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ON THE MARTINGALE PROBLEM FOR GENERATORS OF STABLE PROCESSES WITH PERTURBATIONS

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0. Introduction

We say that a probability measure P_x on $D(\boldsymbol{R}_+\to\boldsymbol{R}^d)$ solves the martingale problem for a Lévy type generator L starting from x at time $t=0$ if

$$M_t^f=f(X_t)-f(x)-\int_0^t Lf(X_s)ds$$

is a P_x-martingale with $M_0^f=0$ for all test functions f on $\boldsymbol{R}^d$. The martingale problem for second order elliptic differential operators was studied by Stroock and Varadhan [7]. It was Grigelionis [3] who first gave a martingale formulation of Markov processes associated with certain integro-differential operators. Komatsu [4], Tsuchiya [9] and Stroock [8] discussed the existence and the uniqueness of solutions for the martingale problem associated with a Lévy type generator L. The existence was proved in [4] and [8] under a certain continuity condition. The uniqueness was shown in [4] and [8] in a context that the principal part of L is a second order elliptic differential operator. Tsuchiya [9] proved the uniqueness in the case where the principal part of L is the generator of an isotropic stable processes with index α ($1\leqq\alpha<2$) and the perturbation part of L has the upper index $\beta<\alpha$ (for the precise meaning, see the remark following assumption [A_2] in section 2). The purpose of this paper is to improve the results in [9] by making use of the theory of singular integrals in Calderôn and Zygmund [1].

Let $\exp[t\psi^{(\alpha)}(\xi)]$ be the characteristic function of a stable process with index α. Then $\psi^{(\alpha)}(\xi)$ is a homogeneous function of index α, and the generator $A^{(\alpha)}$ of the stable process is given by

$$A^{(\alpha)}f(x)=\mathcal{F}^{-1}[\psi^{(\alpha)}\mathcal{F}f](x)\,.$$

In case $1<\alpha<2$, the operator $A^{(\alpha)}$ has the following expression

$$A^{(\alpha)}f(x)=\int[f(x+y)-f(x)-y\cdot\partial f(x)]M^{(\alpha)}(dy)\,,$$

where $M^{(\alpha)}(dy)$ is a measure on $\boldsymbol{R}^d\setminus\{0\}$ such that there is a finite measure

$M_0^{(\alpha)}(d\omega)$ on S^{d-1} satisfying $M^{(\alpha)}(dy)=M_0^{(\alpha)}(d\omega)r^{-1-\alpha}\,dr$ with $y=|y|\omega=r\omega$. We assume that the measure $M^{(\alpha)}(dy)$ has the Radon-Nikodym density $m^{(\alpha)}(y)$ with respect to the Lebesque measure dy which is d-times continuously differentiable on $\boldsymbol{R}^d\setminus\{0\}$ and not identically 0. Let $B^{(\alpha)}$ be the operator

$$B^{(\alpha)}f(x)=\int[f(x+y)-f(x)-I_{(|y|\leq 1)}y\cdot\partial f(x)]N^{(\alpha)}(x,dy)+b^{(\alpha)}(x)\cdot\partial f(x)\,.$$

In case $1<\alpha<2$, the assumption for $B^{(\alpha)}$ is as follows: $b^{(\alpha)}(x)$ is an arbitrary bounded measurable vector and every $|N^{(\alpha)}(x,dy)|$ is dominated by some measure $N_*^{(\alpha)}(dy)$ such that

$$\int|y|^\alpha\wedge 1\,N_*^{(\alpha)}(dy)<\infty\,.$$

We shall consider the martingale problem for the operator

$$L=A^{(\alpha)}+B^{(\alpha)}\,.$$

We say that the operator $A^{(\alpha)}$ is the principal part of L and the operator $B^{(\alpha)}$, the perturbation part of L. In this paper, we shall be exclusively concerned with the uniqueness theorem, for the existence has been already proved in a more general case under some additional conditions on L (see [4] Theorem 5.4 or [8] Theorem 2.2).

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1. Some preliminary results

First of all, we shall list the notations which will be used without further references.

Let $x=(x_1,\cdots,x_d)\in\boldsymbol{R}^d$ and $\partial=(\partial_1,\cdots,\partial_d)$ with $\partial_j=\partial/\partial x_j$. For $\nu=(\nu_1,\cdots,\nu_d)$, $\nu_j\in\boldsymbol{Z}_+$, set $|\nu|=\nu_1+\cdots+\nu_d$ and

$$x^\nu=x_1^{\nu_1}\cdots x_d^{\nu_d}\,,\qquad \partial^\nu=\partial_1^{\nu_1}\cdots\partial_d^{\nu_d}\,.$$

The Fourier transform $\mathcal{F}$ and the inverse Fourier transform $\mathcal{F}^{-1}$ are defined by

$$\mathcal{F}f(\xi)=\int e^{-i\xi\cdot x}f(x)dx\,,\quad \mathcal{F}^{-1}\phi(x)=(2\pi)^{-d}\int e^{i\xi\cdot x}\phi(\xi)d\xi\,.$$

The sup-norm and the L^p-norm of a function $f(x)$ on $\boldsymbol{R}^d$ are denoted respectively by $||f||$ and $||f||_{L^p}$. Let $\mathcal{S}(\boldsymbol{R}^d)$ denote the space of tempered functions on $\boldsymbol{R}^d$. A function $f(\omega)$ on $S^{d-1}=\{\omega\in\boldsymbol{R}^d;\ |\omega|=1\}$ is said to belong to the space $\boldsymbol{C}^n(S^{d-1})$ if there exists an n-times continuously differentiable function $g(x)$ on $\boldsymbol{R}^d$ such that $f(x)=g(x)$ for $|x|=1$. The area element of the surface S^{d-1} is denoted by $\sigma(d\omega)$. Finally set $\Theta(x)=x$ if $|x|\leq 1$ and $\Theta(x)=0$ if $|x|>1$.

We shall next consider the generator $A^{(\alpha)}$ of a stable process with index α, $0<\alpha\leqq 2$. The generator $A^{(\alpha)}$ has the following expression.

$$A^{(\alpha)}f(x)=\int[f(x+y)-f(x)]M^{(\alpha)}(dy) \qquad (0<\alpha<1),$$

$$A^{(1)}f(x)=\int[f(x+y)-f(x)-\Theta(y)\cdot\partial f(x)]M^{(1)}(dy)+\gamma\cdot\partial f(x),$$

$$A^{(\alpha)}f(x)=\int[f(x+y)-f(x)-y\cdot\partial f(x)]M^{(\alpha)}(dy) \qquad (1<\alpha<2),$$

$$A^{(2)}f(x)=\frac{1}{2}\sum_{i,j}a_{ij}\,\partial_i\partial_j f(x),$$

where $a=(a_{ij})\geqq 0$, $\gamma\in\boldsymbol{R}^d$ and $M^{(\alpha)}(dy)$ a measure on $\boldsymbol{R}^d\setminus\{0\}$ such that there is a finite measure $M_0^{(\alpha)}(d\omega)$ on S^{d-1} satisfying

$$M^{(\alpha)}(dy)=M_0^{(\alpha)}(d\omega)r^{-1-\alpha}\,dr \quad\text{with } y=|y|\omega=r\omega,$$

$$\int_{|\omega|=1}\omega\,M_0^{(1)}(d\omega)=0.$$

Let $\exp[t\psi^{(\alpha)}(\xi)]$ be the characteristic function of the α-stable process. The exact form of the function $\psi^{(\alpha)}(\xi)$ was obtained by Lévy [6]: for $0<\alpha<1$ or $1<\alpha<2$,

$$(1.1)\quad \psi^{(\alpha)}(\xi)=-\frac{\sqrt{\pi}\,\Gamma\left(\frac{2-\alpha}{2}\right)}{\alpha 2^{\alpha}\Gamma\left(\frac{1+\alpha}{2}\right)}\int_{|\omega|=1}|\omega\cdot\xi|^{\alpha}\{1-i\tan\frac{\pi\alpha}{2}\times\operatorname{sgn}(\omega\cdot\xi)\}M_0^{(\alpha)}(d\omega);$$

and for $\alpha=1$ or $\alpha=2$,

$$(1.2)\quad \begin{cases}\psi^{(1)}(\xi)=-\dfrac{\pi}{2}\displaystyle\int_{|\omega|=1}(|\omega\cdot\xi|+\frac{2i}{\pi}\omega\cdot\xi\log|\omega\cdot\xi|)M_0^{(1)}(d\omega)+i\gamma\cdot\xi,\\[2mm] \psi^{(2)}(\xi)=-\dfrac{1}{2}\xi\cdot a\xi.\end{cases}$$

It is well known that, for all $f\in\mathcal{S}(\boldsymbol{R}^d)$,

$$(1.3)\qquad A^{(\alpha)}f(x)=\mathcal{F}^{-1}[\psi^{(\alpha)}\mathcal{F}f](x).$$

From now on the measure $M^{(\alpha)}(dy)$ is assumed to have the density function $m^{(\alpha)}(y)$ with respect to the Lebesgue measure dy. Then $m^{(\alpha)}(y)$ is a homogeneous function with index $-d-\alpha$ on $\boldsymbol{R}^d\setminus\{0\}$, *i.e.* $m^{(\alpha)}(ry)=r^{-d-\alpha}m^{(\alpha)}(y)$ for $r>0$, and satisfies

$$M_0^{(\alpha)}(d\omega)=m^{(\alpha)}(\omega)\sigma(d\omega) \quad\text{for } |\omega|=1,$$

$$\int_{|\omega|=1}\omega\,m^{(1)}(\omega)\sigma(d\omega)=0.$$

Here and after we shall assume the following.

