

EXAMPLES OF PROJECTIVE MANIFOLDS NOT ADMITTING KÄHLER METRIC WITH CONSTANT SCALAR CURVATURE

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Introduction. We denote by P^k a complex projective space of dimension k . The direct product $P^m \times P^n$ ($m \leq n$) has a natural imbedding in $P^{m+n+m+n}$, called the Segre imbedding. Let $M_{m,n}$ be a non-singular hyperplane section of $P^m \times P^n$ in $P^{m+n+m+n}$. The first Chern class of $M_{m,n}$ is positive, (the convention here is that the first Chern class of P^k is positive). The purpose of this note is to prove the following

Theorem. *If $m < n$, $M_{m,n}$ does not admit a Kähler metric whose scalar curvature is constant. If $m = n$, $M_{m,n}$ admits a homogeneous Einstein-Kähler metric and hence its scalar curvature is constant.*

This implies immediately that $M_{m,n}$ ($m < n$) can not be Einstein-Kähler. This fact for the cases $m=1, n \geq 2$ and $m=2, n=3$ and 4 have been verified by Y. Sakane [3] and [4]. We show here that some modification of his idea leads to the above theorem.

1. Let $(z_\alpha), (w_\beta)$ and $(\xi_{\alpha\beta})$ be homogeneous coordinates of P^m, P^n and $P^{m+n+m+n}$ respectively. The equations

$$\xi_{\alpha\beta} = z_\alpha w_\beta, \quad \alpha = 0, \dots, m, \quad \beta = 0, \dots, n$$

define the Segre imbedding, which is independent on the choice of these homogeneous coordinates. Let p_1 and p_2 denote the projections of $P^m \times P^n$ into the 1st and 2nd components respectively. Let H_1 and H_2 be the hyperplane bundles of P^m and P^n respectively. Then the line bundle $p_1^* H_1 \otimes p_2^* H_2$ is the restriction of the hyperplane bundle over $P^{m+n+m+n}$ to the submanifold $P^m \times P^n$. We denote by M the intersection of a hyperplane in $P^{m+n+m+n}$ with $P^m \times P^n$. Thus the line bundle $\{M\}$ associated to the divisor M in $P^m \times P^n$ is $p_1^* H_1 \otimes p_2^* H_2$.

Let $K(M)$ and $K(P^m \times P^n)$ be the canonical line bundles for M and for

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$P^m \times P^n$ respectively. Then

$$K(M) = K(P^m \times P^n) \otimes \{M\}|_M,$$

and hence

$$K(M) = p_1^* H_1^{-m} \otimes p_2^* H_2^{-n}|_M.$$

This means that the 1st Chern class of M is positive definite.

2. We shall show that any holomorphic vector field on M extends to a holomorphic vector field on $P^m \times P^n$. Indeed, the following lemma asserts a stronger fact.

Put $X = P^m \times P^n$. Let $T(X)$ and $T(M)$ be the tangent bundles of X and M respectively. Given a holomorphic vector bundle E , we denote by $\Omega^p(E)$ the sheaf of germs of holomorphic E -valued p -forms.

Lemma (Sakane [3], p 356). *The exact sequence*

$$0 \rightarrow \Omega^0(T(X) \otimes \{M\}^{-1}) \rightarrow \Omega^0(T(X)) \rightarrow \Omega^0(T(X)|_M) \rightarrow 0$$

induces the exact sequence

$$0 \rightarrow H^0(X, T(X)) \rightarrow H^0(M, T(X)|_M) \rightarrow 0.$$

Here, we present a simpler proof. The above given sequence implies the following exact sequence:

$$\begin{aligned} 0 &\rightarrow H^0(X, \Omega^0(T(X) \otimes \{M\}^{-1})) \rightarrow H^0(X, \Omega^0(T(X))) \\ &\rightarrow H^0(M, \Omega^0(T(X)|_M)) \rightarrow H^1(X, \Omega^0(T(X) \otimes \{M\}^{-1})) \rightarrow \dots \end{aligned}$$

In order to prove the lemma, it suffices to show that

$$H^q(X, \Omega^0(T(X) \otimes \{M\}^{-1})) = 0, \quad \text{for } q = 0, 1.$$

The cohomology group in question is isomorphic to

$$H^{m+n-q}(X, \Omega^1(K(X) \otimes \{M\}))$$

by the Serre duality. Since

$$K(X) \otimes \{M\} = p_1^* H_1^{-m} \otimes p_2^* H_2^{-n},$$

we can apply the Künneth formula,

$$\begin{aligned} H^{m+n-q}(X, \Omega^1(K(X) \otimes \{M\})) &= \\ &\sum_{p'+p''=m+n-q} \{H^{p'}(P^m, \Omega^0(H_1^{-m})) \otimes H^{p''}(P^n, \Omega^1(H_2^{-n})) + \\ &\quad H^{p'}(P^m, \Omega^1(H_1^{-m})) \otimes H^{p''}(P^n, \Omega^0(H_2^{-n}))\}. \end{aligned}$$

As a result of the Kodaira-Nakano vanishing theorem and the Serre duality,

$$H^p(P^m, \Omega^0(H_1^{-m})) = 0 \quad \text{for all } p.$$

Thus, we see that

$$H^{m+n-q}(X, \Omega^1(K(X) \otimes \{M\})) = 0 \quad \text{for all } q,$$

completing the proof.

