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REMARKS ON THE LIFTING PROPERTY OF SIMPLE MODULES

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Throughout this paper, we assume that R is an associative ring with identity and $\{M_\alpha\}_I$ is an infinite set of completely indecomposable right R-modules. We put $M=\sum_I \oplus M_\alpha$ and $\bar{M}=M/J(M)$, where $J(M)$ $(=\sum_I \oplus J(M_\alpha))$ denotes the Jacobson radical of M.

If each M_α is a cyclic hollow module, then $\bar{M}$ is completely reducible. In this case, M is said to have the *lifting property of simple modules modulo the radical* if every simple submodule of $\bar{M}$ is induced from a direct summand of M ([3]). On the other hand, for the family $\mathcal{M}$ of all maximal submodules of M, M is said to have the *lifting property of modules for* $\mathcal{M}$ if every member A in $\mathcal{M}$ is co-essentially lifted to a direct summand of M, that is, there exists a decomposition $M=A^*\oplus A^{**}$ such that $A^*\subseteq A$ and $A\cap A^{**}$ is small in M ([5]). These two concepts are both dual to '*extending property of simple modules*' mentioned in [4]. Therefore, we must observe whether these two lifting properties coincide or not. In this paper, we study this problem and show the following result: M has the lifting property of modules for $\mathcal{M}$ if and only if it has the lifting property of simple modules modulo the radical and satisfies the following condition: For any $\{M_{\alpha_i}\}_{i=1}^{\infty}\subseteq \{M_\alpha\}_I$ and epimorphisms $\{f_i: M_{\alpha_i}\to M_{\alpha_{i+1}}\}_{i=1}^{\infty}$, there exist n (depending on the sets) and epimorphism $g: M_{\alpha_{n+1}}\to M_{\alpha_n}$ such that $\bar{g}=\bar{f}_n^{-1}$, where $\bar{g}$ and $\bar{f}_n$ are the induced isomorphisms: $\bar{M}_{\alpha_{n+1}}\to \bar{M}_{\alpha_n}$ and $\bar{M}_{\alpha_n}\to \bar{M}_{\alpha_{n+1}}$, respectively (Theorem 10).

NOTATION. By $P(M)$ we denote the set of all submodules X of M such that $X\cap M_\alpha \neq M_\alpha$ for all $\alpha\in I$ and $X=\sum_I \oplus(X\cap M_\alpha)$.

We first show

Theorem 1. *The following conditions are equivalent*:

1) *For any pair $\alpha, \beta\in I$, every epimorphism from M_α to M_β is an isomorphism.*

2) *Let $\{A_\beta\}_J$ be a family of indecomposable direct summands of M. If $A_{\beta_1}+\cdots+A_{\beta_n}+X \not\supseteq A_{\beta_{n+1}}$ for any $X\in P(M)$ and any finite subset $\{\beta_1, \cdots, \beta_{n+1}\}$*

$\subseteq J$, *then* $\sum_J A_\beta$ *is a direct sum* (*and a locally direct summand of* M).

Proof. 1)$\Rightarrow$2). Let $\beta_1, \cdots, \beta_{n+1} \in J$ and assume that $A = A_{\beta_1} + \cdots + A_{\beta_n}$ is a direct sum and a direct summand of M. We may show $A \oplus A_{\beta_{n+1}} \langle \oplus M$. We see from [1] and [6] that every indecomposable direct summand of M satisfies the exchange property and hence we have a subset $I' = \{\alpha_1, \cdots, \alpha_n\} \subseteq I$ satisfying $M = A \oplus \sum_{I-I'} \oplus M_\gamma$. We get either $M = A \oplus A_{\beta_{n+1}} \oplus \sum_{\{I-I'\}-\{\nu\}} \oplus M_\gamma$ for some $\nu \in I-I'$ or $M = A_{\beta_1} \oplus \cdots \oplus A_{\beta_{i-1}} \oplus A_{\beta_{i+1}} \oplus \cdots \oplus A_{\beta_n} \oplus A_{\beta_{n+1}} \oplus \sum_{I-I'} \oplus M_\gamma$ for some i.

In the former case, $A \oplus A_{\beta_{n+1}} \langle \oplus M$ as desired. In the latter case, $M_\alpha \simeq A_{\beta_{n+1}}$ for some $\alpha \in \{\alpha_1, \cdots, \alpha_n\}$. For each $\gamma \in I-I'$, π_γ denotes the projection: $M = A \oplus \sum_{I-I'} \oplus M_\gamma \rightarrow M_\gamma$. If $\pi_\gamma(A_{\beta_{n+1}}) \neq M_\gamma$ for all $\gamma \in I-I'$ then $X = \sum_{I-I'} \oplus \pi_\gamma(A_{\beta_{n+1}}) \in \mathrm{P}(M)$ and $A_{\beta_{n+1}} \subseteq A_{\beta_1} + \cdots + A_{\beta_n} + X$, a contradiction. Therefore, $\pi_{\gamma_0}(A_{\beta_{n+1}}) = M_{\gamma_0}$ for some $\gamma_0 \in I-I'$. Since $M_\alpha \simeq A_{\beta_{n+1}}$ and $\alpha \neq \gamma_0$, $\pi_{\gamma_0} | A_{\beta_{n+1}}$ is an isomorphism by the assumption. Hence it follows that $M = A_{\beta_1} \oplus \cdots \oplus A_{\beta_n} \oplus A_{\beta_{n+1}} \oplus \sum_K \oplus M_\gamma$, where $K = \{I-I'\} - \{\gamma_0\}$.