Assumption [A_1]

For $0<\alpha<2$, $m^{(\alpha)}(\omega)$ *belongs to* $C^d(S^{d-1})$ *and it is not identically* 0. *For* $\alpha=2$, *the matrix* $a=(a_{ij})$ *is positive definite.*

Lemma 1.1. *Under assumption* [A_1], *the function* $\psi^{(\alpha)}(\xi)$ *belongs to* $C^{d+1}(S^{d-1})$ *and* Re $\psi^{(\alpha)}(\xi)<0$ *as long as* $\xi\neq 0$.

Proof. Since

$$S^{d-1}=\bigcup_{k=1}^{d}\left(\left\{\xi\in S^{d-1};\ \xi_k>\frac{1}{2\sqrt{d}}\right\}\cup\left\{\xi\in S^{d-1};\ \xi_k<-\frac{1}{2\sqrt{d}}\right\}\right),$$

it is enough to show that $\psi^{(\alpha)}(\xi)$ is $(d+1)$-times continuously differentiable on each of the above similar $2d$ manifolds. We shall consider only the manifold

$$D=\left\{\xi\in S^{d-1};\ \xi=(\xi_1,\cdots,\xi_d),\ \xi_1>\frac{1}{2\sqrt{d}}\right\}.$$

Let $\xi\in D$ and $2\leqq j\leqq d$. Then $\xi_1(\partial/\partial\xi_j)|\xi\cdot\omega|^\alpha=\omega_j(\partial/\partial\omega_1)|\xi\cdot\omega|^\alpha$, Using Stokes's theorem, we have

$$\begin{aligned}
&\xi_1\frac{\partial}{\partial\xi_j}\int_{|\omega|=1}|\omega\cdot\xi|^\alpha m^{(\alpha)}(\omega)\sigma(d\omega)\\
&=\int_{|\omega|=1}\omega_j\frac{\partial}{\partial\omega_1}|\xi\cdot\omega|^\alpha m^{(\alpha)}(\omega)\sigma(d\omega)\\
&=\int_{|\omega|=1}\frac{\partial}{\partial\omega_1}|\xi\cdot\omega|^\alpha m^{(\alpha)}(\omega)\,(-1)^{j-1}d\omega_1\wedge\cdots\wedge d\omega_{j-1}\wedge d\omega_{j+1}\wedge\cdots\wedge d\omega_d\\
&=-\int_{|\omega|=1}|\xi\cdot\omega|^\alpha\frac{\partial}{\partial\omega_1}m^{(\alpha)}(\omega)\,(-1)^{j-1}d\omega_1\wedge\cdots\wedge d\omega_{j-1}\wedge d\omega_{j+1}\wedge\cdots\wedge d\omega_d\\
&=-\int_{|\omega|=1}|\xi\cdot\omega|^\alpha\omega_j\frac{\partial}{\partial\omega_1}m^{(\alpha)}(\omega)\sigma(d\omega)\,.
\end{aligned}$$

Inductively it is proved that, for $\nu=(0,\nu_2,\cdots,\nu_d)$ with $|\nu|\leqq d$,

$$\begin{aligned}
&(\xi_1\partial)^\nu\int_{|\omega|=1}|\omega\cdot\xi|^\alpha m^{(\alpha)}(\omega)\sigma(d\omega)\\
&=\int_{|\omega|=1}|\xi\cdot\omega|^\alpha\left(-\omega\frac{\partial}{\partial\omega_1}\right)^\nu m^{(\alpha)}(\omega)\sigma(d\omega)\,,
\end{aligned}$$

where $(\xi_1\partial)^\nu=(\xi_1\partial/\partial\xi_2)^{\nu_2}\cdots(\xi_1\partial/\partial\xi_d)^{\nu_d}$. Therefore

$$\frac{\partial}{\partial\xi_j}(\xi_1\partial)^\nu\int_{|\omega|=1}|\omega\cdot\xi|^\alpha m^{(\alpha)}(\omega)\sigma(d\omega)$$

$$= \int_{|\omega|=1} \frac{\partial}{\partial \xi_j} |\omega\cdot\xi|^{\alpha} \left(-\omega \frac{\partial}{\partial \omega_1}\right)^{\nu} m^{(\alpha)}(\omega)\sigma(d\omega)\,.$$

This implies that the function $\int_{|\omega|=1} |\omega\cdot\xi|^{\alpha} m^{(\alpha)}(\omega)\sigma(d\omega)$ is $(d+1)$-times continuously differentiable on D. Similarly it is proved that the functions

$$\int_{|\omega|=1} |\xi\cdot\omega|^{\alpha} \operatorname{sgn}(\xi\cdot\omega) m^{(\alpha)}(\omega)\sigma(d\omega)\,,$$

$$\int_{|\omega|=1} (\xi\cdot\omega) \log|\omega\cdot\xi|\, m^{(1)}(\omega)\sigma(d\omega)$$

are $(d+1)$-times continuously differentiable on D. From (1.1) and (1.2) we see that $\psi^{(\alpha)}(\xi)$ is $(d+1)$-times continuously differentiable on D, and thus $\psi^{(\alpha)}(\xi)$ belongs to $\boldsymbol{C}^{d+1}(S^{d-1})$. It is obvious from (1.1) and (1.2) that $\operatorname{Re} \psi^{(\alpha)}(\xi) < 0$ for $\xi \neq 0$.

Lemma 1.2. *Let $\phi(\xi)$ be a homogeneous function with index* 0 *and $\phi(\xi)$ belong to $\boldsymbol{C}^d(S^{d-1})$. Let $\mathcal{F}^{-1}\phi$ denote the inverse Fourier transform of ϕ in the distribution sense. Then there is a homogeneous function $h(x)$ with index $-d$ such that $h(\omega)$ belongs to $\boldsymbol{C}^0(S^{d-1})$,*

$$\int_{|\omega|=1} h(\omega)\sigma(d\omega) = 0 \tag{1.4}$$

and $\mathcal{F}^{-1}\phi(x) = h(x) + c_\phi \delta(x)$, where $\delta(x)$ is the Dirac δ-function and c_ϕ is the mean value of the function ϕ over the surface S^{d-1}:

$$c_\phi = \int_{|\omega|=1} \phi(\omega)\sigma(d\omega) \Big/ \int_{|\omega|=1} \sigma(d\omega)\,.$$

Proof. In case $d=1$, it is not difficult to see that

$$\mathcal{F}^{-1}\phi(x) = \frac{\phi(1)-\phi(-1)}{2} \frac{i}{\pi x} + \frac{\phi(1)+\phi(-1)}{2} \delta(x)\,.$$

Now suppose that $d \geqq 2$ and $x \neq 0$. Define

$$\begin{aligned} h(x) &= \lim_{\varepsilon \downarrow 0} (2\pi)^{-d} \int_{\boldsymbol{R}^d} \phi(\xi)\, e^{-\varepsilon|\xi| + i\xi\cdot x} d\xi \\ &= \lim_{\varepsilon \downarrow 0} (2\pi)^{-d} \int_{|\omega|=1} \phi(\omega) \left(\int_0^\infty r^{d-1} e^{ir(\omega\cdot x + i\varepsilon)} dr\right) \sigma(d\omega) \\ &= \lim_{\varepsilon \downarrow 0} (2\pi)^{-d} \int_{|\omega|=1} \phi(\omega)\, (-i)^d (d-1)!\, (\omega\cdot x + i\varepsilon)^{-d} \sigma(d\omega)\,. \end{aligned}$$

Let $|\eta|=1$, $S_\eta = \{\zeta \in S^{d-1};\ \zeta\cdot\eta = 0\}$ and $\sigma_{d-2}(d\zeta)$ be the area element on S_η, and define

$$v(\eta, t) = \int_{S_\eta} \phi(t\eta + \sqrt{1-t^2}\zeta)\, \sigma_{d-2}(d\zeta)$$

$$\times(2\pi)^{-d}(-i)^d(d-1)!\,(1-t^2)^{(d-3)/2}I_{(|t|<1)}\,.$$

Then

$$\begin{aligned} h(\eta) &= \lim_{\varepsilon\downarrow 0}\int_{-\infty}^{+\infty} v(\eta,t)\,(t+i\varepsilon)^{-d}\,dt \\ &= \langle \lim_{\varepsilon\downarrow 0}(t+i\varepsilon)^{-d},\ v(\eta,t)\rangle \\ &= \langle t^{-d}+\frac{i\pi(-1)^d}{(d-1)!}\,\delta^{(d-1)}(t),\ v(\eta,t)\rangle\,, \end{aligned}$$

where $\langle t^{-d},\ \cdot\rangle$ is defined by

$$\begin{aligned} \langle t^{-2m}, v(t)\rangle &= \int_0^\infty t^{-2m}\{v(t)+v(-t) \\ &\quad -2(v(0)+\frac{v''(0)}{2!}t^2+\cdots+\frac{v^{(2m-2)}(0)}{(2m-2)!}t^{2m-2})\}\,dt\,, \\ \langle t^{-2m-1}, v(t)\rangle &= \int_0^\infty t^{-2m-1}\{v(t)-v(-t) \\ &\quad -2(v'(0)t+\frac{v^{(3)}(0)}{3!}t^3+\cdots+\frac{v^{(2m-1)}(0)}{(2m-1)!}t^{2m-1})\}\,dt \end{aligned}$$

