Kobayashi, S.
Osaka J. Math.
21 (1984), 679–682

A NOTE ON REGULAR SELF-INJECTIVE RINGS

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(Received March 25, 1983)
(Revised June 17, 1983)

Introduction

Let R be an arbitrary ring and let M be a finitely generated right R-module. Then the dual module $M^*=\mathrm{Hom}_R(M, R)$ of M is a left R-module. If R is a left noetherian ring or finite dimensional algebra, then M^* is also finitely generated. But this property does not hold for a general ring. However, we prove that R has this property if R is a regular right self-injective ring (Proposition).

The purpose of this paper is to prove the following theorems.

Theorem 1. *Let R be a regular ring. Then the following statements are equivalent.*
1) *R is a right and left self-injective ring.*
2) *For every finitely generated non-singular right* (*resp. left*) *R-module, the dual module is a non-zero finitely generated left* (*resp. right*) *R-module.*

In particular, if R is a commutative regular ring, then we have the following theorem.

Theorem 2. *Let R be a commutative regular ring. Then the following conditions are equivalent.*
1) *R is a self-injective ring.*
2) *For every finitely generated R-module, the dual module is also finitely generated.*

Throughout this paper, we assume that R is a ring with identity element and all modules are unitary. We denote the maximal right quotient ring of R by Q.

Let M be a right R-module. Then we denote the right (resp. left) annihilator ideal by $r(M)$ (resp. $l(M)$), i.e. $r(M)=\{r\in R \mid Mr=0\}$, (resp. $l(M)=\{r\in R \mid rM=0\}$).

We denote the category of right R-modules by Mod-R. Let M be a right R-module. Then M is said to be a *cogenerator* in Mod-R if $\mathrm{Hom}_R(-, R)$ is a faithful functor. In particular, if R is an injective cogenerator in Mod-R, then R is said to be a *right PF*-ring.

1. Proofs of the theorems

Proposition. *Let R be a regular right self-injective ring. Then for any finitely generated right (resp. left) R-module M, the dual module M^* is also finitely generated.*

Proof. Let M be a right R-module and let $\{m_1, m_2, \cdots, m_n\}$ be a set of generators of M. Then we prove Proposition by the induction on the number of generators. If $n=1$, then we may assume that $M=R/I$ for some right ideal I of R. In this case, the dual module M^* is isomorphic to $l(I)$. On the other hand, since R is a right self-injective regular ring, R is a *Baer ring*, so that $l(I)=eR$ for some idempotent element e of R. Therefore we have that $M^*\cong Re$, whence Proposition holds in the case $n=1$. Let $n>1$ and assume Proposition holds in the case where the number of generators is less than n. Set $\bar{M}=m_2R+m_3R+\cdots+m_nR$ and take the exact sequence $0\to\bar{M}\to M\to M/\bar{M}\to 0$. Then we have the exact sequence $0\to(M/\bar{M})^*\to M^*\to\bar{M}^*\to 0$, since R is self-injective. From the induction hypothesis, $(M/\bar{M})^*$ and $\bar{M}^*$ are finitely generated. Hence M^* is also finitely generated. Next let M be a finitely generated left R-module and let $0\to K\to F\to M\to 0$ be an exact sequence with a finitely generated and free left R-module F. Then we obtain the exact sequence $0\to M^*\to F^*\to C\to 0$, where C is the cokernel of the map $M^*\to F^*$. Since K^* is a torsionless right R-module, C is a finitely generated torsionless right R-module. We claim that C is projective. Since C is a finitely generated right R-module, C^* is also finitely generated. Note that C^* is non-zero, since C is torsionless. Hence we have that C^* is projective since R is a regular ring. Consequently, C^{**} is also projective and C is isomorphic to a finitely generated submodule of C^{**}. Then [1, Theorem 1.11] shows that C is projective. Thus we obtain that M^* is a direct summand of F^*, which implies that M^* is finitely generated.

Proof of Theorem 1.