2)$\Rightarrow$1). Let $\alpha, \beta \in I$ and consider an epimorphism f: $M_\alpha \rightarrow M_\beta$. Putting $M'_\alpha = \{x + f(x) \mid x \in M_\alpha\}$, we see that $M_\alpha \simeq M'_\alpha \langle \oplus M$ and $M'_\alpha + X \not\supseteq M_\alpha$ for any X in $\mathrm{P}(M)$; whence, by 2), $\ker f = M'_\alpha \cap M_\alpha = 0$ and hence f is an isomorphism.

Theorem 2. *Assume that each* X *in* $\mathrm{P}(M)$ *is small in* M *or each* M_α *is cyclic hollow. Then the following condition is equivalent to each of conditions* 1) *and* 2) *in Theorem* 1.

(K) *If* $M = \sum_J A_\beta$ *is an irredundant sum and each* A_β *is an indecomposable direct summand, then this sum is a direct sum.*

Proof. (K)$\Rightarrow$1) is shown by the same proof as in 2)$\Rightarrow$1) in Theorem 1. Now, assume that 2) holds and let $M = \sum_J A_\beta$ be an irredundant sum and each A_β an indecomposable direct summand. First, if each X in $\mathrm{P}(M)$ is small in M, then we see that $A_{\beta_1} + \cdots + A_{\beta_n} + X \not\supseteq A_{\beta_{n+1}}$ for any X in $\mathrm{P}(M)$ and any finite subset $\{\beta_1, \cdots, \beta_{n+1}\} \subseteq J$. Hence the sum $M = \sum_J A_\beta$ is a direct sum by 2). Next, consider the case when each M_β is cyclic hollow. Assume that there exist a subset $\{\beta_1, \cdots, \beta_n\} \subseteq J$ and X in $\mathrm{P}(M)$ such that $A_{\beta_1} + \cdots + A_{\beta_n} + X \supseteq A_{\beta_{n+1}}$. Then we can take a finite subset $F \subseteq I$ and $Y \subseteq \sum_F \oplus M_\alpha$ such that $A_{\beta_1} + \cdots + A_{\beta_n} + Y \supseteq A_{\beta_{n+1}}$ and $Y \in \mathrm{P}(M)$. Since Y is small in M, this implies that $M = \sum_{J-\{\beta_{n+1}\}} A_\beta$, a contradiction. Therefore, such $\{\beta_1, \cdots, \beta_n\}$ and X do not exist; whence the sum $M = \sum_J A_\beta$ is a direct sum by 2).

Theorem 3. *The following conditions are equivalent*:

1) *For any irredundant sum* $\sum_J A_\beta$ *of direct summands of* M *with the pro-*

perty that $A_{\beta_1}+\cdots+A_{\beta_n}+X \not\supseteq A_{\beta_{n+1}}$ *for any* X *in* $\mathrm{P}(M)$ *and any finite subset* $\{\beta_1, \cdots, \beta_n\} \subseteq J$, *the sum* $\sum_J A_\beta$ *is a direct sum and moreover a direct summand of* M.

2) $\{M_\alpha\}_I$ *is a locally semi-T-nilpotent set and* 2) *in Theorem* 1 *holds.*

Proof. 1)$\Rightarrow$2). We may only show the first condition. Let $\{M_{\alpha_i}\}_{i=1}^{\infty} \subseteq \{M_\alpha\}_I$ and $\{f_i: M_{\alpha_i} \to M_{\alpha_{i+1}}\}_{i=1}^{\infty}$ be a set of non-isomorphisms. Then each f_i is not an epimorphism by Theorem 1. Consider $M'_{\alpha_i}=\{x+f_i(x) \mid x \in M_{\alpha_i}\}$, $i=1, 2, \cdots$. Then, as is easily seen, $\{M'_{\alpha_i}\}_{i=1}^{\infty}$ is a set of indecomposable direct summands of M and satisfies the condition: $M'_{\beta_1}+\cdots+M'_{\beta_n}+X \not\supseteq M'_{\beta_{n+1}}$ for any X in $\mathrm{P}(M)$ and $\{\beta_1, \cdots, \beta_{n+1}\} \subseteq \{\alpha_i\}_{i=1}^{\infty}$. Hence we get $M'=\sum_{i=1}^{\infty} \oplus M'_{\alpha_i} \langle \oplus \sum_{i=1}^{\infty} \oplus M_{\alpha_i}$. We put $N=\sum_{i=1}^{\infty} \oplus M_{\alpha_i}=M' \oplus T$. Assume that T is not indecomposable and non-zero. Then, by the Krull-Remak-Schmidt Azumaya's theorem, we see $M' \cap (M_{\alpha_n} \oplus M_{\alpha_m})=0$ for some $n \neq m$. But we can verify that this is impossible. As a result, T is indecomposable or zero, from which we get $N=M'$ or $N=M' \oplus M_{\alpha_n}$ for some α_n. In either case, we see that for every x in M_{α_1} there exists m such that $f_m f_{m-1} \cdots f_1(x)=0$. 2)$\Rightarrow$1) is clear from Theorem 1 and [2, Theorem 3.2.5].