(see Gel'fand and Shilov [2]). Since $\phi(\omega)\in \boldsymbol{C}^d(S^{d-1})$, it follows that $(d/dt)^k v(\eta,t)$ $(0\leqq k\leqq d)$ are continuous in $(\eta,t)\in S^{d-1}\times(-1,1)$; and thus $h(\eta)\in \boldsymbol{C}^0(S^{d-1})$. It is immediate to show that the function $h(x)$ is homogeneous of index $-d$. Let $\rho(r)$ be a test function on $\boldsymbol{R}_+$ such that $0\leqq\rho(r)\leqq 1$, $\rho(r)=0$ if $r\geqq 2$ and $\rho(r)=1$ for $0\leqq r\leqq 1$. Then

$$\begin{aligned} &\int_0^\infty\int_{|\omega|=1} h(\omega)r^{-d}\{\rho(r)-\rho(2r)\}r^{d-1}dr\,\sigma(d\omega) \\ &= \int h(x)\{\rho(|x|)-\rho(2|x|)\}\,dx \\ &= (2\pi)^{-d}\int\phi(\xi)\{\mathcal{F}[\rho(|\cdot|)](\xi)-\mathcal{F}[\rho(2|\cdot|)](\xi)\}\,d\xi \\ &= (2\pi)^{-d}\int\phi(\xi)\{\mathcal{F}[\rho(|\cdot|)](\xi)-2^d\mathcal{F}[\rho(|\cdot|)](2\xi)\}\,d\xi = 0\,. \end{aligned}$$

Therefore $\int_{|\omega|=1} h(\omega)\,\sigma(d\omega)=0$. Since the distribution $\mathcal{F}^{-1}\phi$ is homogeneous of index $-d$ and since the support of the distribution $\mathcal{F}^{-1}\phi-h$ is concentrated on the origin, there is a constant c_ϕ such that

$$\mathcal{F}^{-1}\phi(x)-h(x) = c_\phi\,\delta(x)\,.$$

We shall show that the constant c_ϕ is the mean value of the function ϕ over the surface S^{d-1}.

$$
\begin{aligned}
c_\phi &= \lim_{\varepsilon\downarrow 0} \langle \mathcal{F}^{-1}\phi - h,\ e^{-|\cdot|^2/2\varepsilon}\rangle \\
&= \lim_{\varepsilon\downarrow 0} \langle \mathcal{F}^{-1}\phi,\ \sqrt{2\pi\varepsilon}^{\,d}\, \mathcal{F}^{-1}[e^{-\varepsilon|\cdot|^2/2}]\rangle \\
&= \lim_{\varepsilon\downarrow 0} (2\pi)^{-d}\int \phi(\xi)\sqrt{2\pi\varepsilon}^{\,d}\, e^{-\varepsilon|\xi|^2/2}d\xi \\
&= \int \phi(\xi)\sqrt{2\pi}^{\,-d}\, e^{-|\xi|^2/2}d\xi \\
&= \int_{|\omega|=1}\phi(\omega)\,\sigma(d\omega)\cdot\int_0^\infty r^{d-1}\sqrt{2\pi}^{\,-d}\, e^{-r^2/2}\,dr \\
&= \int_{|\omega|=1}\phi(\omega)\,\sigma(d\omega)\Big/\int_{|\omega|=1}\sigma(d\omega)\,.
\end{aligned}
$$

The last equality holds because

$$\int_0^\infty r^{d-1}\sqrt{2\pi}^{\,-d}\, e^{-r^2/2}\,dr\cdot\int_{|\omega|=1}\sigma(d\omega) = \int\sqrt{2\pi}^{\,-d}\, e^{-|\xi|^2/2}d\xi = 1\,.$$

Theorem 1. *Let $\phi(\xi)$ be a homogeneous function with index* 0, *and $\phi(\omega) \in \boldsymbol{C}^d(S^{d-1})$. Fix a constant p, $1<p<\infty$. Then there is a constant C_p such that*

$$\|\mathcal{F}^{-1}[\phi\mathcal{F}f]\|_{L^p}\leqq C_p\|f\|_{L^p} \qquad \text{for all } f\in\mathcal{S}(\boldsymbol{R}^d)\,. \tag{1.5}$$

Proof. Set $h(x)=\mathcal{F}^{-1}\phi(x)-c_\phi\,\delta(x)$, where c_ϕ is the mean value of ϕ over S^{d-1}. Define

$$(h*f)\,(x) = \lim_{\varepsilon\downarrow 0}\int_{|y|>\varepsilon} h(y)f(x-y)\,dy\,.$$

By Lemma 1.2, $h*f$ is a singular integral. From a theorem of Calderon and Zygmund [1], there exists a constant c_p such that $\|h*f\|_{L^p}\leqq c_p\|f\|_{L^p}$ for all $f\in\mathcal{S}(\boldsymbol{R}^d)$. Note that

$$\mathcal{F}^{-1}[\phi\mathcal{F}f] = \mathcal{F}^{-1}\phi*f = h*f+c_\phi f \qquad \text{for each } f\in\mathcal{S}(\boldsymbol{R}^d)\,.$$

Set $C_p=c_p+|c_\phi|$. Then we have (1.5).

The function $\phi_2(\xi)=\xi_i\xi_j/\psi^{(2)}(\xi)$ satisfies the condition of the above theorem. In Stroock and Varadhan [7], inequality (1.5) associated with the function $\phi_2(\xi)$ played an essential role in the proof of the uniqueness theorem. We shall apply Theorem 1 for the function $\phi(\xi)=|\xi|^\alpha/\psi^{(\alpha)}(\xi)$ in case $0<\alpha<1$, and for the function $\phi(\xi)=\xi_j|\xi|^{\alpha-1}/\psi^{(\alpha)}(\xi)$ in case $1\leqq\alpha<2$. From Lemma 1.1, these functions satisfy the condition of Theorem 1. Inequality (1.5) is also essential in this paper.

2. Perturbation of generator of stable processes

Let $p^{(\alpha)}(t, x-y)$ be the transition function of the α-stable process with the generator $A^{(\alpha)}$. Then, for $t>0$, we have

$$p^{(\alpha)}(t, x) = \mathcal{F}^{-1}[e^{t\psi^{(\alpha)}}](x). \tag{2.1}$$

Since $\psi^{(\alpha)}(\xi)$ is a homogeneous function with index α, we have

$$p^{(\alpha)}(t, x) = t^{-d/\alpha}p^{(\alpha)}(1, t^{-1/\alpha}x). \tag{2.2}$$

The λ-potential operator $G_\lambda^{(\alpha)}$ of the α-stable process is given by

$$\begin{aligned} G_\lambda^{(\alpha)}f(x) &= \int_0^\infty\int e^{-\lambda t}p^{(\alpha)}(t, x-y)f(y)\,dtdy \\ &= \int_0^\infty e^{-\lambda t}\mathcal{F}^{-1}[e^{t\psi^{(\alpha)}}\mathcal{F}f](x)\,dt \\ &= \mathcal{F}^{-1}[(\lambda-\psi^{(\alpha)})^{-1}\mathcal{F}f](x) \end{aligned} \tag{2.3}$$

for each $f\in\mathcal{S}(\boldsymbol{R}^d)$. Let

$$g_\lambda^{(\alpha)}(x) = \int_0^\infty e^{-\lambda t}p^{(\alpha)}(t, x)\,dt. \tag{2.4}$$

Then $\|g_\lambda^{(\alpha)}\|_{L^1}=\lambda^{-1}$ and $G_\lambda^{(\alpha)}f=g_\lambda^{(\alpha)}*f$. From Young's inequality we have

$$\|G_\lambda^{(\alpha)}f\|_{L^p}\leqq\lambda^{-1}\|f\|_{L^p},\ \|G_\lambda^{(\alpha)}f\|\leqq\lambda^{-1}\|f\|. \tag{2.5}$$

For $0<\delta<1$, define a pseudo-differential operator $|\partial|^\delta$ by

$$\begin{aligned} |\partial|^\delta f(x) &= \mathcal{F}^{-1}[|\xi|^\delta\mathcal{F}f(\xi)](x) \\ &= \left(2^\delta\sqrt{\pi}^{-d}\Gamma\left(\frac{d+\delta}{2}\right)/\Gamma\left(-\frac{\delta}{2}\right)\right)\int(f(x+y)-f(x))|y|^{-d-\delta}dy. \end{aligned} \tag{2.6}$$

Lemma 2.1. *Let $0<\delta<1$. There are constants c_1 and c_2 such that*

$$\int||x+y|^{\delta-d}-|y|^{\delta-d}|\,dy = c_1|x|^\delta, \tag{2.7}$$

$$f(x+z)-f(x) = c_2\int(|y+z|^{\delta-d}-|y|^{\delta-d})|\partial|^\delta f(x-y)\,dy \tag{2.8}$$

for each smooth bounded function f on $\boldsymbol{R}^d$.