1)$\Rightarrow$2). Since R is a right and left self-injective regular ring, by the above Proposition and [1; Theorem 9.2], 1) implies 2).
2)$\Rightarrow$1). Let I be a non-essential right ideal of R. We consider the right R-module $M=R/I$. Then there exists an exact sequence $0\to(M/Z_r(M))^*\to M^*$ where $Z_r(M)$ is the singular submodule of M. Since I is a non-essential right ideal of R, $M\neq Z_r(M)$. Therefore by our assumption, $(M/Z_r(M))^*$ is a non-zero and finitely generated left R-module, whence M^* is also non-zero. The same is true for any non-essential left ideal. In this case, Theorem of *Utumi* [4; Theorem 3.3] shows that the maximal right quotient ring of and the maximal left quotient ring of R coincide. Next let a be any element of the maximal quotient ring Q of R. Then we set $J=\{r\in R\,|\,ar\in R\}$ and $K=\{r\in R\,|\,ra\in R\}$.

Clearly J is an essential right ideal and K is an essential left ideal of R. We set $N=aR+R\subset Q$, then by the assumption, N^* is non-zero finitely generated. We claim that $N^*\cong K$ as left R-modules. For any element f of N^*, we put $f(a)=x$ and $f(1)=y$. Then we have, for any element z of I, $f(az)=yaz$. Hence we obtain that $x=ya$ since Q is a non-singular right R-module. Now we define an R-homomorphism φ from N^* to K by $\varphi(f)=f(1)$. Evidently, φ is a well-defined R-homomorphism. We shall show that φ is injective. If $\varphi(f)=f(1)=0$, then from $x=ya$, it follows that $f(a)=0$, whence φ is injective. Finally, we shall show that φ is surjective. Let k be any element of K. Then we define the element f_k in N^* by the left multiplication by k. Clearly, $\varphi(f_k)=k$. Consequently, we have $N^*\cong K$ as claimed. This implies that K is finitely generated since N^* is finitely generated. On the other hand, since K is an essential left ideal of R, K must be equal to R. Therefore we obtain that a is in R, which shows that R is a right and left self-injective ring.

Proof of Theorem 2.

1)$\Rightarrow$2). This is a direct consequence of the above Proposition.
2)$\Rightarrow$1). Let Q be a maximal quotient ring of R and let a be any element of Q. We set $M=aR+R$. By the proof of Theorem 1, it suffices to prove that M^* is non-zero. We set $I=\{r\in R \mid ar\in R\}$. For any element i of I, we can define R-homomorphisms f_i: $M\to R$ by the multiplication map by i. Hence $f_i\in M^*$, so that M^* is non-zero. Therefore R is a self-injective ring.

REMARK. In the above Theorems, we can not drop the assumption that R is a regular ring. For example, let $Z_{(p)}$ be the *Prufer group* (p a prime), and $Z_{p\infty}$ be the ring of the p-adic integers, considered as the endomorphism ring of $Z_{p\infty}$. Define a multiplication by $(a, x)\ (b, y)=(ab, ay+xb)$, for a, $b\in Z_{(p)}$ and x, $y\in Z_{p\infty}$, on the additive group $R=Z_{(p)}\oplus Z_{p\infty}$. Then by [3], R is a commutative quasi-local PF-ring but not a noetherian ring. Furthermore we shall show that R does not have the property: For any finitely generated R-module M, the dual module M^* is also finitely generated. In order to show this fact, we prove the following result. "Let R be a right PF-ring and suppose that R satisfies the above property. Then R is a QF-ring". In fact, let M be a finitely generated right R-module. Then since R is a right PF-ring, M is torsionless. From the assumption, M^* is also finitely generated. Now since we have an exact sequence $F\to M^*\to 0$ with a finitely generated and free left R-module F, we obtain an exact sequence $0\to M^{**}\to F^*$. Since M is torsionless, M can be embedded in a free R-module. In this case, by the Theorem of *Faith* and *Walker* [2], R is a QF-ring. Therefore in our example, R does not satisfy this property. Hence R is a self-injective ring but not a regular ring, and does not satisfy the property: For any finitely generated R-module M, the dual module M^* is also finitely generated.

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