DEFINITION ([5]). Let $\{A_1, \cdots, A_n\}$ be a family of submodules of M. We say that the family is *co-independent* if the canonical map: $M \to \sum_{i=1}^{n} \oplus (M/A_i)$ is an epimorphism.

Theorem 4. *The following conditions are equivalent:*

1) *For any* $\alpha \in I$, *every epimorphism from* $\sum_{I-\{\alpha\}} \oplus M_\beta$ *to* M_α *splits.*

2) *If* $\{A_1, \cdots, A_n\}$ *is a co-independent family of direct summands of* M *such that* M/A_i *is indecomposable, then* $\bigcap_{i=1}^{n} A_i$ *is a direct summand of* M.

Proof. By [1] and [6], we see that every indecomposable direct summand of M is isomorphic to some member in $\{M_\alpha\}_I$ and hence satisfies the finite exchange property.

2)$\Rightarrow$1). Let $\alpha \in I$ and f: $T=\sum_{I-\{\alpha\}} \oplus M_\beta \to M_\alpha$ be an epimorphism. Putting $N=\{x+f(x) \mid x \in T\}$, we see that $M=N+T$, whence $\{N, T\}$ is co-independent. Thus $\ker f=T \cap N \langle \oplus M$.

1)$\Rightarrow$2). We show this by induction. So, let $\{A_1, \cdots, A_n, A\}$ be a co-independent family of direct summands of M such that each M/A_i and M/A are indecomposable, and assume $B=\bigcap_{i=1}^{n} A_i \langle \oplus M$. Setting

$$M = A \oplus A^*$$

$$= B \oplus B^*$$

we see, by the above remark, that either

$$M = B \oplus X \oplus A^*$$

for some $X \subseteq B^*$ or

$$M = B' \oplus A^* \oplus B^*$$

for some $B' \subseteq B$.

We first assume the former case, and let π_A: $M = A \oplus A^* \to A$ and π_{A^*}: $M = A \oplus A^* \to A^*$ be the projections. Since $M = A + B$ and $B \oplus A^* \langle \oplus M$ we see $\pi_{A^*}(B) = A^*$ and $B \subseteq \pi_A(B) \oplus \pi_{A^*}(B) = B \oplus A^* \langle \oplus M$; so $\pi_A(B) \langle \oplus M$. Since $B \cap A^* = 0$, the mapping f: $\pi_A(B) \to A^*$ given by $\pi_A(b) \to \pi_{A^*}(b)$ is well defined and an epimorphism. As a result, $B \cap A = \ker f \langle \oplus M$ by the condition 1).

Next consider the latter case:

$$M = B' \oplus A^* \oplus B^*$$

where $B' \subseteq B$. Since $B^* \simeq M/B \simeq M/A_1 \oplus \cdots \oplus M/A_n$, B^* has the exchange property (cf. [1], [6]) and so does $A^* \oplus B^*$. Therefore

$$M = B' \oplus A^* \oplus B^*$$
$$= A' \oplus A^* \oplus B^*$$

for some $A' \subseteq A$. Consider the projections:

$$\pi_{A^*}: M \to A^*, \ \pi_{B^*}: M \to B^*$$

with respect to $M = A' \oplus A^* \oplus B^*$, and

$$\tau_{A^*}: M \to A^*, \ \tau_{B^*}: M \to B^*$$

with respect to $M = B' \oplus A^* \oplus B^*$.

Here the mapping f: $B^* \to A^*$ given by $\pi_{B^*}(a) \to \pi_{A^*}(a)$ for $a \in A$ and g: $A^* \to B^*$ given by $\tau_{A^*}(b) \to \tau_{B^*}(b)$ for $b \in B$ are well defined. Put

$$X = \{\pi_{B^*}(a) + \pi_{A^*}(a) \mid a \in A\} ,$$
$$Y = \{\tau_{A^*}(b) + \tau_{B^*}(b) \mid b \in B\} .$$

Then $A = A' \oplus X$, $B = B' \oplus Y$, $X \oplus A^* = Y \oplus B^* = A^* \oplus B^*$ and

$$M = A' \oplus X \oplus A^*$$
$$= B' \oplus Y \oplus B^* .$$

If $X \oplus A^* = X \oplus T$ for some $T \subseteq B^*$, then $B = \{\delta(b) + \delta'(b) \mid b \in B\}$ where δ and

δ' are the projections: $M \to A' \oplus X$ and $M \to T$, respectively with respect to $M = A' \oplus X \oplus T$. Noting $M = A + B$ and $B \cap T = 0$, we see $\delta(B) = A$ and $\delta'(B) = T$, and further the mapping ϕ: $A \to T$ given by $\delta(b) \to \delta'(b)$ is well defined and an epimorphism. Consequently $A \cap B = \ker \phi \langle \oplus M$.