Proof. Since the function $\Phi(x)=\int||x+y|^{\delta-d}-|y|^{\delta-d}|\,dy$ is isotropic and homogeneous with index δ, (2.7) follows if $\Phi(x_0)<\infty$ for some $x_0\neq0$. If $x_0=(2, 0, \cdots, 0)$, then

$$\begin{aligned} &\int||x_0+y|^{\delta-d}-|y|^{\delta-d}|\,dy \\ &= 2\int_{y_1>-1}||x_0+y|^{\delta-d}-|y|^{\delta-d}|\,dy_1\cdots dy_d \end{aligned}$$

$$= 2 \lim_{N\to\infty} \int_{-1<y_1<N} (|y|^{\delta-d} - |x_0+y|^{\delta-d}) dy_1 \cdots dy_d$$
$$= 2 \lim_{N\to\infty} (\int_{-1<y_1<N} |y|^{\delta-d} dy_1 \cdots dy_d - \int_{-1<y_1<N-2} |x_0+y|^{\delta-d} dy_1 \cdots dy_d)$$
$$= 2\int_{-1}^{+1} dy_1 \int |y|^{\delta-d} dy_2 \cdots dy_d < \infty .$$

We shall next prove (2.8). Note that

$$\mathcal{F}^{-1}[|\xi|^{-\delta}](x) = c_2 |x|^{\delta-d} ,$$

where $c_2 = 2^{-\delta}\sqrt{\pi}^{-d}\Gamma((d-\delta)/2)/\Gamma(\delta/2)$. Suppose that $f \in \mathcal{S}(\boldsymbol{R}^d)$. Then

$$\begin{aligned} f(x+z)-f(x) &= \mathcal{F}^{-1}\mathcal{F}f(x+z) - \mathcal{F}^{-1}\mathcal{F}f(x) \\ &= c_2(|\cdot|^{\delta-d} * |\partial|^{\delta} f)(x+z) - c_2(|\cdot|^{\delta-d} * |\partial|^{\delta} f)(x) \\ &= c_2\int (|y+z|^{\delta-d} - |y|^{\delta-d}) |\partial|^{\delta} f(x-y) dy . \end{aligned}$$

Next suppose that f is a smooth bounded function on $\boldsymbol{R}^d$. There is a sequence $\{f_n\} \subset \mathcal{S}(\boldsymbol{R}^d)$ such that $f_n(x) \to f(x)$,

$$\sup_n \| |\partial|^{\delta} f_n \| < \infty \text{ and } |\partial|^{\delta} f_n(x) \to |\partial|^{\delta} f(x) .$$

Since (2.8) is satisfied for each f_n, from Lebesque's convergence theorem, we see that (2.8) also holds for f.

In the following two lemmas, let $1<p<\infty$ and $f \in \mathcal{S}(\boldsymbol{R}^d)$, and let c's denote constants independent of λ and f.

Lemma 2.2. (i) *If* $0<\alpha\leqq 1$, *then*

$$\|G_\lambda^{(\alpha)} f(\cdot+y) - G_\lambda^{(\alpha)} f(\cdot)\|_{L^p} \leqq c|y|^{\alpha} \|f\|_{L^p} . \tag{2.9}$$

(ii) *If* $1<\alpha\leqq 2$, *then*

$$\|\partial_j G_\lambda^{(\alpha)} f(\cdot+y) - \partial_j G_\lambda^{(\alpha)} f(\cdot)\|_{L^p} \leqq c|y|^{\alpha-1} \|f\|_{L^p} . \tag{2.10}$$

Proof. Let $0<\alpha<1$ and $f \in \mathcal{S}(\boldsymbol{R}^d)$. From (2.8) we have

$$\begin{aligned} &G_\lambda^{(\alpha)} f(x+y) - G_\lambda^{(\alpha)} f(x) \\ &= c_1(|\cdot+y|^{\alpha-d} - |\cdot|^{\alpha-d}) * |\partial|^{\alpha} G_\lambda^{(\alpha)} f(x) . \end{aligned}$$

Therefore by (2.7) and Young's inequality

$$\|G_\lambda^{(\alpha)} f(\cdot+y) - G_\lambda^{(\alpha)} f(\cdot)\|_{L^p} \leqq c_2 |y|^{\alpha} \| |\partial|^{\alpha} G_\lambda^{(\alpha)} f \|_{L^p} .$$

Note that

$$|\partial|^{\alpha} G_\lambda^{(\alpha)} f = \mathcal{F}^{-1}[|\xi|^{\alpha}(\lambda - \psi^{(\alpha)}(\xi))^{-1} \mathcal{F} f(\xi)]$$

$$= \mathcal{F}^{-1}\left[\frac{|\xi|^\alpha}{\psi^{(\alpha)}(\xi)}\left(\frac{\lambda}{\lambda-\psi^{(\alpha)}(\xi)}-1\right)\mathcal{F}f(\xi)\right]$$

$$= \mathcal{F}^{-1}\left[\frac{|\xi|^\alpha}{\psi^{(\alpha)}(\xi)}\mathcal{F}[\lambda G_\lambda^{(\alpha)}f-f](\xi)\right].$$

Thus, by Theorem 1, we have

$$|||\partial|^\alpha G_\lambda^{(\alpha)}f||_{L_p} \leqq c_3||\lambda G_\lambda^{(\alpha)}f-f||_{L^p}.$$

Since $||\lambda G_\lambda^{(\alpha)}f-f||_{L^p} \leqq 2||f||_{L^p}$ by (2.5), we conclude that

$$||G_\lambda^{(\alpha)}f(\cdot+y)-G_\lambda^{(\alpha)}f(\cdot)||_{L^p} \leqq 2c_2c_3|y|^\alpha||f||_{L^p}.$$

Next, let $\alpha=1$ and $f\in\mathcal{S}(\boldsymbol{R}^d)$. It is easy to show that

$$||G_\lambda^{(1)}f(\cdot+y)-G_\lambda^{(1)}f(\cdot)||_{L^p} \leqq |y|\sum_{j=1}^d ||\partial_j G_\lambda^{(1)}f||_{L^p}.$$

We have, as in case $0<\alpha<1$,

$$\partial_j G_\lambda^{(1)}f = \mathcal{F}^{-1}\left[\frac{i\xi_j}{\psi^{(1)}(\xi)}\mathcal{F}[\lambda G_\lambda^{(1)}f-f](\xi)\right].$$

From Theorem 1 and (2.5), we have

$$||\partial_j G_\lambda^{(1)}f||_{L^p} \leqq c_4||\lambda G_\lambda^{(1)}f-f||_{L^p} \leqq 2c_4||f||_{L^p}.$$

Therefore

$$||G_\lambda^{(1)}f(\cdot+y)-G_\lambda^{(1)}f(\cdot)||_{L^p} \leqq 2c_4 d|y|\,||f||_{L^p}.$$

Hence (i) is proved. The proof of (ii) is similar to (i), so it is omitted.

Lemma 2.3. (i) *If $\alpha>1$, then*

(2.11) $$||\partial_j G_\lambda^{(\alpha)}f||_{L^p} \leqq c\lambda^{-1+1/\alpha}||f||_{L^p}.$$

(ii) *If $\alpha p<d$, then $||G_\lambda^{(\alpha)}f|| \leqq c\lambda^{-1+d/\alpha p}||f||_{L^p}$; and if $(\alpha-1)p>d$, then $||\partial_j G_\lambda^{(\alpha)}f|| \leqq c\lambda^{-1+1/\alpha+d/\alpha p}||f||_{L^p}$.*

Proof. Since, by Lemma 1.1, $\psi^{(\alpha)}(\omega)\in \boldsymbol{C}^{d+1}(S^{d-1})$, we have

$$|\partial^\nu(e^{\psi^{(\alpha)}(\xi)}\xi_j)| \leqq \text{const.}\,(|\xi|^{\alpha-d}+|\xi|^{(d+1)\alpha-d})e^{\mathrm{Re}(\psi^{(\alpha)}(\xi))}$$

as long as $|\nu|\leqq d+1$. Therefore

$$|x^\nu \partial_j p^{(\alpha)}(1, x)| = |\mathcal{F}^{-1}[\partial^\nu(e^{\psi^{(\alpha)}(\xi)}\xi_j)](x)| \leqq \text{const.}$$

This implies that $\partial_j p^{(\alpha)}(1, x)=0(|x|^{-d-1})$ as $|x|\to\infty$, and thus $||\partial_j p^{(\alpha)}(1, \cdot)||_{L^1} <\infty$. From (2.2) we have

$$||\partial_j g_\lambda^{(\alpha)}||_{L^1} \leqq \int_0^\infty e^{-\lambda t}\left(\int|\partial_j p^{(\alpha)}(t, x)|dx\right)dt$$

$$= \int_0^\infty t^{-1/\alpha} e^{-\lambda t} \left(\int |\partial_j p^{(\alpha)}(1, y)| dy \right) dt = c\lambda^{-1+1/\alpha} .$$

Since $\partial_j G_\lambda^{(\alpha)} f = \partial_j g_\lambda^{(\alpha)} * f$, (i) is proved using Young's inequality. Similarly we have

$$\|p^{(\alpha)}(1, \cdot)\|_{L^q} < \infty \text{ and } \|\partial_j p^{(\alpha)}(1, \cdot)\|_{L^q} < \infty ,$$

where $p^{-1} + q^{-1} = 1$. Using (2.2), it is easy to show that

$$\|g_\lambda^{(\alpha)}\|_{L^q} = c_1 \lambda^{-1+d/\alpha p} \qquad \text{if } \alpha p > d ,$$
$$\|\partial_j g_\lambda^{(\alpha)}\|_{L^q} = c_2 \lambda^{-1+1/\alpha+d/\alpha p} \qquad \text{if } (\alpha-1)p > d .$$

Since $G_\lambda^{(\alpha)} f = g_\lambda^{(\alpha)} * f$ and $\partial_j G_\lambda^{(\alpha)} f = \partial_j g_\lambda^{(\alpha)} * f$, (ii) follows from Hölder's inequality.

