambda((\nu_\delta)^\alpha) = \nu$. *Hence the images* $\Lambda(((\mathfrak{g}_\delta)^\alpha)_\mu)$, $\mu = 0, \pm 1$, $\Lambda((\mathfrak{k}_\delta)^\alpha)$, $\Lambda(((\mathfrak{k}_\delta)^\alpha)_0)$ *are given by*

$$\begin{aligned} \Lambda(((\mathfrak{g}_\delta)^\alpha)_\mu) &= \{X \in \mathfrak{g}_\mu; \bar{\alpha}^C \circ \bar{\delta}(X) = X\} , \\ \Lambda((\mathfrak{k}_\delta)^\alpha) &= \{X \in \mathfrak{g}; \bar{\alpha}^C \circ \bar{\delta}(X) = \bar{\delta} \circ \rho(X) = X\} , \\ \Lambda(((\mathfrak{k}_\delta)^\alpha)_0) &= \{X \in \mathfrak{g}_0; \bar{\alpha}^C \circ \bar{\delta}(X) = \bar{\delta} \circ \rho(X) = X\} . \end{aligned}$$

Proof. We show the first claim. Denote by τ the conjugation of $\mathfrak{g}^C$ with respect to $\mathfrak{g}_\delta$. Then, by Lemma 3.4, it follows that

$$\Lambda((\mathfrak{g}_\delta)^\alpha) = \{X \in \mathfrak{g}; \Lambda \circ \bar{\alpha}^C \circ \tau \circ \Lambda^{-1}(X) = X\} .$$

Here, by easy calculations, we can see that $\Lambda \circ \tau \circ \Lambda^{-1} = \tau$, $\Lambda \circ \bar{\alpha}^C \circ \Lambda^{-1} = \bar{\alpha}^C$, $\tau = \bar{\delta}$, and thus $\Lambda \circ \bar{\alpha}^C \circ \tau \circ \Lambda^{-1} = \bar{\alpha}^C \circ \bar{\delta}$. Moreover, by Lemma 3.4, (3.6), it is verified that $\Lambda \circ (\rho_\delta)^\alpha \circ \Lambda^{-1} = \bar{\delta} \circ \rho$ on $\Lambda((\mathfrak{g}_\delta)^\alpha)$ and it is obvious by Lemma 3.4 that $\Lambda((\nu_\delta)^\alpha) = \nu$.

The other claims are obvious. Q.E.D.

Now we start to list up our objects. Compact simple hermitian JTS's are devided into six types. They are denoted by notations: I, II, III, IV, V, VI according to the classical ones. Compact simple JTS's which are not hermitian are realized as all real forms of compact simple hermitian JTS's.

Let $M(p, q; \boldsymbol{C})$ be the vector space of complex (p, q)-matrices and denote by $\kappa(*) = \bar{*}$, $\upsilon(*) = {*}^t$ the ordinary conjugation, transpose of $*$ respectively. Put

$$S(p, j) = \begin{pmatrix} 1_j & 0 \\ 0 & -1_{p-j} \end{pmatrix}, \quad S(p, j; q, k) = \begin{pmatrix} S(p, j) & 0 \\ 0 & S(q, k) \end{pmatrix},$$

$$J(2p)=\begin{pmatrix} 0 & 1_p \\ -1_p & 0 \end{pmatrix}, \quad I(2p)=\begin{pmatrix} 0 & 1_p \\ 1_p & 0 \end{pmatrix}.$$

Let $\mathfrak{gl}(p;\, \boldsymbol{K})$ be the Lie algebra of square matrices of degree p with elements in $\boldsymbol{K}$. The Lie algebra $\mathfrak{gl}(p;\, \boldsymbol{C})$ is identified with a subalgebra of $\mathfrak{gl}(2p;\, \boldsymbol{R})$ by the injection: $X+\sqrt{-1}\,Y \rightarrow \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}$, $X, Y \in \mathfrak{gl}(p;\, \boldsymbol{R})$. For a subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(p;\, \boldsymbol{C})$, denote by $\hat{\mathfrak{g}}$ the image by this identification.

TYPE I. Put

$$V=M(p, q;\, \boldsymbol{C}), \quad \{X, Y, Z\} = X\bar{Y}^t Z + Z\bar{Y}^t X.$$

This compact simple hermitian JTS is called *of type* I_{pq}. The trace form β is given by

$$\text{(I.a)} \qquad \beta(X, Y) = (p+q)\,\mathrm{Trace}\, X\bar{Y}^t.$$

For $S \in M(p, p;\, \boldsymbol{C})$, $T \in M(q, q;\, \boldsymbol{C})$, denote by (S, T) a linear endomorphism of V defined by $(S, T)(Z)=SZ+ZT$ for $Z\in V$. Then we have

$$\text{(I.b)} \qquad L = \{(S, T) \in \mathrm{End}\, V;\ \mathrm{Tr}\, S = \mathrm{Tr}\, T\}, \quad (S, T)^* = (\bar{S}^t, \bar{T}^t).$$

Let $(\mathfrak{g}, \rho)$ be the SGLA constructed from $(V, \{\ \})$. Then the Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}(p+q;\, \boldsymbol{C})$ by the correspondence:

$$\text{(I.c)} \qquad (X, (S, T), Y) \leftrightarrow \begin{pmatrix} S & (1/\sqrt{2})X \\ -(1/\sqrt{2})\bar{Y}^t & -T \end{pmatrix}.$$

Under this correspondence, we have

$$\text{(I.d)} \qquad \rho \leftrightarrow X \rightarrow -\bar{X}^t, \quad \nu \leftrightarrow \begin{pmatrix} -(q/p+q)\cdot 1_p & 0 \\ 0 & (p/p+q)\cdot 1_q \end{pmatrix}.$$

Moreover, identifying $\mathfrak{g}^C$ with $\widehat{\mathfrak{sl}}(p+q;\, \boldsymbol{C})^C \subset \mathfrak{gl}(2(p+q);\, \boldsymbol{C})$, we have

$$\text{(I.e)} \qquad \begin{array}{l} \rho^C \leftrightarrow X \rightarrow -X^t, \\ \tau \leftrightarrow X \rightarrow \bar{X}, \end{array} \quad \nu \leftrightarrow \begin{pmatrix} -(q/p+q)\cdot 1_p & & & \\ & (p/p+q)\cdot 1_q & & 0 \\ & & -(q/p+q)\cdot 1_p & \\ 0 & & & (p/p+q)\cdot 1_q \end{pmatrix}.$$

Now we give examples of how to decide our objects. By (I.d), Lemma 3.3, it follows that

$$(\mathfrak{g}, \mathfrak{k}, \mathfrak{k}_0) = (\mathfrak{sl}(p+q;\, \boldsymbol{C}),\ \mathfrak{su}(p+q),\ \mathfrak{su}(p)\oplus\mathfrak{su}(q)\oplus\boldsymbol{R})$$
$$(\mathfrak{g}^C, \mathfrak{k}^C, (\mathfrak{k}^C)_0) = (\mathfrak{sl}(p+q;\, \boldsymbol{C})\oplus\mathfrak{sl}(p+q;\, \boldsymbol{C}),\ \mathfrak{sl}(p+q;\, \boldsymbol{C}),\ \mathfrak{sl}(p;\, \boldsymbol{C})\oplus\mathfrak{sl}(q;\, \boldsymbol{C})\oplus\boldsymbol{C}).$$