If the case: $X \oplus A^* = X \oplus T$ for some $T \subseteq B^*$ does not occur, we must have $A^* \oplus B^* = X \oplus Y$, so

$$\begin{aligned} M &= A' \oplus X \oplus Y \\ &= B' \oplus Y \oplus X . \end{aligned}$$

Then let $\eta_{A'}$: $M \to A'$ and η_X: $M \to X$ be the projections with respect to $M = A' \oplus X \oplus Y$. Putting $Z = \{\eta_{A'}(b') + \eta_X(b') \mid b' \in B'\}$, we get $Z \langle \oplus A = A' \oplus X$ and $A \cap B = Z \langle \oplus M$. The proof is now completed.

REMARK. a) Under the assumptions 'each M_α is cyclic hollow' and '$\mathrm{J}(M)$ is small in M' the equivalence of 1) in Theorem 1 and (K) in Theorem 2 was shown in [3]. Theorem 2 says that this second assumption is supperfluous. b) In the case when each M_α is cyclic hollow, the condition 1) in Theorem 1 and 1) in Theorem 4 are clearly equivalent and hence all conditions in Theorems 1, 2 and 4 are equivalent. We also know from [3] that the following condition is also an equivalent condition: If $\{A_\alpha\}_J$ is a family of direct summands of M such that $\{\bar{A}_\beta\}_J$ is independent in $\bar{M} = M/\mathrm{J}(M)$, then the sum $\sum_J A_\beta$ is a direct sum and a locally direct summand.

Theorem 5. *The following conditions are equivalent*:

1) *For any independent family* $\{A_\beta\}_J$ *of indecomposable direct summands of* M, $\sum_J \oplus A_\beta$ *is a locally direct summand.*

2) *For any* $\alpha \in I$ *and any monomorphism* f: $M_\alpha \to \sum_{I-\{\alpha\}} \oplus M_\beta$, $f(M_\alpha)$ *is a direct summand of* $\sum_{I-\{\alpha\}} \oplus M_\beta$.

Proof. The proof is done as in the proof of [4, Theorem 13].

1)$\Rightarrow$2). Let $\alpha \in I$ and consider a monomorphism f: $M_\alpha \to T = \sum_{I-\{\alpha\}} \oplus M_\beta$. Put $M'_\alpha = \{x + f(x) \mid x \in M_\alpha\}$. Then $M'_\alpha \cap T = 0$ and $M'_\alpha \oplus T = M_\alpha \oplus T$; whence $M'_\alpha \simeq M_\alpha$ and M'_α is a direct summand of $M_\alpha \oplus M_\beta$. Further $M'_\alpha \cap M_\alpha = 0$ and hence it follows from 1) that $M'_\alpha \oplus M_\alpha = M_\alpha \oplus \mathrm{Im}\, f \langle \oplus M$; so $\mathrm{Im}\, f \langle \oplus M$.

2)$\Rightarrow$1). We may show the following: If $\{A_1, \cdots, A_n\}$ is an independent set of indecomposable direct summands of M, $A_1 \oplus \cdots \oplus A_n$ is also a direct summand of M.

If $n = 1$, this is clear. Assume $n > 1$ and $A = A_1 \oplus \cdots \oplus A_{n-1} \langle \oplus M$. Since each member of $\{A_1, \cdots, A_{n-1}\}$ is isomorphic to some member in $\{M_\alpha\}_I$ (cf. [1]), A has the exchange property (cf. [6]), so

$$M = A\oplus \sum_J \oplus M_\gamma$$

for some subset $J\subseteq I$. Since A_n has the exchange property,

$$M = A_1\oplus \cdots \oplus A_{k-1}\oplus A_{k+1}\oplus \cdots \oplus A_{n-1}\oplus A_n\oplus \sum_J \oplus M_\gamma \cdots (*)$$

for some k or

$$M= A\oplus A_n\oplus \sum_{J-\{\sigma\}} \oplus M_\gamma$$

for some $\sigma\in J$. In the latter case the proof is completed. In the former case, $A_k\simeq M_\lambda$ for some $\lambda\in I-J$ and $f=\pi|A_k$: $A_k\to\sum_J\oplus M_\gamma$ is a monomorphism, where π denotes the projection: $M\to\sum_J\oplus M_\gamma$ with respect to $(*)$. By 1), $f(A_k)<\oplus M$ and hence we see that $A\oplus A_n<\oplus M$.