We shall next introduce the operator

$$(2.12) \quad B^{(\alpha)} f(x) = \int [f(x+y) - f(x) - \Theta(y) \cdot \partial f(x)] N^{(\alpha)}(x, dy) + b^{(\alpha)}(x) \cdot \partial f(x)$$

as a perturbation of $A^{(\alpha)}$. Recall that $\Theta(y) = y$ for $|y| \leqq 1$ and $\Theta(y) = 0$ for $|y| > 1$. We shall be concerned with this operator under the following assumption.

Assumption [$\mathbf{A}_2$]

(1) *$M^{(\alpha)}(dy) + N^{(\alpha)}(x, dy) \geqq 0$ in case $0 < \alpha < 2$, and $N^{(\alpha)}(x, dy) \geqq 0$ in case $\alpha = 2$.*

(2) *There exists a measure $N_*^{(\alpha)}(dy)$ such that*

$$\int |y|^\alpha \wedge 1 \, N_*^{(\alpha)}(dy) < \infty \textit{ and } |N^{(\alpha)}(x, dy)| \leqq N_*^{(\alpha)}(dy) \textit{ for all } x.$$

(3) *$b^{(\alpha)}(x)$ is an arbitrary bounded measurable vector in case $1 < \alpha \leqq 2$, and*

$$b^{(\alpha)}(x) = \int \Theta(y) N^{(\alpha)}(x, dy) \qquad \textit{in case } 0 < \alpha \leqq 1 .$$

REMARK. The constant β:

$$\beta = \inf \{\beta' ; \sup_x \int |y|^{\beta'} \wedge 1 \, |N^{(\alpha)}(x, dy)| < \infty\}$$

is called the upper index of the operator $B^{(\alpha)}$. If assumption [A_2] is satisfied, then the upper index β is equal to or less than α. Tsuchiya [9] considered the case where $0 < \alpha' < \alpha$ and

$$N^{(\alpha)}(x, dy) = k(x, y) |y|^{-d-\alpha'} dy, \quad \sup_{x,y} |k(x, y)| < \infty .$$

In his case, the upper index β of the operator $B^{(\alpha)}$ does not exceed α', so that $\beta < \alpha$. As was shown in [9], if $\beta < \alpha$, Lemma 2.2 can be replaced by some weaker inequalities due to Motoo. Since Motoo's inequalities can be proved without the theory of singular integrals, the whole argument as that in this paper becomes much easier.

Define a non-linear operator

$$(2.13)\qquad \begin{cases} B_*^{(\alpha)}f(x) = \int |f(x+y)-f(x)-\Theta(y)\cdot\partial f(x)|N_*^{(\alpha)}(dy) \\ \qquad\qquad +\| |b^{(\alpha)}| \|\cdot|\partial f(x)| \qquad \text{for } 1<\alpha\leqq 2; \\ B_*^{(\alpha)}f(x) = \int |f(x+y)-f(x)|N_*^{(\alpha)}(dy) \qquad \text{for } 0<\alpha\leqq 1\,. \end{cases}$$

Theorem 2. *Let $1<p<\infty$. There is a function $k_p(\lambda)$ on $(0, \infty)$ such that $k_p(\lambda)\downarrow 0$ as $\lambda\uparrow\infty$ and, for each $f\in\mathcal{S}(\boldsymbol{R}^d)$,*

$$\|B^{(\alpha)}G_\lambda^{(\alpha)}f\|_{L^p}\leqq\|B_*^{(\alpha)}G_\lambda^{(\alpha)}f\|_{L^p}\leqq k_p(\lambda)\|f\|_{L^p}\,.$$

Proof. 1° Let $0<\alpha\leqq 1$. From (2.5) we have

$$\begin{aligned} &\|\int_{\lambda|y|^\alpha>1}|G_\lambda^{(\alpha)}f(\cdot+y)-G_\lambda^{(\alpha)}f(\cdot)|N_*^{(\alpha)}(dy)\|_{L^p} \\ &\leqq (2\int_{\lambda|y|^\alpha>1}N_*^{(\alpha)}(dy))\|G_\lambda^{(\alpha)}f\|_{L^p}\leqq(2\lambda^{-1}\int_{\lambda|y|^\alpha>1}N_*^{(\alpha)}(dy))\|f\|_{L^p}\,. \end{aligned}$$

On the other hand, by (2.9), we have

$$\begin{aligned} &\|\int_{\lambda|y|^\alpha\leqq 1}|G_\lambda^{(\alpha)}f(\cdot+y)-G_\lambda^{(\alpha)}f(\cdot)|N_*^{(\alpha)}(dy)\|_{L^p} \\ &\leqq (\int_{\lambda|y|^\alpha\leqq 1}|y|^\alpha N_*^{(\alpha)}(dy))\sup_y(|y|^{-\alpha}\|G_\lambda^{(\alpha)}f(\cdot+y)-G_\lambda^{(\alpha)}f(\cdot)\|_{L^p}) \\ &\leqq c_1(\int_{\lambda|y|^\alpha\leqq 1}|y|^\alpha N_*^{(\alpha)}(dy))\|f\|_{L^p}\,. \end{aligned}$$

Let $k_p(\lambda)=(2\vee c_1)\int |y|^\alpha\wedge\lambda^{-1}N_*^{(\alpha)}(dy)$. Then $k_p(\lambda)\downarrow 0$ as $\lambda\uparrow\infty$ and $\|B_*^{(\alpha)}G_\lambda^{(\alpha)}f\|_{L^p}\leqq k_p(\lambda)\|f\|_{L^p}$.

2° Let $1<\alpha\leqq 2$. It follows from (2.5) and (2.11) that

$$\begin{aligned} &\|\int_{|y|>1}|G_\lambda^{(\alpha)}f(\cdot+y)-G_\lambda^{(\alpha)}f(\cdot)|N_*^{(\alpha)}(dy)\|_{L^p} \\ &\qquad\qquad +\| |b^{(\alpha)}| \|\cdot\| |\partial G_\lambda^{(\alpha)}f| \|_{L^p} \\ &\leqq 2\lambda^{-1}\int_{|y|>1}N_*^{(\alpha)}(dy)\cdot\|f\|_{L^p}+c_2\| |b^{(\alpha)}| \|\lambda^{-1+1/\alpha}\|f\|_{L^p}\,. \end{aligned}$$

Note that

$$\begin{aligned} &\|\int_{|y|\leqq 1}|G_\lambda^{(\alpha)}f(\cdot+y)-G_\lambda^{(\alpha)}f(\cdot)-y\cdot\partial G_\lambda^{(\alpha)}f(\cdot)|N_\lambda^{(\alpha)}(dy)\|_{L^p} \\ &\leqq\|\int_0^1 d\theta\int_{|y|\leqq 1}|y|\cdot|\partial G_\lambda^{(\alpha)}f(\cdot+\theta y)-\partial G_\lambda^{(\alpha)}f(\cdot)|N_*^{(\alpha)}(dy)\|_{L^p} \\ &\leqq\sup_{0\leqq\theta\leqq 1}\|\int|\partial G_\lambda^{(\alpha)}f(\cdot+\theta y)-\partial G_\lambda^{(\alpha)}f(\cdot)|\cdot|\Theta(y)|N_*^{(\alpha)}(dy)\|_{L^p}\,. \end{aligned}$$

Using (2.10) and (2.11), in a similar way to step 1°, we have

$$\sup_{0\leqq\theta\leqq 1}\|\int|\partial G_\lambda^{(\alpha)}f(\cdot+\theta y)-\partial G_\lambda^{(\alpha)}f(\cdot)|\cdot|\Theta(y)|N_*^{(\alpha)}(dy)\|_{L^p}$$
$$\leqq c_3\int(|y|^{\alpha-1}\wedge\lambda^{-1+1/\alpha})|\Theta(y)|N_*^{(\alpha)}(dy)\cdot\|f\|_{L^p}$$
$$=c_3\int_{|y|\leqq 1}|y|(|y|\wedge\lambda^{-1/\alpha})^{\alpha-1}N_*^{(\alpha)}(dy)\cdot\|f\|_{L^p}.$$

Set

$$k_p(\lambda)=2\lambda^{-1}\int_{|y|>1}N_*^{(\alpha)}(dy)+c_2\||b^{(\alpha)}|\|\lambda^{-1+1/\alpha}$$
$$+c_3\int_{|y|\leqq 1}|y|(|y|\wedge\lambda^{-1/\alpha})^{\alpha-1}N_*^{(\alpha)}(dy).$$

Then $k_p(\lambda)\downarrow 0$ as $\lambda\uparrow\infty$ and $\|B_*^{(\alpha)}G_\lambda^{(\alpha)}f\|_{L^p}\leqq k_p(\lambda)\|f\|_{L^p}$.