Let δ be a conjugation of V defined by $\delta(X)=\bar{X}$. Then, under the identification (I.e), we have

$$\bar{\delta}^{C}(X) = S(2p, p)XS(2q, q)$$

and thus, by (I.e),

$$\bar{\delta}^{C}\circ\tau(X) = S(2p, p)\bar{X}S(2q, q)$$

for $X\in\widehat{\mathfrak{sl}}(p+q; \boldsymbol{C})^{C}$. By Lemma 3.4, it follows that

$$(\mathfrak{g}^{\delta}, \mathfrak{k}^{\delta}, (\mathfrak{k}^{\delta})_0) = (\mathfrak{sl}(p+q; \boldsymbol{R})\oplus\mathfrak{sl}(p+q; \boldsymbol{R}), \mathfrak{sl}(p+q; \boldsymbol{R}), \mathfrak{sl}(p; \boldsymbol{R})\oplus\mathfrak{sl}(q; \boldsymbol{R})\oplus\boldsymbol{R}).$$

Suppose that $p=q$ and let α be the involutive automorphism of V_{δ} defined by $\alpha(X)=X^{t}$. Then, under the identification (I.c), we have

$$\bar{\delta}(X) = \bar{X}, \ \bar{\alpha}^{C}(X) = J(2p)X^{t}J(2p)$$

and thus, by (I.d),

$$\bar{\alpha}^{C}\circ\bar{\delta}(X) = J(2p)\bar{X}^{t}J(2p), \ \bar{\delta}\circ\rho(X) = -\bar{X}^{t}$$

for $X\in\mathfrak{sl}(2p; \boldsymbol{C})$. By Lemma 3.5, it follows that

$$((\mathfrak{g}_{\delta})^{\alpha}, (\mathfrak{k}_{\delta})^{\alpha}, ((\mathfrak{k}_{\delta})^{\alpha})_0) = (\mathfrak{su}(p, p), \mathfrak{so}^*(2p), \mathfrak{so}(p; \boldsymbol{C}))\,.$$

Objects for other cases of classical types I$\sim$IV are also decided in this way. For $S\in M(p, p; \boldsymbol{C})$, $T\in M(q, q; \boldsymbol{C})$, denote by $[S; T]$ a linear endomorphism of $M(p, q; \boldsymbol{C})$ defined by $[S; T](Z)=SZT$.

Table I

δ	$\mathfrak{g}$	$\mathfrak{k}$	$\mathfrak{k}_0$	Remark
1_V	$\mathfrak{sl}(p+q;\boldsymbol{C})$	$\mathfrak{su}(p+q)$	$\mathfrak{su}(p)\oplus\mathfrak{su}(q)\oplus\boldsymbol{R}$	I_{pq}
—	$\mathfrak{g}^{C}$	$\mathfrak{k}^{C}$	$(\mathfrak{k}^{C})_0$	—
—	$\mathfrak{sl}(p+q;\boldsymbol{C})\oplus\mathfrak{sl}(p+q:\boldsymbol{C})$	$\mathfrak{sl}(p+q;\boldsymbol{C})$	$\mathfrak{sl}(p;\boldsymbol{C})\oplus\mathfrak{sl}(q;\boldsymbol{C})\oplus\boldsymbol{C}$	$I_{pq}{}^{C}$
δ	$\mathfrak{g}^{\delta}$	$\mathfrak{k}^{\delta}$	$(\mathfrak{k}^{\delta})_0$	—
υ	$\mathfrak{sl}(2p;\boldsymbol{C})$	$\mathfrak{su}^*(2p)$	$\mathfrak{sl}(p;\boldsymbol{C})\oplus\boldsymbol{R}$	I_{pp}
$-\upsilon$	$\mathfrak{sl}(2p;\boldsymbol{C})$	$\mathfrak{sl}(2p;\boldsymbol{R})$	$\mathfrak{sl}(p;\boldsymbol{C})\oplus\boldsymbol{R}$	I_{pp}
$[S(p,j);S(q,k)]$	$\mathfrak{sl}(p+q;\boldsymbol{C})$	$\mathfrak{su}(p+q-j-k, j+k)$	$\mathfrak{su}(p-j,j)\oplus\mathfrak{su}(q-k,k)\oplus\boldsymbol{R}$	I_{pq}
κ	$\mathfrak{sl}(p+q;\boldsymbol{R})\oplus\mathfrak{sl}(p+q;\boldsymbol{R})$	$\mathfrak{sl}(p+q;\boldsymbol{R})$	$\mathfrak{sl}(p;\boldsymbol{R})\oplus\mathfrak{sl}(q;\boldsymbol{R})\oplus\boldsymbol{R}$	I_{pq}, real form
$-\kappa[J(2p);J(2q)]$	$\mathfrak{su}^*(2p+2q)\oplus\mathfrak{su}^*(2p+2q)$	$\mathfrak{su}^*(2p+2q)$	$\mathfrak{su}^*(2p)\oplus\mathfrak{su}^*(2q)\oplus\boldsymbol{R}$	$I_{2p,2q}$, real form
$\kappa\upsilon$	$\mathfrak{su}(p,p)\oplus\mathfrak{su}(p,p)$	$\mathfrak{su}(p,p)$	$\mathfrak{sl}(p;\boldsymbol{C})\oplus\boldsymbol{R}$	I_{pp}, real form

Conjugation: $\delta=\kappa$.

α	$\mathfrak{g}_\delta$	$\mathfrak{k}_\delta$	$(\mathfrak{k}_\delta)_0$	Remark
1_{V_δ}	$\mathfrak{sl}(p+q;\boldsymbol{R})$	$\mathfrak{so}(p+q)$	$\mathfrak{so}(p)\oplus\mathfrak{so}(q)$	I_{pq}
—	$(\mathfrak{g}_\delta)^C$	$(\mathfrak{k}_\delta)^C$	$((\mathfrak{k}_\delta)^C)_0$	—
—	$\mathfrak{sl}(p+q;\boldsymbol{C})$	$\mathfrak{so}(p+q;\boldsymbol{C})$	$\mathfrak{so}(p;\boldsymbol{C})\oplus\mathfrak{so}(q;\boldsymbol{C})$	$I_{pq}{}^C$
α	$(\mathfrak{g}_\delta)^\alpha$	$(\mathfrak{k}_\delta)^\alpha$	$((\mathfrak{k}_\delta)^\alpha)_0$	—
υ	$\mathfrak{su}(p,p)$	$\mathfrak{so}^*(2p)$	$\mathfrak{so}(p;\boldsymbol{C})$	I_{pp}
$-\upsilon$	$\mathfrak{su}(p,p)$	$\mathfrak{so}(p,p)$	$\mathfrak{so}(p,\boldsymbol{C})$	I_{pp}
$[S(p,j);S(q,k)]$	$\mathfrak{sl}(p+q;\boldsymbol{R})$	$\mathfrak{so}(p+q-j-k, j+k)$	$\mathfrak{so}(p-j,j)\oplus \mathfrak{so}(q-k,k)$	I_{pq}
$-[J(2p);J(2q)]$	$\mathfrak{su}^*(2p+2q)$	$\mathfrak{so}^*(2p+2q)$	$\mathfrak{so}^*(2p)\oplus\mathfrak{so}^*(2q)$	$I_{2p,2q}$

Conjugation: $\delta=-\kappa[J(2p);J(2q)]$.