3. The group of all holomorphic automorphisms of $P^m \times P^n$ is $PSL(m+1, C) \times PSL(n+1, C)$. By the previous lemma the group G of all holomorphic automorphisms of $P^m \times P^n$ which map M onto itself contains the identity connected component of the group of all holomorphic automorphisms of M .

Let \tilde{G} be the inverse image of G under the projection $SL(m+1, C) \times SL(n+1, C) \rightarrow PSL(m+1, C) \times PSL(n+1, C)$. The isomorphism $C^{m+1} \otimes C^{n+1} = C^{(m+1)(n+1)}$ defines a canonical imbedding of $SL(m+1, C) \times SL(n+1, C)$ into $SL((m+1)(n+1), C)$. Thus, an element in $SL(m+1, C) \times SL(n+1, C)$ is a linear transformation of $C^{(m+1)(n+1)}$ and maps a hyperplane onto another hyperplane, obviously.

Let H be the hyperplane in P^{m+n+1} such that $M = (P^m \times P^n) \cap H$. Then, \tilde{G} consists of elements $(\sigma, \tau) \in SL(m+1, C) \times SL(n+1, C)$ which leave the hyperplane H invariant. The hyperplane H is the zero of a non-zero linear form A on $C^{m+1} \otimes C^{n+1}$. If we put

$$a(z, w) = A(z \otimes w), \quad z \in C^{m+1}, w \in C^{n+1},$$

a is a bilinear form on $C^{m+1} \times C^{n+1}$. Thus, \tilde{G} is the set of all $(\sigma, \tau) \in SL(m+1, C) \times SL(n+1, C)$ with the property

$$a(\sigma(z), \tau(w)) = \lambda_{\sigma, \tau} a(z, w), \quad z \in C^{m+1}, w \in C^{n+1},$$

for some non-zero constant $\lambda_{\sigma, \tau}$.

Now, choose homogeneous coordinates on C^{m+1} and C^{n+1} so that

$$a(z, w) = z_0 w_0 + \dots + z_k w_k.$$

where $0 \leq k \leq m(\leq n)$. Suppose $k < m$. Take a point (z, w) with $z_0 = \dots = z_{k-1} = w_0 = \dots = w_{k-1} = 0$. Then the point is in M , but is a singular point, as

$$dA|_{C^{m+1} \times C^{n+1}} = \sum_{\alpha=0}^{k-1} w_\alpha dz_\alpha + z_\alpha dw_\alpha$$

vanishes at this point. Therefore k must be m , and

$$a(z, w) = z_0 w_0 + \dots + z_m w_m.$$

Write an $(n+1) \times (n+1)$ matrix τ in the form

$$(*) \quad \begin{array}{|c|c|} \hline \tau_1 & \beta \\ \hline \gamma & \delta \\ \hline \end{array}$$

where τ_1 is an $(m+1) \times (m+1)$ matrix. For $(\sigma, \tau) \in SL(m+1, \mathbf{C}) \times SL(n+1, \mathbf{C})$,

$${}^t\sigma \begin{array}{|c|c|} \hline I_{m+1} & 0 \\ \hline \end{array} \tau = \begin{array}{|c|c|} \hline \lambda I_{m+1} & 0 \\ \hline \end{array}$$

if and only if $\tau_1 = \lambda {}^t\sigma^{-1}$ and, $\beta = 0$.

Let L be the subgroup of $SL(n+1, \mathbf{C})$ consisting of all matrices τ with $\beta = 0$ in the above expression (*). The map $(\sigma, \tau) \rightarrow \tau$ defines a covering homomorphism of \tilde{G} onto L . The subset N in L consisting of all matrices of the form

$$\begin{array}{|c|c|} \hline I_{m+1} & 0 \\ \hline * & I_{n-m} \\ \hline \end{array}$$

is a normal subgroup in L . The center Z of L is the set of all matrices of the form

$$\begin{array}{|c|c|} \hline \alpha I_{m+1} & 0 \\ \hline 0 & \beta I_{n-m} \\ \hline \end{array}$$

with $\alpha^{m+1}\beta^{n+1}=1$. The radical of L is $N \cdot Z$ and $N \cap Z = \{\text{identity}\}$. Thus, we conclude that if $m < n$, the Lie algebra of all holomorphic vector fields is not reductive, and if $m = n$, the identity connected component of G is isomorphic to $PSL(m+1, \mathbf{C})$, which acts transitively on M ;

$$M = SU(m+1)/S(U(1) \cdot U(1) \cdot U(m+1)).$$

The complex Lie algebra of all holomorphic vector fields on a compact complex manifold is known to be reductive if M admits a Kähler metric with constant scalar curvature (A. Lichnerowicz [2], see [1]). Applying this result to our case, we obtain our non-existence theorem. When $m = n$, M is a compact homogeneous space of kählerian type and admits an Einstein-Kähler metric (see M. Takeuchi [5], Corollary in p. 195).

Bibliography

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