Let $\boldsymbol{L}^p=\boldsymbol{L}^p(\boldsymbol{R}^d)=\{f;\ \|f\|_{L^p}<\infty\}$. From (2.5), the operator $G_\lambda^{(\alpha)}$ can be extended to the bounded operator on $\boldsymbol{L}^p$, which is also denoted by $G_\lambda^{(\alpha)}$. It is easy to show that

$$(\lambda-A^{(\alpha)})G_\lambda^{(\alpha)}f=G_\lambda^{(\alpha)}(\lambda-A^{(\alpha)})f=f\qquad\text{for } f\in\mathcal{S}(\boldsymbol{R}^d).\tag{2.14}$$

Note that if $f\in\mathcal{S}(\boldsymbol{R}^d)$, then the function $A^{(\alpha)}f$ is smooth and $A^{(\alpha)}f(x)=0(|x|^{-d})$ as $|x|\to\infty$, so that $(\lambda-A^{(\alpha)})f\in\boldsymbol{L}^p$. Let L be the operator

$$L=A^{(\alpha)}+B^{(\alpha)}.\tag{2.15}$$

From (2.14) we have

$$(\lambda-L)G_\lambda^{(\alpha)}f=(I-B^{(\alpha)}G_\lambda^{(\alpha)})f$$

for $f\in\mathcal{S}(\boldsymbol{R}^d)$. Let $k_p(\lambda)$ be the function of the above theorem, and choose a constant λ_p so that $k_p(\lambda_p)\leqq 1/2$. If $\lambda\geqq\lambda_p$, since $\|B^{(\alpha)}G_\lambda^{(\alpha)}f\|_{L^p}\leqq 1/2\|f\|_{L^p}$ for all $f\in\mathcal{S}(\boldsymbol{R}^d)$, the operator $B^{(\alpha)}G_\lambda^{(\alpha)}$ can be extended to the bounded operator on $\boldsymbol{L}^p$, which is denoted by $[B^{(\alpha)}G_\lambda^{(\alpha)}]$. Then the operator

$$(I-[B^{(\alpha)}G_\lambda^{(\alpha)}])^{-1}:\ \boldsymbol{L}^p\to\boldsymbol{L}^p$$

is well defined, and its operator norm is equal to or less than 2. For $\lambda\geqq\lambda_p$, we shall define the operator

$$R_\lambda=G_\lambda^{(\alpha)}(I-[B^{(\alpha)}G_\lambda^{(\alpha)}])^{-1}:\ \boldsymbol{L}^p\to G_\lambda^{(\alpha)}(\boldsymbol{L}^p).\tag{2.16}$$

From the resolvent equation:

$$G_\lambda^{(\alpha)}-G_\mu^{(\alpha)}=(\mu-\lambda)G_\lambda^{(\alpha)}G_\mu^{(\alpha)}\qquad\text{on } \boldsymbol{L}^p,$$

the space $G_\lambda^{(\alpha)}(\boldsymbol{L}^p)$ is independent of $\lambda>0$. From (2.14) we see that if $f\in\mathcal{S}(\boldsymbol{R}^d)$ and $\lambda\geqq\lambda_p$, then

$$(2.17)\qquad R_\lambda(\lambda-L)f=R_\lambda(\lambda-L)G_\lambda^{(\alpha)}(\lambda-A^{(\alpha)})f \\ =R_\lambda(I-[B^{(\alpha)}G_\lambda^{(\alpha)}])(\lambda-A^{(\alpha)})f=G_\lambda^{(\alpha)}(\lambda-A^{(\alpha)})f=f\,.$$

The operator R_λ gives the λ-potential operator associated with L. Namely, the operator $(L, \mathcal{S}(\boldsymbol{R}^d))$ is closable with respect to the L^p-norm, and its closed extension $(\tilde{L}, \boldsymbol{D}(\tilde{L}))$ satisfies

$$(2.18)\qquad (\lambda-\tilde{L})R_\lambda f=f \quad \text{for each } f\in \boldsymbol{L}^p \text{ and } \lambda\geqq\lambda_p$$

with $\boldsymbol{D}(\tilde{L})=G_\lambda^{(\alpha)}(\boldsymbol{L}^p)$. This fact is not necessary in the proof of the uniqueness theorem of the martingale problem associated with L. However it will be indispensable to the construction of the Markov semi-group with the pre-generator $(L, \mathcal{S}(\boldsymbol{R}^d))$ which does not always satisfy the continuity condition as in [4]. We shall discuss this problem elsewhere, so that the proof of (2.18) is omitted.

3. Uniqueness for the martingale problem

Let W be the space $D(\boldsymbol{R}_+\to\boldsymbol{R}^d)$: the space of right continuous functions having left hand side limits. Given w in W, let $X_t(w)$ denote the position of w at time t. Set $\mathcal{W}=\sigma(X_s; 0\leqq s<\infty)$ and $\mathcal{W}_t=\bigcap_{\varepsilon>0}\sigma(X_s; 0\leqq s\leqq t+\varepsilon)$. Let L be the operator defined by (2.15). We shall say that a probability measure P_x on the space $(W, \mathcal{W})$ solves the martingale problem for the operator L starting from x if, for each $f\in\mathcal{S}(\boldsymbol{R}^d)$,

$$(3.1)\qquad M_t^f=f(X_t)-f(x)-\int_0^t Lf(X_s)\,ds$$

is a P_x-martingale with $M_0^f=0$. We shall prove the following uniqueness theorem.

Theorem 3. *Under assumptions $[A_1]$ and $[A_2]$, there is at most one probability measure P_x which solves the martingale problem associated with the operator $L=A^{(\alpha)}+B^{(\alpha)}$ starting from x.*

In this section, the constant p is chosen so that $p>d/\alpha$ in case $0<\alpha\leqq 1$ and that $p>d/(\alpha-1)$ in case $1<\alpha\leqq 2$. Let $\boldsymbol{B}(\boldsymbol{R}^d)$ be the space of bounded Borel measurable functions and $\boldsymbol{C}^0(\boldsymbol{R}^d)$ the completion of the space $\mathcal{S}(\boldsymbol{R}^d)$ by the sup-norm $\|\cdot\|$. Spaces $\boldsymbol{L}^p(\boldsymbol{R}^d)$, $\boldsymbol{B}(\boldsymbol{R}^d)$ and $\boldsymbol{C}^0(\boldsymbol{R}^d)$ are simply denoted by $\boldsymbol{L}^p$, $\boldsymbol{B}$ and $\boldsymbol{C}^0$. By Lemma 2.3 (ii), we have $G_\lambda^{(\alpha)}(\boldsymbol{L}^p)\subset\boldsymbol{C}^0$. Let $\lambda\geqq\lambda_p$, where λ_p is the constant defined at the end of the previous section. Since $(I-[B^{(\alpha)}G_\lambda^{(\alpha)}])^{-1}$ is a bounded operator on $\boldsymbol{L}^p$ and $R_\lambda=G_\lambda^{(\alpha)}(I-[B^{(\alpha)}G_\lambda^{(\alpha)}])^{-1}$, we have

$$(3.2)\qquad R_\lambda(\boldsymbol{L}^p)\subset\boldsymbol{C}^0 \qquad \text{for } \lambda\geqq\lambda_p\,.$$

Lemma 3.1. *Let P^1 and P^2 be probability measures on $(W, \mathcal{W})$. Let $E^i[\cdot]$ denotes the expectation by P^i. If*

$$P^1[X_0\in dx] = P^2[X_0\in dx], \text{ and}$$
$$E^i\Big[\int_s^\infty e^{-\lambda(t-s)}f(X_t)dt \mid \mathcal{W}_s\Big] = R_\lambda f(X_s)\ P^i\text{—}a.e.\ (i=1,2)$$

for all $s\geqq 0$, $\lambda\geqq\lambda_p$ *and* $f\in \boldsymbol{L}^p\cap \boldsymbol{B}$, *then we have* $P^1=P^2$ *on* $\mathcal{W}$.

Proof. It suffices to show that the proposition

$$[\mathcal{P}_n]:\begin{cases}\text{for each } 0=s_0<s_1<\cdots<s_n \text{ and } f_0, f_1, \cdots, f_n\in \boldsymbol{C}^0\,,\\ E^1[f_0(X_{s_0})\cdots f_n(X_{s_n})] = E^2[f_0(X_{s_0})\cdots f_n(X_{s_n})]\end{cases}$$

holds for each $n\in \boldsymbol{Z}_+$. Obviously $[\mathcal{P}_0]$ holds. Suppose that $[\mathcal{P}_k]$ holds. For each $0=s_0<s_1<\cdots<s_k$; $f_0, f_1, \cdots, f_k\in \boldsymbol{C}^0$ and $f\in \boldsymbol{L}^p\cap \boldsymbol{C}^0$, we have

$$\begin{aligned}&E^i\Big[f_0(X_{s_0})\cdots f_k(X_{s_k})\int_{s_k}^\infty e^{-\lambda(t-s_k)}f(X_t)dt\Big]\\ &= E^i\Big[f_0(X_{s_0})\cdots f_k(X_{s_k})E^i\Big[\int_{s_k}^\infty e^{-\lambda(t-s_k)}f(X_t)dt \mid \mathcal{W}_{s_k}\Big]\Big]\\ &= E^i[f_0(X_{s_0})\cdots f_k(X_{s_k})R_\lambda f(X_{s_k})]\,.\end{aligned}$$

From (3.2), the function $f_k\cdot R_\lambda f$ belongs to $\boldsymbol{C}^0$. Since $[\mathcal{P}_k]$,

$$\begin{aligned}\int_{s_k}^\infty e^{-\lambda(t-s_k)}\{&E^1[f_0(X_{s_0})\cdots f_k(X_{s_k})f(X_t)]\\ &-E^2[f_0(X_{s_0})\cdots f_k(X_{s_k})f(X_t)]\}\,dt = 0\end{aligned}$$

for all $\lambda\geqq\lambda_p$. Since the integrand

$$E^1[f_0(X_{s_0})\cdots f(X_t)]-E^2[f_0(X_{s_0})\cdots f(X_t)]$$

is right continuous in t, it is identically equal to 0. Proposition $[\mathcal{P}_{k+1}]$ follows immediately.