α	$\mathfrak{g}_\delta$	$\mathfrak{k}_\delta$	$(\mathfrak{k}_\delta)_0$	Remark
1_{V_δ}	$\mathfrak{su}^*(2p+2q)$	$\mathfrak{sp}(p+q)$	$\mathfrak{sp}(p)\oplus\mathfrak{sp}(q)$	$I_{2p,2q}$
—	$(\mathfrak{g}_\delta)^C$	$(\mathfrak{k}_\delta)^C$	$((\mathfrak{k}_\delta)^C)_0$	—
—	$\mathfrak{sl}(2p+2q;\boldsymbol{C})$	$\mathfrak{sp}(p+q;\boldsymbol{C})$	$\mathfrak{sp}(p;\boldsymbol{C})\oplus\mathfrak{sp}(q;\boldsymbol{C})$	$I_{2p,2q}{}^C$
α	$(\mathfrak{g}_\delta)^\alpha$	$(\mathfrak{k}_\delta)^\alpha$	$((\mathfrak{k}_\delta)^\alpha)_0$	—
$-\upsilon[J(2p);J(2p)]$	$\mathfrak{su}(2p,2p)$	$\mathfrak{sp}(2p;\boldsymbol{R})$	$\mathfrak{sp}(p;\boldsymbol{C})$	$I_{2p,2p}$
$\upsilon[J(2p);J(2p)]$	$\mathfrak{su}(2p,2p)$	$\mathfrak{sp}(p,p)$	$\mathfrak{sp}(p;\boldsymbol{C})$	$I_{2p,2p}$
$-[S(2p,p);S(2q,q)]$	$\mathfrak{sl}(2p+2q;\boldsymbol{R})$	$\mathfrak{sp}(p+q;\boldsymbol{R})$	$\mathfrak{sp}(p;\boldsymbol{R})\oplus\mathfrak{sp}(q;\boldsymbol{R})$	$I_{2p,2q}$
$[S(p,j;p,j);S(q,k;q,k)]$	$\mathfrak{su}^*(2p+2q)$	$\mathfrak{sp}(p+q-j-k, j+k)$	$\mathfrak{sp}(p-j,j)\oplus\mathfrak{sp}(q-k,k)$	$I_{2p,2q}$

Conjugation: $\delta=\kappa\upsilon$.

α	$\mathfrak{g}_\delta$	$\mathfrak{k}_\delta$	$(\mathfrak{k}_\delta)_0$	Remark
1_{V_δ}	$\mathfrak{su}(p,p)$	$\mathfrak{su}(p)\oplus\mathfrak{su}(p)\oplus\boldsymbol{R}$	$\mathfrak{su}(p)$	I_{pp}
—	$(\mathfrak{g}_\delta)^C$	$(\mathfrak{k}_\delta)^C$	$((\mathfrak{k}_\delta)^C)_0$	—
—	$\mathfrak{sl}(2p;\boldsymbol{C})$	$\mathfrak{sl}(p;\boldsymbol{C})\oplus \mathfrak{sl}(p;\boldsymbol{C})\oplus\boldsymbol{C}$	$\mathfrak{sl}(p;\boldsymbol{C})$	$I_{pp}{}^C$
α	$(\mathfrak{g}_\delta)^\alpha$	$(\mathfrak{k}_\delta)^\alpha$	$((\mathfrak{k}_\delta)^\alpha)_0$	—
υ	$\mathfrak{sl}(2p;\boldsymbol{R})$	$\mathfrak{sl}(p;\boldsymbol{C})\oplus\boldsymbol{R}$	$\mathfrak{sl}(p;\boldsymbol{R})$	I_{pp}
$-\upsilon$	$\mathfrak{sl}(2p;\boldsymbol{R})$	$\mathfrak{sl}(p;\boldsymbol{R})\oplus \mathfrak{sl}(p;\boldsymbol{R})\oplus\boldsymbol{R}$	$\mathfrak{sl}(p;\boldsymbol{R})$	I_{pp}
$[S(p,j);S(p,j)]$	$\mathfrak{su}(p,p)$	$\mathfrak{su}(j,p-j)\oplus \mathfrak{su}(j,p-j)\oplus\boldsymbol{R}$	$\mathfrak{su}(j,p-j)$	I_{pp}
$-[S(p,j);S(p,j)]$	$\mathfrak{su}(p,p)$	$\mathfrak{sl}(p;\boldsymbol{C})\oplus\boldsymbol{R}$	$\mathfrak{su}(j,p-j)$	I_{pp}
$\upsilon[J(2p);J(2p)]$	$\mathfrak{su}^*(4p)$	$\mathfrak{su}^*(2p)\oplus \mathfrak{su}^*(2p)\oplus\boldsymbol{R}$	$\mathfrak{su}^*(2p)$	$I_{2p,2p}$
$-\upsilon[J(2p);J(2p)]$	$\mathfrak{su}^*(4p)$	$\mathfrak{sl}(2p;\boldsymbol{C})\oplus\boldsymbol{R}$	$\mathfrak{su}^*(2p)$	$I_{2p,2p}$

TYPE II. Let $V=O(p;\, \boldsymbol{C})$ be the vector space of skew symmetric complex matrices of degree p and put

$$\{X,\, Y,\, Z\} = -X\bar{Y}Z - Z\bar{Y}X\,.$$

This compact simple hermitian JTS is called *of type* II_p. The trace form β is given by

$$\text{(II.a)} \qquad \beta(X,\, Y) = (p-1)\,\mathrm{Trace}\, X\bar{Y}^t\,.$$

For $S \in M(p,\, p;\, \boldsymbol{C})$, put $(S)=(S,\, S^t)$. Then we have

$$\text{(II.b)} \qquad L = \{(S) \in \mathrm{End}\, V;\; S \in M(p,\, p;\, \boldsymbol{C})\}\,, \qquad (S)^* = (\bar{S}^t)\,.$$