Theorem 6. *Assume that each M_α is uniform. Then the following conditions are equivalent*:

1) *For any pair α, $\beta\in I$, every monomorphism from M_α to M_β is an isomorphism.*

2) *For any $\alpha\in I$ and any monomorphism f from M_α to $\sum_{I-\{\alpha\}}\oplus M_\beta$, the image $f(M_\alpha)$ is a direct summand.*

Proof. 2)$\Rightarrow$1) is clear. Assume 1). Let $\alpha\in I$ and consider a monomorphism f: $M_\alpha\to\sum_{I-\{\alpha\}}\oplus M_\beta$. Put $T=f(M_\alpha)$. Since each M_γ is uniform, we can take $\beta\in I-\{\alpha\}$ such that $T\cap\sum_{I-\{\beta\}}\oplus M_\gamma=0$. Let π be the projection: $M=\sum_I\oplus M_\alpha\to M_\beta$. Then $g=\pi|T$: $T\to M_\beta$ is a monomorphism and hence gf: $M_\alpha\to M_\beta$ is a monomorphism. Therefore g is an isomorphism by 1) and it follows that $M=T\oplus\sum_{I-\{\beta\}}\oplus M_\gamma$.

REMARK. Under the assumption that each M_α is uniform, all conditions in Theorems 5 and 6 are equivalent (cf. [4, Theorem 13]).

DEFINITION. Let $\mathcal{A}$ be a family of submodules of M. M is said to have the *lifting property of modules for* $\mathcal{A}$ if, for any A in $\mathcal{A}$, there exists a decomposition $M=A^*\oplus A^{**}$ such that $A^*\subseteq A$ and $A\cap A^{**}$ is small in M (see [5]).

NOTATION. By $\mathcal{H}(M)$, we denote the set of all submodules A of M such that M/A is a cyclic hollow module and define $\mathcal{H}^*(M)=\{A\in\mathcal{H}(M)\,|\,A$ contains almost all M_α but finit$\}$.

Theorem 7. *Assume that each M_α is cyclic hollow. Then the following conditions are equivalent*:

1) *M has the lifting property of modules for $\mathcal{H}^*(M)$.*

2) *For any pair α, $\beta\in I$, any $X\subsetneqq M_\beta$ and any epimorphism f: $M_\alpha\to M_\beta/X$, there exists either g: $M_\alpha\to M_\beta$ or h: $M_\beta\to M_\alpha$ such that*

$$\begin{tikzcd} M_\alpha \arrow[r, "g"] \arrow[dr, "f"'] & M_\beta \arrow[d, "\phi"] \\ & M_\beta/X \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} M_\alpha \arrow[dr, "f"'] & M_\beta \arrow[l, "h"'] \arrow[d, "\phi"] \\ & M_\beta/X \end{tikzcd}$$

is commutative, where ϕ is the canonical map.

Proof. 1)$\Rightarrow$2). Let α, $\beta\in I$ and consider submodules $X_\alpha\subsetneqq M_\alpha$ and $X_\beta\subsetneqq M_\beta$. Put $\bar{\bar{M}}=M/(X_\alpha\oplus X_\beta\oplus \sum_{I-\{\alpha,\beta\}}\oplus M_\gamma)$ and let f: $\bar{\bar{M}}_\alpha\to\bar{\bar{M}}_\alpha$ be an isomorphism. If we put $A=\{x\in M_\alpha\oplus M_\beta\,|\,\bar{\bar{x}}\in\{\bar{\bar{y}}+f(\bar{\bar{y}})\,|\,y\in M_\alpha\}\}$, then $M/A\simeq\bar{\bar{M}}/\bar{\bar{A}}\simeq\bar{\bar{M}}_\alpha$ and hence $A\oplus\sum_{I-\{\alpha,\beta\}}\oplus M_\gamma\in\mathcal{H}^*(M)$. So, by 1), there exists a decomposition $M=A^*\oplus A^{**}$ such that $A^*\subseteq A$ and $A\cap A^{**}$ is small in M. Since $M/A\simeq A^{**}/(A\cap A^{**})$ is cyclic hollow, A^{**} is also cyclic hollow. Hence A^{**} can be exchanged by some member in $\{M_\alpha\}_I$. Since $\bar{\bar{M}}=\bar{\bar{A}}^*$, A^{**} must be in fact exchanged by M_α or M_β; whence we get either $M=A^*\oplus M_\alpha$ or $M=A^*\oplus M_\beta$. In the former case, let π: $M=A^*\oplus M_\beta\to M_\beta$ be the projection. Then the diagram

$$\begin{tikzcd} \bar{\bar{M}}_\alpha \arrow[r, phantom, "\overset{f}{\simeq}"] & \bar{\bar{M}}_\alpha \\ M_\alpha \arrow[u, "\varphi_\alpha"] \arrow[r, "f'"] & M_\alpha \arrow[u, "\varphi_\beta"'] \end{tikzcd}$$

is commutative, where $f'=-\pi|M_\alpha$ and φ_α and φ_β are the canonical maps. In the latter case, we can obtain the desired epimorphism: $M_\beta\to M_\alpha$ by considering the projection: $M=A^*\oplus M_\alpha\to M_\alpha$.