Let P_x be a probability measure on $(W.\ \mathcal{W})$ solving the martingale problem for L starting from x. From the above lemma, in order to prove Theorem 3, it suffices to show that

$$E_x\Big[\int_s^\infty e^{-\lambda(t-s)}f(X_t)dt \mid \mathcal{W}_s\Big] = R_\lambda f(X_s)\quad P_x\text{—a.e.} \tag{3.3}$$

for each $s\geqq 0, f\in \boldsymbol{L}^p\cap \boldsymbol{B}$ and $\lambda\geqq\lambda_p$. $E_x[\cdot]$ denotes the expectation by P_x. Because of (2.17) and (2.18), relation (3.3) would seem to be valid. But the fact that R_λ is the λ-potential operator for L is not used in the proof of Theorem 3. In case $\alpha=2$, Theorem 3 is a special case of Theorem 4.4 in Komatsu [4]. Thus we shall consider the case $0<\alpha<2$.

Let $J_X(dt, dy)$ denote the number of times s such that $s\in dt$ and $\Delta X_s=X_s-X_{s-}\in dy\backslash\{0\}$. By Theorem 2.1 in Komatsu [4],

$$ {}^{c}J_X(dt, dy) = J_X(dt, dy) - (M^{(\alpha)}(dy) + N^{(\alpha)}(X_t, dy))dt $$

is a P_x-martingale measure. Namely, for each non-negative measurable function $h(t, x, y)$ and each stopping time T,

$$ \begin{aligned} &E_x[\int_0^T\int h(t, X_t, y)J_X(dt, dy)] \\ &= E_x[\int_0^T\int h(t, X_t, y)\,(M^{(\alpha)}(dy) + N^{(\alpha)}(X_t, dy))dt]\,. \end{aligned} $$

And the process $\{X_t, P_x\}$ is decomposed as follows:

$$ (3.4)\qquad \begin{cases} X_t = x + \int_0^t\int yJ_X(ds, dy) & \text{in case } 0<\alpha<1\,, \\ X_t = x + \int_0^t(\gamma + \int_{|y|\leq 1} yN^{(1)}(X_s, dy))ds & \\ \qquad + \int_0^t\int_{|y|\leq 1} y\,{}^{c}J_X(ds, dy) + \int_0^t\int_{|y|>1} yJ_X(ds, dy) & \\ & \text{in case } \alpha = 1\,, \\ X_t = x + \int_0^t(b^{(\alpha)}(X_s) - \int_{|y|>1} yM^{(\alpha)}(dy))ds & \\ \qquad + \int_0^t\int_{|y|\leq 1} y\,{}^{c}J_X(ds, dy) + \int_0^t\int_{|y|>1} yJ_X(ds, dy) & \\ & \text{in case } 1<\alpha<2\,. \end{cases} $$

For a moment let $g\in\mathcal{S}(\boldsymbol{R}^d)$. Then the function $G_\lambda^{(\alpha)}g$ is smooth and its derivatives are bounded. Applying the formula of change of variables of semi-martingales (see Kunita and Watanabe [5]) for the process X_t which is decomposed in the form (3.4), we see that the process

$$ \begin{aligned} &e^{-\lambda(t-s)}G_\lambda^{(\alpha)}g(X_t) - G_\lambda^{(\alpha)}g(X_s) \\ &\qquad\qquad + \int_s^t e^{-\lambda(\tau-s)}(\lambda - L)G_\lambda^{(\alpha)}g(X_\tau)d\tau \end{aligned} $$

is a P_x-martingale with mean 0. Therefore

$$ \begin{aligned} G_\lambda^{(\alpha)}g(X_s) &= E_x[\int_s^\infty e^{-\lambda(t-s)}(\lambda - L)G_\lambda^{(\alpha)}g(X_t)dt\,|\,\mathcal{W}_s] \\ &= E_x[\int_s^\infty e^{-\lambda(t-s)}(I - [B^{(\alpha)}G_\lambda^{(\alpha)}])g(X_t)dt\,|\,\mathcal{W}_s]\,. \end{aligned} $$

Let $\lambda\geqq\lambda_p$. Since $G_\lambda^{(\alpha)}g = R_\lambda(I - [B^{(\alpha)}G_\lambda^{(\alpha)}])g$, the equality

$$ E_x[\int_s^\infty e^{-\lambda(t-s)}f(X_t)dt\,|\,\mathcal{W}_s] = R_\lambda f(X_s) \qquad P_x\text{—a.e.} $$

holds good for the function $f = (I - [B^{(\alpha)}G_\lambda^{(\alpha)}])g$. Since the class $(I - [B^{(\alpha)}G_\lambda^{(\alpha)}])$ $(\mathcal{S}(\boldsymbol{R}^d))$ of functions is dense in $\boldsymbol{L}^p\cap\boldsymbol{B}$ with respect to the L^p-norm, relation

(3.3) is a consequence of the inequality: for each $f\in \boldsymbol{L}^p\cap \boldsymbol{B}$,

$$|E_x[\int_s^\infty e^{-\lambda(t-s)}f(X_t)dt\,|\,\mathcal{W}_s]|\leqq c_\lambda\|f\|_{L^p}\quad P_x\text{—a.e.,}\tag{3.5}$$

where (s, x) is fixed and c_λ is a constant independent of f.

Hereafter we shall prove inequality (3.5). Unfortunately the proof is not so easy. Let $\{W, \mathcal{W}, \mathcal{W}_t, Q, X_t\}$ be a stable process such that

$$\int J_X(dt, dy)Q(dw) = M^{(\alpha)}(dy)dt\,.$$

Set $\tilde{W}=W\times W$, $\tilde{\mathcal{W}}=\mathcal{W}\times\mathcal{W}$, $\tilde{\mathcal{W}}_t=\mathcal{W}_t\times\mathcal{W}_t$ and $\tilde{P}_x=P_x\times Q$. Given $\tilde{w}=(w_1, w_2)\in\tilde{W}$, let $Y_t(\tilde{w})=w_2(t)$ and $X_t(\tilde{w})=w_1(t)$. (We dare to use the same symbol X_t as before.) Let $J_X(dt, dy)$ and ${}^cJ_X(dt, dy)$ be the same objects as before, and let

$$\begin{aligned}J_Y(dt, dy) &= \#\{s;\, s\in dt \text{ and } \Delta Y_s= Y_s-Y_{s-}\in dy\backslash\{0\}\}\,,\\ {}^cJ_Y(dt, dy) &= J_Y(dt, dy)-M^{(\alpha)}(dy)dt\,.\end{aligned}$$

For $\delta>0$, define the process Z_t^δ on the space $(\tilde{W}, \tilde{\mathcal{W}}, \tilde{P}_x)$ by

$$\begin{cases} Z_t^\delta = X_t-\int_0^t\int_{|y|\leqq\delta} y(J_X(ds, dy)-J_Y(ds, dy)) \\ \qquad\qquad\qquad\qquad\qquad\qquad\text{in case } 0<\alpha<1\,, \\ Z_t^\delta = X_t-\int_0^t\int_{|y|\leqq\delta} y({}^cJ_X(ds, dy)-{}^cJ_Y(ds, dy) \\ \qquad\qquad +N^{(1)}(X_s, dy)ds) \qquad \text{in case } \alpha=1\,, \\ Z_t^\delta = X_t-\int_0^t\int_{|y|\leqq\delta} y({}^cJ_X(ds, dy)-{}^cJ_Y(ds, dy)) \\ \qquad\qquad\qquad\qquad\qquad\qquad\text{in case } 1<\alpha<2\,. \end{cases}\tag{3.6}$$

It is easy to show that

$$\lim_{\delta\downarrow 0}\tilde{P}_x[\sup_{0\leqq t\leqq T}|Z_t^\delta-X_t|>\varepsilon]=0$$

for each $\varepsilon>0$ and $T<\infty$. Therefore we have

$$\begin{aligned}&\tilde{E}_x[\int_s^\infty e^{-\lambda(t-s)}f(X_t)dt\,|\,\tilde{\mathcal{W}}_s]\\ &=\lim_{\delta\downarrow 0}\tilde{E}_x[\int_s^\infty e^{-\lambda(t-s)}f(Z_t^\delta)dt\,|\,\tilde{\mathcal{W}}_s]\quad\text{in } L^1(\tilde{W}, \tilde{\mathcal{W}}, \tilde{P}_x)\end{aligned}\tag{3.7}$$

for each $f\in\boldsymbol{C}^0$, where $\tilde{E}_x[\cdot]$ denotes the expectation by the product measure $\tilde{P}_x=P_x\times Q$. Fix $(s, x)\in\boldsymbol{R}_+\times\boldsymbol{R}^d$, and define

$$V_\lambda^\delta f(\tilde{w})=\tilde{E}_x[\int_s^\infty e^{-\lambda(t-s)}f(Z_t^\delta)dt\,|\,\tilde{\mathcal{W}}_s]\,.\tag{3.8}$$

Lemma 3.2. *There exists a constant c_λ^δ such that*

$$|V_\lambda^\delta f(\tilde{w})| \leqq c_\lambda^\delta \|f\|_{L^p} \quad \tilde{P}_x\text{—a.e.}$$

for any $f \in \boldsymbol{L}^p \cap \boldsymbol{B}$ and $\lambda > 0$.