Let $(\mathfrak{g},\, \rho)$ be the SGLA constructed from $(V,\, \{\ \})$. Let $\bar{\mathfrak{so}}(2p;\, \boldsymbol{C})$ be the Lie algebra of complex matrices X of degree $2p$ such that $I(2p)X^t + XI(2p) = 0$. This is isomorphic to $\mathfrak{so}(2p;\, \boldsymbol{C})$. Then, the Lie algebra $\mathfrak{g}$ is isomorphic to $\bar{\mathfrak{so}}(2p;\, \boldsymbol{C})$ by the correspondence:

$$\text{(II.c)} \qquad (X,\, (S),\, Y) \leftrightarrow \begin{pmatrix} S & (1/\sqrt{2})X \\ (1/\sqrt{2})\bar{Y} & -S^t \end{pmatrix}.$$

Under this correspondence, we have

$$\text{(II.d)} \qquad \rho \leftrightarrow X \to -\bar{X}^t,\ \nu \leftrightarrow \begin{pmatrix} -(1/2)\cdot 1_p & 0 \\ 0 & (1/2)\cdot 1_p \end{pmatrix}.$$

Moreover, identifying $\mathfrak{g}^C$ with $\hat{\bar{\mathfrak{so}}}(2p;\, \boldsymbol{C})^C \subset \mathfrak{gl}(4p;\, \boldsymbol{C})$, we have

$$\text{(II.e)} \qquad \begin{matrix} \rho^C \leftrightarrow X \to -X^t, \\ \tau \leftrightarrow X \to \bar{X}, \end{matrix} \quad \nu \leftrightarrow \begin{pmatrix} -(1/2)\cdot 1_p & & & \\ & (1/2)\cdot 1_p & & \\ & & -(1/2)\cdot 1_p & \\ & & & (1/2)\cdot 1_p \end{pmatrix}$$

Put $[S]=[S^t;\, S]$ for $S \in M(p,\, p;\, \boldsymbol{C})$.

TABLE II

δ	$\mathfrak{g}$	$\mathfrak{k}$	$\mathfrak{k}_0$	Remark
1_V	$\bar{\mathfrak{so}}(2p;\boldsymbol{C})$	$\bar{\mathfrak{so}}(2p)$	$\mathfrak{su}(p)\oplus\boldsymbol{R}$	II_p
—	$\mathfrak{g}^C$	$\mathfrak{k}^C$	$(\mathfrak{k}^C)_0$	—
—	$\bar{\mathfrak{so}}(2p;\boldsymbol{C})\oplus\bar{\mathfrak{so}}(2p;\boldsymbol{C})$	$\bar{\mathfrak{so}}(2p;\boldsymbol{C})$	$\mathfrak{sl}(p;\boldsymbol{C})\oplus\boldsymbol{C}$	$II_p{}^C$
δ	$\mathfrak{g}^\delta$	$\mathfrak{k}^\delta$	$(\mathfrak{k}^\delta)_0$	—
$[S(p,j)]$	$\bar{\mathfrak{so}}(2p;\boldsymbol{C})$	$\bar{\mathfrak{so}}(2j,2p-2j)$	$\mathfrak{su}(j,p-j)\oplus\boldsymbol{R}$	II_p
$-[S(p,j)]$	$\bar{\mathfrak{so}}(2p;\boldsymbol{C})$	$\bar{\mathfrak{so}}^*(2p)$	$\mathfrak{su}(j,p-j)\oplus\boldsymbol{R}$	II_p
κ	$\bar{\mathfrak{so}}(p,p)\oplus\bar{\mathfrak{so}}(p,p)$	$\bar{\mathfrak{so}}(p,p)$	$\mathfrak{sl}(p;R)\oplus\boldsymbol{R}$	II_p, real form
$\kappa[J(2p)]$	$\bar{\mathfrak{so}}^*(4p)\oplus\bar{\mathfrak{so}}^*(4p)$	$\bar{\mathfrak{so}}^*(4p)$	$\mathfrak{su}^*(2p)\oplus\boldsymbol{R}$	II_{2p}, real form

Conjugation: $\delta=\kappa$.

α	$\mathfrak{g}_\delta$	$\mathfrak{k}_\delta$	$(\mathfrak{k}_\delta)_0$	Remark
1_{V_δ}	$\mathfrak{so}(p,p)$	$\mathfrak{so}(p)\oplus\mathfrak{so}(p)$	$\mathfrak{so}(p)$	II_p
—	$(\mathfrak{g}_\delta)^C$	$(\mathfrak{k}_\delta)^C$	$((\mathfrak{k}_\delta)^C)_0$	—
—	$\mathfrak{so}(2p;\boldsymbol{C})$	$\mathfrak{so}(p;\boldsymbol{C})\oplus\mathfrak{so}(p;\boldsymbol{C})$	$\mathfrak{so}(p;\boldsymbol{C})$	$II_p{}^C$
α	$(\mathfrak{g}_\delta)^\alpha$	$(\mathfrak{k}_\delta)^\alpha$	$((\mathfrak{k}_\delta)^\alpha)_0$	—
$[S(p,j)]$	$\mathfrak{so}(p,p)$	$\mathfrak{so}(j,p-j)\oplus\mathfrak{so}(j,p-j)$	$\mathfrak{so}(j,p-j)$	II_p
$-[S(p,j)]$	$\mathfrak{so}(p,p)$	$\mathfrak{so}(p;\boldsymbol{C})$	$\mathfrak{so}(j,p-j)$	II_p
$[J(2p)]$	$\mathfrak{so}^*(4p)$	$\mathfrak{so}^*(2p)\oplus\mathfrak{so}^*(2p)$	$\mathfrak{so}^*(2p)$	II_{2p}
$-[J(2p)]$	$\mathfrak{so}^*(4p)$	$\mathfrak{so}(2p;\boldsymbol{C})$	$\mathfrak{so}^*(2p)$	II_{2p}

Conjugation: $\delta=\kappa[J(2p)]$.

α	$\mathfrak{g}_\delta$	$\mathfrak{k}_\delta$	$(\mathfrak{k}_\delta)_0$	Remark
1_{V_δ}	$\mathfrak{so}^*(4p)$	$\mathfrak{su}(2p)\oplus\boldsymbol{R}$	$\mathfrak{sp}(p)$	II_{2p}
—	$(\mathfrak{g}_\delta)^C$	$(\mathfrak{k}_\delta)^C$	$((\mathfrak{k}_\delta)^C)_0$	—
—	$\mathfrak{so}(4p;\boldsymbol{C})$	$\mathfrak{sl}(2p;\boldsymbol{C})\oplus\boldsymbol{C}$	$\mathfrak{sp}(p;\boldsymbol{C})$	$II_{2p}{}^C$
α	$(\mathfrak{g}_\delta)^\alpha$	$(\mathfrak{k}_\delta)^\alpha$	$((\mathfrak{k}_\delta)^\alpha)_0$	—
$[S(p,j;p,j)]$	$\mathfrak{so}^*(4p)$	$\mathfrak{su}(2j,2p-2j)\oplus\boldsymbol{R}$	$\mathfrak{sp}(j,p-j)$	II_{2p}
$-[S(p,j;p,j)]$	$\mathfrak{so}^*(4p)$	$\mathfrak{su}^*(2p)\oplus\boldsymbol{R}$	$\mathfrak{sp}(j,p-j)$	II_{2p}
$[S(2p,p)]$	$\mathfrak{so}(2p,2p)$	$\mathfrak{sl}(2p,\boldsymbol{R})\oplus\boldsymbol{R}$	$\mathfrak{sp}(p;\boldsymbol{R})$	II_{2p}
$-[S(2p,p)]$	$\mathfrak{so}(2p,2p)$	$\mathfrak{su}(p,p)\oplus\boldsymbol{R}$	$\mathfrak{sp}(p;\boldsymbol{R})$	II_{2p}