2)$\Rightarrow$1). Let $A\in\mathcal{H}^*(M)$. Then we can take $F=\{\alpha_1,\cdots,\alpha_n\}\subseteq I$ and submodule $T\subseteq M_{\alpha_1}\oplus\cdots\oplus M_{\alpha_n}$ such that $A=\sum_{I-F}\oplus M_\beta\oplus T$ and $M=A+M_{\alpha_i}$, $i=1,\cdots,n$. We put $X=(A\cap M_{\alpha_1})\oplus\cdots\oplus(A\cap M_{\alpha_n})\oplus\sum_{I-E}\oplus M_\beta$ and $\tilde{M}=M/X$. Then

$$\tilde{M}=\tilde{A}\oplus\tilde{M}_{\alpha_1}=\cdots=\tilde{A}\oplus\tilde{M}_{\alpha_n}$$
$$\tilde{M}_{\alpha_i}\simeq M_{\alpha_i}/(A\cap M_{\alpha_i})\ \text{(canonically)},\ i=1,\cdots,n.$$

Let π_i: $\tilde{M}=\tilde{A}\oplus\tilde{M}_{\alpha_i}\to\tilde{M}_{\alpha_i}$ be the projection, $i=1,\cdots,n$. Then $\pi_i(\tilde{M}_{\alpha_j})=\tilde{M}_{\alpha_i}$ and $\{\tilde{x}+\pi_i(\tilde{x})\,|\,x\in M_{\alpha_j}\}\subseteq\tilde{A}$ for $j\neq i$. Here, using 2), we can take $i_0\in\{1,\cdots,n\}$ and mappings $\{f_j\colon M_{\alpha_j}\to M_{\alpha_{i_0}}\,|\,j\neq i_0\}$ such that

$$\widetilde{f_j(x)}=\pi_{i_0}(\tilde{x})$$

for all $x\in M_{\alpha_j}$ and $j\neq i_0$. Putting $A_j=\{x+f_j(x)\mid x\in M_{\alpha_j}\}$ and $T=A_1\oplus\cdots\oplus A_{i_0-1}\oplus A_{i_0+1}\oplus\cdots\oplus A_n\oplus\sum_J\oplus M_\beta$, we see that $T\subseteq A$ and $M=T\oplus M_{\alpha_{i_0}}$.

NOTATION. By $\mathcal{M}(M)$ we denote the set of all maximal submodules of M and put $\mathcal{M}^*(M)=\{A\in\mathcal{M}(M)\mid A$ contains almost all M_α but finite$\}$.

Theorem 8. *Assume that each M_α is a cyclic hollow module. Then the following conditions are equivalent*:

1) *M has the lifting property of simple modules modulo the radical.*

2) *M has the lifting property of modules for $\mathcal{M}^*(M)$.*

3) *For any pair α, β in I such that $\bar{M}_\alpha\simeq\bar{M}_\beta$ and any isomorphism f: $\bar{M}_\alpha\to\bar{M}_\beta$ (where $\bar{M}=M/J(M)$) there exists an epimorphism g of either M_α onto M_β or M_β onto M_α such that $\bar{g}=f$ or $\bar{g}=f^{-1}$, where $\bar{g}$ is the induced isomorphism.*

Proof. 1)$\Leftrightarrow$3) is due to Harada ([3]). 2)$\Leftrightarrow$3) is shown by the quite same argument as in the proof of Theorem 7.

NOTATION. Let $\{M_{\alpha_i}\}_{i=1}^\infty\subseteq\{M_\alpha\}_I$ and let $\{f_i\colon M_{\alpha_i}\to M_{\alpha_{i+1}}\}$ be a set of epimorphisms. By X_i we denote the set of all x in M_{α_i} such that $f_nf_{n-1}\cdots f_i(x)=0$ for some n (depending on x). Put $X=\sum_{i=1}^\infty\oplus X_i$ and $\hat{M}=M/X$. Then, as is easily seen, f_i induces an isomorphism $\hat{f}_i$: $\hat{M}_{\alpha_i}\to\hat{M}_{\alpha_{i+1}}$. Here we shall consider the following condition:

(*) For any such $\{M_{\alpha_i}\}_{i=1}^\infty$, epimorphisms $\{f_i\colon M_{\alpha_i}\to M_{\alpha_{i+1}}\}_{i=1}^\infty$ and $\hat{M}$, there exist n (depending on the sets) and epimorphism $g\colon M_{\alpha_{n+1}}\to M_{\alpha_n}$ such that g induces $\hat{f}_n^{-1}$

Theorem 9. *Assume that each M_α is cyclic hollow. Then the following conditions are equivalent*:

1) *M has the lifting property of modules for $\mathcal{H}(M)$.*

2) *M has the lifting property of modules for $\mathcal{H}^*(M)$ and satisfies the condition* (*).