Proof. Suppose that $1<\alpha<2$. Let $f \in \mathcal{S}(\boldsymbol{R}^d)$ and $v = G_\lambda^{(\alpha)} f$. Applying the formula of change of variables of semi-martingales for the process Z_t^δ, we have

$$\begin{aligned} & e^{-\lambda(t-s)} v(Z_t^\delta) - v(Z_s^\delta) \\ &= \int_s^t e^{-\lambda(\tau-s)} \{(L-\lambda) v(Z_\tau^\delta) \\ &\quad - \int_{|y| \leqq \delta} (v(Z_\tau^\delta + y) - v(Z_\tau^\delta) - y \cdot \partial v(Z_\tau^\delta)) N^{(\alpha)}(Z_\tau^\delta, dy)\} d\tau \\ &\quad + [\text{a } \tilde{P}_x\text{-martingale with mean } 0]. \end{aligned}$$

Therefore

$$\begin{aligned} (3.9) \qquad -v(Z_s^\delta) &= \tilde{E}_x [\int_s^\infty e^{-\lambda(t-s)} \{-f(Z_t^\delta) + b^{(\alpha)}(Z_t^\delta) \cdot \partial v(Z_t^\delta) \\ &\quad + \int_{|y|>\delta} (v(Z_t^\delta + y) - v(Z_t^\delta) - \Theta(y) \cdot \partial v(Z_t^\delta)) N^{(\alpha)}(Z_t^\delta, dy)\} dt \,|\, \tilde{\mathcal{W}}_s]. \end{aligned}$$

Using Lemma 2.3 (ii),

$$\begin{aligned} |V_\lambda^\delta f(\tilde{w})| &\leqq \|v\| + \int_s^\infty e^{-\lambda(t-s)} \{\|b^{(\alpha)} \cdot \partial v\| \\ &\quad + (2 \int_{|y|>1} N_*^{(\alpha)}(dy)) \|v\| + (2 \int_{\delta < |y| \leqq 1} \|y \cdot \partial v\| N_*^{(\alpha)}(dy))\} dt \\ &\leqq (1 + \frac{2}{\lambda} \int_{|y|>1} N_*^{(\alpha)}(dy)) c_1 \lambda^{-1+d/\alpha p} \|f\|_{L^p} \\ &\quad + \frac{1}{\lambda} (\||b^{(\alpha)}|\| + 2 \int_{\delta < |y| \leqq 1} |y| N_*^{(\alpha)}(dy)) c_2 \lambda^{-1+1/\alpha + d/\alpha p} \|f\|_{L^p}, \end{aligned}$$

where c_1 and c_2 are constants independent of f. Thus there is a constant c_λ^δ such that $|V_\lambda^\delta f(\tilde{w})| \leqq c_\lambda^\delta \|f\|_{L^p}$ for all $f \in \mathcal{S}(\boldsymbol{R}^d)$. Since V_λ^δ is a positive bounded operator on $\boldsymbol{B}$, by making use of the Egorov theorem, it is easy to show that

$$|V_\lambda^\delta f(\tilde{w})| \leqq c_\lambda^\delta \|f\|_{L^p} \qquad \text{for all } f \in \boldsymbol{L}^p \cap \boldsymbol{B}.$$

Next suppose that $0<\alpha \leqq 1$. Let $f \in \mathcal{S}(\boldsymbol{R}^d)$ and $v = G_\lambda^{(\alpha)} f$. Then it is proved in much the same manner as was (3.9) that

$$\begin{aligned} (3.10) \qquad -v(Z_s^\delta) &= \tilde{E}_x [\int_s^\infty e^{-\lambda(t-s)} \{-f(Z_t^\delta) \\ &\quad + \int_{|y|>\delta} (v(Z_t^\delta + y) - v(Z_t^\delta)) N^{(\alpha)}(Z_t^\delta, dy)\} dt \,|\, \tilde{\mathcal{W}}_s]. \end{aligned}$$

And the proof of the lemma in the case $0<\alpha\leqq 1$ is completed in a similar way to the case $1<\alpha<2$.

Lemma 3.3. *Let c_λ be a constant such that*

$$\|G_\lambda^{(\alpha)}f\|\leqq 2^{-1}c_\lambda\|f\|_{L^p} \qquad \text{for all } f\in \boldsymbol{L}^p .$$

Then, for each $f\in \boldsymbol{L}^p\cap \boldsymbol{B}$ and $\lambda\geqq\lambda_p$,

$$|V_\lambda^\delta f(\tilde{w})|\leqq c_\lambda\|f\|_{L^p} \qquad \tilde{P}_x\text{—a.e.} \tag{3.11}$$

Proof. From Lemma 3.2, the constant

$$c_\lambda^\delta=\inf\{c;\ \tilde{P}_x[|V_\lambda^\delta f|>c\|f\|_{L^p}]=0 \quad \text{for all } f\in \boldsymbol{L}^p\cap \boldsymbol{B}\}$$

is finite. Let $f\in\mathcal{S}(\boldsymbol{R}^d)$ and $\lambda\geqq\lambda_p$. By (3.9) and (3.10) we have

$$|V_\lambda^\delta f(\tilde{w})|\leqq\|G_\lambda^{(\alpha)}f\|+|V_\lambda^\delta(B_*^{(\alpha)}G_\lambda^{(\alpha)}f)(\tilde{w})| .$$

Since the function $G_\lambda^{(\alpha)}f$ is smooth and its derivatives are bounded on $\boldsymbol{R}^d$, the function $B_*^{(\alpha)}G_\lambda^{(\alpha)}f$ belongs to $\boldsymbol{L}^p\cap\boldsymbol{B}$. Therefore

$$|V_\lambda^\delta(B_*^{(\alpha)}G_\lambda^{(\alpha)}f)(\tilde{w})|\leqq c_\lambda^\delta\|B_*^{(\alpha)}G_\lambda^{(\alpha)}f\|_{L^p} \qquad \tilde{P}_x\text{—a.e.}$$

From Theorem 2, $\|B_*^{(\alpha)}G_\lambda^{(\alpha)}f\|_{L^p}\leqq(1/2)\|f\|_{L^p}$ for $\lambda\geqq\lambda_p$. Then

$$|V_\lambda^\delta f(\tilde{w})|\leqq\frac{1}{2}(c_\lambda+c_\lambda^\delta)\|f\|_{L^p} \qquad \tilde{P}_x\text{—a.e.}$$

Since V_λ^δ is a positive bounded operator on $\boldsymbol{B}$, the last inequality holds for each $f\in\boldsymbol{L}^p\cap\boldsymbol{B}$. Therefore we have $c_\lambda^\delta\leqq(c_\lambda+c_\lambda^\delta)/2$, which implies that $c_\lambda^\delta\leqq c_\lambda$. This completes the proof.

We now proceed the proof of proposition (3.5). From (3.7) and (3.11), if $f\in\boldsymbol{L}^p\cap\boldsymbol{C}^0$, then

$$|\tilde{E}_x[\int_s^\infty e^{-\lambda(t-s)}f(X_t)dt\,|\,\tilde{\mathcal{W}}_s]|=\lim_{\delta\downarrow 0}|V_\lambda^\delta f(\tilde{w})| \qquad \text{in } L^1(\tilde{W},\tilde{\mathcal{W}},\tilde{P}_x),$$
$$|V_\lambda^\delta f(\tilde{w})|\leqq c_\lambda\|f\|_{L^p} \qquad \tilde{P}_x\text{—a.e.}$$

Therefore

$$|\tilde{E}_x[\int_s^\infty e^{-\lambda(t-s)}f(X_t)dt\,|\,\tilde{\mathcal{W}}_s]|\leqq c_\lambda\|f\|_{L^p} \qquad \tilde{P}_x\text{—a.e.}$$

This implies that

$$|E_x[\int_s^\infty e^{-\lambda(t-s)}f(X_t)dt\,|\,\mathcal{W}_s]|\leqq c_\lambda\|f\|_{L^p} \qquad P_x\text{—a.e.,}$$

because $\tilde{P}_x$ is the direct product of P_x and Q. Since the operator

$$g \rightsquigarrow E_x\left[\int_s^\infty e^{-\lambda(t-s)} g(X_t)dt \mid \mathcal{W}_s\right]$$

is a positive and bounded one on $\boldsymbol{B}$, using the Egorov theorem, we see that the last inequality holds for each $f \in \boldsymbol{L}^p \cap \boldsymbol{B}$. Therefore proposition (3.5) is valid. Hence the proof of Theorem 3 is completed.

References

[1] D.P. Calderón and A. Zygmund: *On the existence of certain singular integrals,* Acta Math. **88** (1952), 85–139.

[2] I.M. Gel'fand and G.E. Shilov: Generalized functions I, 1959, (Russian).

[3] B. Grigelionis: *On Markov property of stochastic processes,* Lietuvos Mathematikos Rinkinys **8** (1968), 489–502, (Russian).

[4] T. Komatsu: *Markov processes associated with certain integro-differential operators,* Osaka J. Math. **10** (1973), 271–303.

[5] H. Kunita and S. Watanabe: *On square integrable martingales,* Nagoya Math. J. **30** (1967), 209–245.

[6] P. Lévy: Théorie de l'addition des variables aléatoires, Gauthier-Villars, Paris, 1937, (2^{e} édition, 1954).

[7] D. Stroock and S.R.S. Varadhan: *Diffusion processes with continuous coefficients* I and II, Comm. Pure Appl. Math. **22** (1969), 345–400 and 479–530.

[8] D. Stroock: *Diffusion processes associated with Lévy generators,* Z. Wahrsch. Verw. Gebiete **32** (1975), 209–244.

[9] M. Tsuchiya: *On some perturbations of stable processes,* Lecture Notes in Math. **330** (1973), Proc. 2nd Japan-USSR Symposium on Prob. Theory, 490–497.