TYPE III. Let $V=S(p;\ \boldsymbol{C})$ be the vector space of complex symmetric matrices of degree p and put

$$\{X,\ Y,\ Z\} = X\bar{Y}Z+Z\bar{Y}X\,.$$

This compact simple hermitian JTS is called *of type* III_p. The trace form β is given by

(III.a) $\quad \beta(X,\ Y) = (p+1)\,\mathrm{Trace}\,X\bar{Y}^t\,.$

Moreover we have

(III.b) $\quad L = \{(S)\in\mathrm{End}\,V;\ S\in M(p,\ p;\ \boldsymbol{C})\}\,, \qquad (S)^* = (\bar{S}^t)\,.$

Let $(\mathfrak{g},\ \rho)$ be the SGLA constructed from $(V,\ \{\ \})$. Then the Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{sp}(p;\ \boldsymbol{C})$ by the correspondence:

TABLE III

δ	$\mathfrak{g}$	$\mathfrak{k}$	$\mathfrak{k}_0$	Remark
1_V	$\mathfrak{sp}(p;\boldsymbol{C})$	$\mathfrak{sp}(p)$	$\mathfrak{su}(p)\oplus\boldsymbol{R}$	III_p
—	$\mathfrak{g}^C$	$\mathfrak{k}^C$	$(\mathfrak{k}^C)_0$	—
—	$\mathfrak{sp}(p;\boldsymbol{C})\oplus\mathfrak{sp}(p;\boldsymbol{C})$	$\mathfrak{sp}(p;\boldsymbol{C})$	$\mathfrak{sl}(p;\boldsymbol{C})\oplus\boldsymbol{C}$	$\mathrm{III}_p{}^C$
δ	$\mathfrak{g}^\delta$	$\mathfrak{k}^\delta$	$(\mathfrak{k}^\delta)_0$	—
$[S(p,j)]$	$\mathfrak{sp}(p;\boldsymbol{C})$	$\mathfrak{sp}(j,\ p-j)$	$\mathfrak{su}(j,p-j)\oplus\boldsymbol{R}$	III_p
$-[S(p,j)]$	$\mathfrak{sp}(p;\boldsymbol{C})$	$\mathfrak{sp}(p;\boldsymbol{R})$	$\mathfrak{su}(j,p-j)\oplus\boldsymbol{R}$	III_p
κ	$\mathfrak{sp}(p;\boldsymbol{R})\oplus\mathfrak{sp}(p;\boldsymbol{R})$	$\mathfrak{sp}(p;\boldsymbol{R})$	$\mathfrak{sl}(p;\boldsymbol{R})\oplus\boldsymbol{R}$	III_p, real form
$\kappa[J(2p)]$	$\mathfrak{sp}(p,p)\oplus\mathfrak{sp}(p,p)$	$\mathfrak{sp}(p,p)$	$\mathfrak{su}^*(2p)\oplus\boldsymbol{R}$	III_{2p}, real form

Conjugation: $\delta=\kappa$.

α	$\mathfrak{g}_\delta$	$\mathfrak{k}_\delta$	$(\mathfrak{k}_\delta)_0$	Remark
1_{V_δ}	$\mathfrak{sp}(p;\boldsymbol{R})$	$\mathfrak{su}(p)\oplus\boldsymbol{R}$	$\mathfrak{so}(p)$	III_p
—	$(\mathfrak{g}_\delta)^C$	$(\mathfrak{k}_\delta)^C$	$((\mathfrak{k}_\delta)^C)_0$	—
—	$\mathfrak{sp}(p;\boldsymbol{C})$	$\mathfrak{sl}(p;\boldsymbol{C})\oplus\boldsymbol{C}$	$\mathfrak{so}(p;\boldsymbol{C})$	$\mathrm{III}_p{}^C$
α	$(\mathfrak{g}_\delta)^\alpha$	$(\mathfrak{k}_\delta)^\alpha$	$((\mathfrak{k}_\delta)^\alpha)_0$	—
$[S(p,j)]$	$\mathfrak{sp}(p;\boldsymbol{R})$	$\mathfrak{su}(j,p-j)\oplus\boldsymbol{R}$	$\mathfrak{so}(j,p-j)$	III_p
$-[S(p,j)]$	$\mathfrak{sp}(p;\boldsymbol{R})$	$\mathfrak{sl}(p;\boldsymbol{R})\oplus\boldsymbol{R}$	$\mathfrak{so}(j,p-j)$	III_p
$[J(2p)]$	$\mathfrak{sp}(p,p)$	$\mathfrak{su}(p,p)\oplus\boldsymbol{R}$	$\mathfrak{so}^*(2p)$	III_{2p}
$-[J(2p)]$	$\mathfrak{sp}(p,p)$	$\mathfrak{su}^*(2p)\oplus\boldsymbol{R}$	$\mathfrak{so}^*(2p)$	III_{2p}

Conjugation: $\delta=\kappa[J(2p)]$.

α	$\mathfrak{g}_\delta$	$\mathfrak{k}_\delta$	$(\mathfrak{k}_\delta)_0$	Remark
1_{V_δ}	$\mathfrak{sp}(p,p)$	$\mathfrak{sp}(p)\oplus\mathfrak{sp}(p)$	$\mathfrak{sp}(p)$	III_{2p}
—	$(\mathfrak{g}_\delta)^C$	$(\mathfrak{k}_\delta)^C$	$((\mathfrak{k}_\delta)^C)_0$	—
—	$\mathfrak{sp}(2p;\boldsymbol{C})$	$\mathfrak{sp}(p;\boldsymbol{C})\oplus\mathfrak{sp}(p;\boldsymbol{C})$	$\mathfrak{sp}(p;\boldsymbol{C})$	$\mathrm{III}_{2p}{}^C$
α	$(\mathfrak{g}_\delta)^\alpha$	$(\mathfrak{k}_\delta)^\alpha$	$((\mathfrak{k}_\delta)^\alpha)_0$	—
$S(p,j;p,j)]$	$\mathfrak{sp}(p,p)$	$\mathfrak{sp}(j,p-j)\oplus$ $\mathfrak{sp}(j,p-j)$	$\mathfrak{sp}(j,p-j)$	III_{2p}
$-[S(p,j;p,j)]$	$\mathfrak{sp}(p,p)$	$\mathfrak{sp}(p;\boldsymbol{C})$	$\mathfrak{sp}(j,p-j)$	III_{2p}
$[S(2p,p)]$	$\mathfrak{sp}(2p;\boldsymbol{R})$	$\mathfrak{sp}(p;\boldsymbol{C})$	$\mathfrak{sp}(p;\boldsymbol{R})$	III_{2p}
$-[S(2p,p)]$	$\mathfrak{sp}(2p;\boldsymbol{R})$	$\mathfrak{sp}(p;\boldsymbol{R})\oplus\mathfrak{sp}(p;\boldsymbol{R})$	$\mathfrak{sp}(p;\boldsymbol{R})$	III_{2p}

(III.c) $$(X, (S), Y) \leftrightarrow \begin{pmatrix} S & (1/\sqrt{2})X \\ -(1/\sqrt{2})\bar{Y} & -S^t \end{pmatrix}.$$

Moreover $\mathfrak{g}^C$ is identified with $\widehat{\mathfrak{sp}}(p; C)^C \subset \mathfrak{gl}(4p, C)$. Under these correspondences, we have the same relations (III.d), (III.e) as relations (II.d), (II.e) respectively.