Proof. 1)$\Rightarrow$2). The first part is clear. Let $\{M_{\alpha_i}\}_{i=1}^\infty\subseteq\{M_\alpha\}_I$ and let $\{f_i\colon M_{\alpha_i}\to M_{\alpha_{i+1}}\}_{i=1}^\infty$ be a set of epimorphisms. To verify (*) for these sets we can assume that $\{M_\alpha\}_I=\{M_{\alpha_i}\}_{i=1}^\infty$, since $\sum_{i=1}^\infty\oplus M_{\alpha_i}$ also has the lifting property of modules for $\mathcal{H}(\sum_{i=1}^\infty\oplus M_{\alpha_i})$. Now, we put $X_i=\{x\in M_{\alpha_i}\mid\exists n\colon f_nf_{n-1}\cdots f_i(x)=0\}$, $X=\sum_{i=1}^\infty\oplus X_i$ and $\hat{M}=M/X$. Since each M_{α_i} is cyclic hollow, we can put $M_{\alpha_i}=m_iR$ with $f_i(m_i)=m_{i+1}$ for some $\{m_i\}_{i=1}^\infty$. Putting $A=\sum_{i=1}^\infty(m_i+m_{i+1})R$, we see that $M=m_iR+A$ and $m_iR\cap A=X_i$, $i=1, 2, \cdots$. Since $M/A=(m_iR+A)/A\simeq m_iR/(A\cap m_iR)$, A lies in $\mathcal{H}(M)$. Hence there exists a decomposition $M=A^*\oplus A^{**}$ such that $A^*\subseteq A$ and $A\cap A^{**}$ is small in M. Since $M/A\simeq$

$A^{**}/(A\cap A^{**})$, $A^{**}/(A\cap A^{**})$ is cyclic hollow and hence so is A^{**}. As a result, we can assume that A^{**} coincides with some member in $\{M_{\alpha_i}\}_{i=1}^{\infty}$ by the Krull-Remak-Schmidt-Azumaya's theorem: say

$$M = A^{*}\oplus M_{\alpha_n}$$

with $A^{*}\subseteq A$. We express m_{n+1} as

$$m_{n+1} = -m_n r_n + (m_n + m_{n+1}) r_n + m_{n+1} r_{n+1}$$

with $m_{n+1}r_{n+1}\in X_{n+1}$.

Now the mapping $g\colon M_{\alpha_{n+1}}\to M_{\alpha_n}$ given by the rule $m_{n+1}r\to m_n r_n r$ is well defined and an epimorphism. We claim that $\hat{g}=\hat{f}_n^{-1}$. In fact, it is easy to see that $m_n r_n r\in X_n$ if and only if $m_{n+1}r\in X_{n+1}$; whence g induces an isomorphism $\hat{g}$ from $\hat{M}_{\alpha_{n+1}}$ to $\hat{M}_{\alpha_n}$ and moreover $\hat{m}_{n+1}=\hat{m}_{n+1}r_n=\hat{f}_n(\hat{m}_n r_n)=\hat{f}_n\hat{g}(\hat{m}_{n+1})$ and hence $\hat{g}=\hat{f}_n^{-1}$.

2)$\Rightarrow$1). We fix $\alpha_0\in I$ and put $M_{\alpha_0}=m_{\alpha_0}R$. Let $A\in\mathcal{H}(M)$. To show that A can be co-essentially lifted to a direct summand of M, we may assume that each M_α is not contained in A, namely, $M=M_\alpha+A$ for all $\alpha\in I$. Put $Y_\alpha=M_\alpha\cap A$ for all $\alpha\in I$, $Y=\sum_I\oplus Y_\alpha$ and $\tilde{M}=M/Y$. For any $\beta\in I-\{\alpha_0\}$, we see

$$\begin{aligned}\tilde{M} &= \tilde{M}_{\alpha_0}\oplus\tilde{A}\\ &= \tilde{M}_\beta\oplus\tilde{A}\,.\end{aligned}$$

So, there exist $m_\beta\in M_\beta$ and $a_\beta\in A$ such that

$$\tilde{m}_{\alpha_0} = \tilde{m}_\beta + \tilde{a}_\beta\,.$$

Clearly the rule $\tilde{m}_{\alpha_0}r\to\tilde{m}_\beta r$ defines an isomorphism from $\tilde{M}_{\alpha_0}$ to $\tilde{M}_\beta$. Therefore the rule $\tilde{m}_\beta r\leftrightarrow\tilde{m}_{\beta'}r$ define an isomorphism $\eta_\beta^{\beta'}\colon\tilde{M}_\beta\to\tilde{M}_{\beta'}$ for any pair β, β' in I. Here we shall show that there does not exist the following subset $\{\alpha_i\}_{i=1}^{\infty}\subseteq I-\{\alpha_0\}$:

i) there exists a set $\{f_i\colon M_{\alpha_i}\to M_{\alpha_{i+1}}\}_{i=1}^{\infty}$ of epimorphisms such that each f_i induces the isomorphism $\eta_{\alpha_i}^{\alpha_{i+1}}$

ii) but for all i there does not exist any epimorphism $g\colon M_{\alpha_{i+1}}\to M_{\alpha_i}$ which induces the isomorphism $(\eta_{\alpha_i}^{\alpha_{i+1}})^{-1}$.