TYPE IV. Put

$$V = C^n, \quad \{X, Y, Z\} = 2\{(X^t\bar{Y})Z + (Z^t\bar{Y})X - (X^tZ)\bar{Y}\}.$$

This compact simple hermitian JTS is called *of type* IV_n. The trace form β is given by

(IV.a) $$\beta(X, Y) = n(X^t\bar{Y}),$$

and moreover we have

(IV.b) $$\begin{cases} L = \{S + a\cdot 1_n \in \mathrm{End}\, V;\ S \in \mathfrak{so}(n; C),\ a \in C\}, \\ (S + a\cdot 1_n)^* = (\bar{S}^t + \bar{a}\cdot 1_n). \end{cases}$$

Let $(\mathfrak{g}, \rho)$ be the SGLA constructed from $(V, \{\ \})$. Let $\mathfrak{so}(n+1, 1; C)$ be the Lie algebra of complex matrices X of degree $n+2$ such that $S(n+2, n+1)X^t + XS(n+2, n+1) = 0$. This is isomorphic to $\mathfrak{so}(n+2; C)$. Then, the Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(n+1, 1; C)$ by the correspondence:

(IV.c) $$(X, S + a1_n, Y) \leftrightarrow \begin{pmatrix} 0 & -(X+\bar{Y})^t/\sqrt{2} & -a \\ (X+\bar{Y})/\sqrt{2} & S & (X-\bar{Y})/\sqrt{2} \\ -a & (X-\bar{Y})^t/\sqrt{2} & 0 \end{pmatrix}.$$

Under this correspondence, we have

(IV.d) $$\rho \leftrightarrow X \to -\bar{X}^t, \quad \nu \leftrightarrow \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Moreover, identifying $\mathfrak{g}^C$ with $\widehat{\mathfrak{so}}(n+1, 1; C)^C \subset \mathfrak{gl}(2(n+2); C)$, we have

(IV.e) $$\begin{matrix} \rho^C \leftrightarrow X \to -X^t, \\ \tau \leftrightarrow X \to \bar{X}, \end{matrix} \quad \nu \leftrightarrow \begin{pmatrix} \begin{matrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{matrix} \end{pmatrix}.$$

For $S \in M(n, n; C)$ denote by the same notation S a linear endomorphism of C^n defined by $S(Z) = S\cdot Z$ for $Z \in C^n$.

TABLE IV

δ	$\mathfrak{g}$	$\mathfrak{k}$	$\mathfrak{k}_0$	Remark
1_V	$\mathfrak{so}(n+2;\boldsymbol{C})$	$\mathfrak{so}(n+2)$	$\mathfrak{so}(n)\oplus\boldsymbol{R}$	IV_n
—	$\mathfrak{g}^C$	$\mathfrak{k}^C$	$(\mathfrak{k}^C)_0$	—
—	$\mathfrak{so}(n+2;\boldsymbol{C})\oplus\mathfrak{so}(n+2;\boldsymbol{C})$	$\mathfrak{so}(n+2;\boldsymbol{C})$	$\mathfrak{so}(n,\boldsymbol{C})\oplus\boldsymbol{C}$	$\mathrm{IV}_n{}^C$
δ	$\mathfrak{g}^\delta$	$\mathfrak{k}^\delta$	$(\mathfrak{k}^\delta)_0$	—
$s(n,j)$	$\mathfrak{so}(n+2;\boldsymbol{C})$	$\mathfrak{so}(j+2,n-j)$	$\mathfrak{so}(j,n-j)\oplus\boldsymbol{R}$	IV_n
$\sqrt{-1}J(2n)$	$\mathfrak{so}(2n+2;\boldsymbol{C})$	$\mathfrak{so}^*(2n+2)$	$\mathfrak{so}^*(2n)\oplus\boldsymbol{R}$	IV_{2n}
$\kappa s(n,j)$	$\mathfrak{so}(j+1,n-j+1)\oplus\mathfrak{so}(j+1,n-j+1)$	$\mathfrak{so}(j+1,n-j+1)$	$\mathfrak{so}(j,n-j)\oplus\boldsymbol{R}$	IV_n, real form

Conjugation: $\delta=\kappa S(n,j)$.

α	$\mathfrak{g}_\delta$	$\mathfrak{k}_\delta$	$(\mathfrak{k}_\delta)_0$	Remark
1_{V_δ}	$\mathfrak{so}(j+1,n-j+1)$	$\mathfrak{so}(j+1)\oplus\mathfrak{so}(n-j+1)$	$\mathfrak{so}(j)\oplus\mathfrak{so}(n-j)$	IV_n
—	$(\mathfrak{g}_\delta)^C$	$(\mathfrak{k}_\delta)^C$	$((\mathfrak{k}_\delta)^C)_0$	—
—	$\mathfrak{so}(n+2;\boldsymbol{C})$	$\mathfrak{so}(j+1;\boldsymbol{C})\oplus\mathfrak{so}(n-j+1;\boldsymbol{C})$	$\mathfrak{so}(j;\boldsymbol{C})\oplus\mathfrak{so}(n-j;\boldsymbol{C})$	$\mathrm{IV}_n{}^C$
α	$(\mathfrak{g}_\delta)^\alpha$	$(\mathfrak{k}_\delta)^\alpha$	$((\mathfrak{k}_\delta)^\alpha)_0$	—
$s(j,r;n-j,k)$	$\mathfrak{so}(r+n-j-k+1,\ j-r+k+1)$	$\mathfrak{so}(r+1,j-r)\oplus\mathfrak{so}(n-j-k,k+1)$	$\mathfrak{so}(r,j-r)\oplus\mathfrak{so}(n-j-k,k)$	IV_n
$-\sqrt{-1}J(2n)$	$\mathfrak{so}^*(2n+2)$	$\mathfrak{so}(n+1;\boldsymbol{C})$	$\mathfrak{so}(n;\boldsymbol{C})$	$\mathrm{IV}_{2n}(j=n)$