In fact, assume, on the contrary, that such $\{\alpha_i\}_{i=1}^{\infty}$ exists. Put $X_i=\{x\in M_{\alpha_i}\mid f_n f_{n-1}\cdots f_i(x)=0$ for some $n\geq i\}$, $X=\sum_{i=1}^{\infty}\oplus X_i$ and $\hat{M}=M/X$. Then clearly $X_i\subseteq Y_i$ and $f_i(X_i)=X_{i+1}$ for all i. By $\hat{f}_i$ we denote the induced isomorphism: $\hat{M}_{\alpha_i}\to\hat{M}_{\alpha_{i+1}}$. Here using the condition 2) we can take k and an epimorphism $g\colon M_{\alpha_k}\to M_{\alpha_{k-1}}$ such that g induces $\hat{f}_k^{-1}$. Then $\hat{m}_k=\hat{g}(\hat{m}_{k+1})$ and it follows that $\tilde{m}_k=\widetilde{g(m_{k+1})}$. As a result, g induces $(\eta_{\alpha_k}^{\alpha_{k+1}})^{-1}$, a contradiction.

Now, by this fact and Theorem 8, we may consider the following two cases.

*) For any $\alpha \in I-\{\alpha_0\}$ there exists an epimorphism f_α: $M_\alpha \to M_{\alpha_0}$ such that f_α induces the isomorphism $\eta^{\alpha_0}_{\alpha}$; $\tilde{M}_\alpha \to \tilde{M}_{\alpha_0}$.

**) There exist $J=\{\alpha_1, \cdots, \alpha_t\} \subseteq I-\{\alpha_0\}$ and sets $\{f_i^{i+1}\colon M_{\alpha_i} \to M_{\alpha_{i+1}} | i=0, \cdots, t-1\}$ and $\{f^{\alpha_t}_{\beta}\colon M_\beta \to M_{\beta_t} | \beta \in I-\{J \cup \{\alpha_0\}\}$ of epimorphisms such that f_i^{i+1} and $f^{\alpha_t}_{\beta}$ induce $\eta^{\alpha_{i+1}}_{\alpha_i}$ and $\eta^{\alpha_t}_{\beta}$, respectively. Then

$$\tilde{m}_{\alpha_{i+1}} = \widetilde{f_i^{i+1}(m_{\alpha_i})}$$

for all $i=1, 2, \cdots, t-1$, and

$$\tilde{m}_\beta = \widetilde{f^{\alpha_t}_{\beta}(m_{\alpha_t})}$$

for all $\beta \in K = I-\{J \cup \{\alpha_0\}\}$.

In the first case, consider the map $f=\sum_{I-\{\alpha_0\}} f^{\alpha_0}_{\alpha}\colon \sum_{I-\{\alpha_0\}} \oplus M_\alpha \to M_{\alpha_0}$ and put $A^*=\{x+f(x) | x \in \sum_{I-\{\alpha_0\}} \oplus M_\alpha\}$. Then $M=A^* \oplus M_{\alpha_0}$ and it follows from $\tilde{A}^*=\sum_I \oplus \tilde{a}_\alpha R$ that $A^* \subseteq A$ as desired. In the second case we put $M'_{\alpha_i}=\{x+f_i^{i+1}(x) | x \in m_{\alpha_i}R\}$ for $i=0, 1, \cdots, t-1$ and $T=\{x+g(x) | x \in \sum_K \oplus m_\beta R\}$ where $g=\sum_K f^{\alpha_t}_{\beta}$. Then

$$M = \sum_{i=0}^{t-1} \oplus M'_{\alpha_i} \oplus T \oplus M_{\alpha_t},$$
$$\tilde{M}'_{\alpha_i} = \tilde{a}_{\alpha_i} R \text{ for } i = 1, \cdots, t-1, \text{ and}$$
$$\tilde{T} = (\tilde{a}_\beta - \tilde{a}_{\beta_t})R \text{ for all } \beta \in K.$$

Hence putting $A^*=\sum_{i=0}^{t-1} \oplus M'_{\alpha_i} \oplus T$ we see that $A^* \subseteq A$ and $M=A^* \oplus M_{\alpha_t}$. Our proof is now completed.

By a similar proof as in the proof of the above theorem, we can obtain the following result which is mentioned in introduction of this paper.

Theorem 10. *Assume that each M_α is cyclic hollow. Then the following conditions are equivalent*:

1) *M has the lifting property of modules for $\mathcal{M}(M)$.*

2) *M has the lifting property of modules for $\mathcal{M}^*(M)$ and satisfies the following condition: For any subfamily $\{M_{\alpha_i}\}_{i=1}^{\infty} \subseteq \{M_\alpha\}_I$ and epimorphisms $\{f_i\colon M_{\alpha_i} \to M_{\alpha_{i+1}}\}_{i=1}^{\infty}$, there exist n and epimorphism $g\colon M_{\alpha_{n+1}} \to M_{\alpha_n}$ satisfying $\bar{f}_n^{-1}=\bar{g}$ on $\bar{M}=M/J(M)$ where $\bar{f}_n$ and $\bar{g}$ are the induced isomorphisms.*

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