Now we consider exceptional types V, VI. In principle, objects in these cases can be also listed up in the same way as in the classical cases. But we use another way here. The objects $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{k}_0$ associated with compact simple JTS's of these types are listed up in [Kobayashi–Nagano [6]]. Hence we can easily list up the objects $\mathfrak{g}^C$, $\mathfrak{k}^C$, $(\mathfrak{k}^C)_0$ associated with their complexifications by Lemma 3.3. Next fix a compact simple JTS of these types. Then the number of modifications of its JTS is finite (Neher [14]). Let $\mathfrak{g}^\alpha$, $\mathfrak{k}^\alpha$, $(\mathfrak{k}^\alpha)_0$ be the objects associated with the modification of its JTS by an involutive automorphism α. Then $\mathfrak{g}^\alpha$, $\mathfrak{k}^\alpha$, $(\mathfrak{k}^\alpha)_0$ are real forms of $\mathfrak{g}^C$, $\mathfrak{k}^C$, $(\mathfrak{k}^C)_0$ respectively. Moreover, we can easily see that Lie algebras $\mathfrak{g}^\alpha$, $(\mathfrak{k}^\alpha)_0$ are isomorphic to Lie algebras $\mathfrak{g}^{-\alpha}$, $(\mathfrak{k}^{-\alpha})_0$ respectively, where $\mathfrak{g}^{-\alpha}$, $(\mathfrak{k}^{-\alpha})_0$ denote objects associated with the modification by the involutive automorphism $-\alpha$. Under these notes we apply for pairs $(\mathfrak{g}^\alpha, \mathfrak{k}^\alpha)$, $(\mathfrak{k}^\alpha, (\mathfrak{k}^\alpha)_0)$ the Berger's classification [1] of simple symmetric spaces. We describe only results in the following tables V, VI.

REMARK. In this occasion I would like to correct two errors in my papers [9], [11] (Part I) about parallel submanifolds.

(1) [9]: P 427 ↑ 7 submanifolds→Kählerian or totally real submanifolds
P 435 ↓ 13 planer→Kählerian or totally real planer.

(2) [11] (Part I): P 95 ↑ 11 $(n+2)c$, $(n+2)c/(n+1)$→$3c$, $3c/2$.

These errors have no influence on contents of above papers.

TABLE V

$\mathfrak{g}$	$\mathfrak{k}$	$\mathfrak{k}_0$	Remark
$E_6{}^{\boldsymbol{C}}$	E_6	$\mathfrak{so}(10)\oplus\boldsymbol{R}$	
$\mathfrak{g}^{\boldsymbol{C}}$	$\mathfrak{k}^{\boldsymbol{C}}$	$(\mathfrak{k}^{\boldsymbol{C}})_0$	—
$E_6{}^{\boldsymbol{C}}\oplus E_6{}^{\boldsymbol{C}}$	$E_6{}^{\boldsymbol{C}}$	$\mathfrak{so}(10;\boldsymbol{C})\oplus\boldsymbol{C}$	
$\mathfrak{g}^{\delta}$	$\mathfrak{k}^{\delta}$	$(\mathfrak{k}^{\delta})_0$	—
$F_6{}^{\boldsymbol{C}}$	E_6^3	$\mathfrak{so}(10)\oplus\boldsymbol{R}$	
$E_6{}^{\boldsymbol{C}}$	E_6^2	$\mathfrak{so}^*(10)\oplus\boldsymbol{R}$	
$F_6{}^{\boldsymbol{C}}$	E_6^2	$\mathfrak{so}(4,6)\oplus\boldsymbol{R}$	
$E_6{}^{\boldsymbol{C}}$	E_6^3	$\mathfrak{so}^*(10)\oplus\boldsymbol{R}$	
$E_6{}^{\boldsymbol{C}}$	E_6^3	$\mathfrak{so}(2,8)\oplus\boldsymbol{R}$	
$E_6^1\oplus E_6^1$	E_6^1	$\mathfrak{so}(5,5)\oplus\boldsymbol{R}$	real form 1
$E_6^4\oplus E_6^4$	E_6^4	$\mathfrak{so}(1,9)\oplus\boldsymbol{R}$	real form 2

Real form 1

$\mathfrak{g}_{\delta}$	$\mathfrak{k}_{\delta}$	$(\mathfrak{k}_{\delta})_0$	Remark
E_6^1	$\mathfrak{sp}(4)$	$\mathfrak{sp}(2)\oplus\mathfrak{sp}(2)$	
$(\mathfrak{g}_{\delta})^{\boldsymbol{C}}$	$(\mathfrak{k}_{\delta})^{\boldsymbol{C}}$	$((\mathfrak{k}_{\delta})^{\boldsymbol{C}})_0$	—
$E_6{}^{\boldsymbol{C}}$	$\mathfrak{sp}(4;\boldsymbol{C})$	$\mathfrak{sp}(2;\boldsymbol{C})\oplus\mathfrak{sp}(2;\boldsymbol{C})$	
$(\mathfrak{g}_{\delta})^{\alpha}$	$(\mathfrak{k}_{\delta})^{\alpha}$	$((\mathfrak{k}_{\delta})^{\alpha})_0$	—
E_6^1	$\mathfrak{sp}(2,2)$	$\mathfrak{sp}(2)\oplus\mathfrak{sp}(2)$	
E_6^1	$\mathfrak{sp}(2,2)$	$\mathfrak{sp}(2;\boldsymbol{C})$	
F_6^1	$\mathfrak{sp}(2,2)$	$\mathfrak{sp}(1,1)\oplus\mathfrak{sp}(1,1)$	
E_6^1	$\mathfrak{sp}(4;\boldsymbol{R})$	$\mathfrak{sp}(2;\boldsymbol{R})\oplus\mathfrak{sp}(2;\boldsymbol{R})$	
E_6^1	$\mathfrak{sp}(4;\boldsymbol{R})$	$\mathfrak{sp}(2;\boldsymbol{C})$	
E_6^4	$\mathfrak{sp}(1,3)$	$\mathfrak{sp}(1,1)\oplus\mathfrak{sp}(2)$	

Real form 2

$\mathfrak{g}_\delta$	$\mathfrak{k}_\delta$	$(\mathfrak{k}_\delta)_0$	Remark
E_6^4	F_4	$\mathfrak{so}(9)$	
$(\mathfrak{g}_\delta)^C$	$(\mathfrak{k}_\delta)^C$	$((\mathfrak{k}_\delta)^C)_0$	—
E_6^C	F_4^C	$\mathfrak{so}(9; \boldsymbol{C})$	
$(\mathfrak{g}_\delta)^{\alpha\delta}$	$(\mathfrak{k}_\delta)^{\alpha\delta}$	$((\mathfrak{k}_\delta)^{\alpha\delta})_0$	—
E_6^4	F_4^2	$\mathfrak{so}(9)$	
E_6^1	F_4^1	$\mathfrak{so}(4,5)$	
E_6^4	F_4^2	$\mathfrak{so}(1,8)$	

TABLE VI

$\mathfrak{g}$	$\mathfrak{k}$	$\mathfrak{k}_0$	Remark
E_7^C	E_7	$E_6 \oplus \boldsymbol{R}$	
$\mathfrak{g}^C$	$\mathfrak{k}^C$	$(\mathfrak{k}^C)_0$	—
$E_7^C \oplus E_7^C$	E_7^C	$E_6^C \oplus \boldsymbol{C}$	
$\mathfrak{g}^\delta$	$\mathfrak{k}^\delta$	$(\mathfrak{k}^\delta)_0$	—
E_7^C	E_7^3	$E_6 \oplus \boldsymbol{R}$	
E_7^C	E_7^1	$E_6^2 \oplus \boldsymbol{R}$	
E_7^C	E_7^2	$E_6^2 \oplus \boldsymbol{R}$	
E_7^C	E_7^2	$E_6^3 \oplus \boldsymbol{R}$	
E_7^C	E_7^3	$E_6^3 \oplus \boldsymbol{R}$	
$E_7^1 \oplus E_7^1$	E_7^1	$E_6^1 \oplus \boldsymbol{R}$	real form 1
$E_7^3 \oplus E_7^3$	E_7^3	$E_6^4 \oplus \boldsymbol{R}$	real form 2

Real form 1

$\mathfrak{g}_\delta$	$\mathfrak{k}_\delta$	$(\mathfrak{k}_\delta)_0$	Remark
E_7^1	$\mathfrak{su}(8)$	$\mathfrak{sp}(4)$	
$(\mathfrak{g}_\delta)^{\boldsymbol{C}}$	$(\mathfrak{k}_\delta)^{\boldsymbol{C}}$	$((\mathfrak{k}_\delta)^{\boldsymbol{C}})_0$	—
$E_7{}^{\boldsymbol{C}}$	$\mathfrak{sl}(8;\boldsymbol{C})$	$\mathfrak{sp}(4;\boldsymbol{C})$	
$(\mathfrak{g}_\delta)^{\alpha}$	$(\mathfrak{k}_\delta)^{\alpha}$	$((\mathfrak{k}_\delta)^{\alpha})_0$	—
E_7^1	$\mathfrak{su}^*(8)$	$\mathfrak{sp}(4)$	
E_7^1	$\mathfrak{su}(4,4)$	$\mathfrak{sp}(4;\boldsymbol{R})$	
E_7^1	$\mathfrak{sl}(8;\boldsymbol{R})$	$\mathfrak{sp}(4;\boldsymbol{R})$	
E_7^1	$\mathfrak{su}(4,4)$	$\mathfrak{sp}(2,2)$	
E_7^1	$\mathfrak{su}^*(8)$	$\mathfrak{sp}(2,2)$	
E_7^3	$\mathfrak{su}^*(8)$	$\mathfrak{sp}(1,3)$	
E_7^3	$\mathfrak{su}(2,6)$	$\mathfrak{sp}(1,3)$	

Real form 2

$\mathfrak{g}_\delta$	$\mathfrak{k}_\delta$	$(\mathfrak{k}_\delta)_0$	Remark
E_7^3	$E_6 \oplus \boldsymbol{R}$	F_4	
$(\mathfrak{g}_\delta)^{\boldsymbol{C}}$	$(\mathfrak{k}_\delta)^{\boldsymbol{C}}$	$((\mathfrak{k}_\delta)^{\boldsymbol{C}})_0$	—
$E_7{}^{\boldsymbol{C}}$	$E_6{}^{\boldsymbol{C}} \oplus \boldsymbol{C}$	$F_4{}^{\boldsymbol{C}}$	
$(\mathfrak{g}_\delta)^{\alpha}$	$(\mathfrak{k}_\delta)^{\alpha}$	$((\mathfrak{k}_\delta)^{\alpha})_0$	—
E_7^3	$E_6^4 \oplus \boldsymbol{R}$	F_4	
E_7^1	$E_6^1 \oplus \boldsymbol{R}$	F_4^1	
E_7^1	$E_6^2 \oplus \boldsymbol{R}$	F_4^1	
E_7^3	$E_6^3 \oplus \boldsymbol{R}$	F_4^2	
E_7^3	$E_6^4 \oplus \boldsymbol{R}$	F_4^2	

References

[1] M. Berger: *Les Espaces symétriques non compacts,* Ann. Sci. Ecole Norm. Sup. **74** (1957), 85–177.

[2] D. Ferus: *Immersions with parallel second fundamental form,* Math. Z. **140** (1974), 87–92.

[3] ———: *Immersionen mit paralleler zweiter Fundamentalform: Beispiele und Nicht-Beispiele,* Manuscripta Math. **12** (1974), 153–162.

[4] ———: *Symmetric submanifolds of euclidean space,* Math. Ann. **247** (1980), 81–93.

[5] S. Helgason: Differential geometry, Lie groups and symmetric spaces, Academic Press, New York, 1978.

[6] S. Kobayashi-T. Nagano: On filtered Lie algebras and geometric structures I, J. Math. Mech. **13** (1964), 875–907.

[7] S. Kobayashi-K. Nomizu: Foundations of differential geometry I, II, Wiley (Interscience), 1963 and 1969.

[8] M. Koecher: An elementary approach to bounded symmetric domains, Lect. Notes, Rice Univ., Houston, 1969.

[9] H. Naitoh: *Isotropic submanifolds with parallel second fundamental form in* $P^m(c)$, Osaka J. Math. **18** (1981), 427–464.

[10] ———: *Totally real parallel submanifolds in* $P^n(c)$, Tokyo J. Math. **4** (1981), 279–306.

[11] ———: *Parallel submanifolds of complex space forms* I, II, Nagoya Math. J. **90** (1983), 85–117 and **91** (1983), 119–149.

[12] E. Neher: *Cartan-Involutionen von halbeinfachen reellen Jordan-Tripelsystemen,* Math. Z. **169** (1979), 271–292.

[13] ———: *Klassifikation der einfachen reellen speziellen Jordan-Tripelsysteme,* Manuscripta Math. **31** (1980), 197–215.

[14] ———: *Klassifikation der einfachen reellen Ausnahme-Jordan-Tripelsysteme,* J. Reine Angew. Math. **322** (1981), 145–169.

[15] K. Nomizu: *Uniqueness of the normal connections and congruence of isometric immersions,* Tôhoku Math. J. **28** (1976), 613–617.

[16] I. Satake: Algebraic structures of symmetric domains, Iwanami Shoten, Publishers and Princeton Univ. Press, 1981.

[17] R.H. Szczarba: *On existence and rigidity of isometric immersions,* Bull. Amer. Math. Soc. **75** (1969), 783–787.

[18] M. Takeuchi: *Parallel submanifolds of space forms,* Manifolds and Lie groups: Papers in honor of Yozô Matsushima, Birkhäuser, 1981, 429–447.

[19] J. Vilms: Submanifolds of euclidean space with parallel second fundamental form, Proc. Amer. Math. Soc. **32** (1972), 263–267.

Department of Mathematics
Yamaguchi University
Yamaguchi 753, Japan