Hill, D. A.
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QUASI-PROJECTIVE MODULES OVER HEREDITARY NOETHERIAN PRIME RINGS

DAVID A. HILL

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Introduction

The structure theory of quasi-projective modules over hereditary Noetherian prime (HNP) rings has been developed by a number of authors, [10], [13], [16], [17], [18]. A left module M over a ring R is said to be *projective modulo its annihilator* when M considered as a $R/l_R(M)$ module is projective where the left annihilator is defined to be $l_R(M)=\{x\in R\,|\,x\cdot M=0\}$. In a similar fashion one may make the same definition for right R-modules. The quasi-projective modules, projective modulo their annihilator generally have a more manageable structure than those quasi-projectives which fail this condition. For example, quasi-projectivity is preserved under direct sums of copies of such modules. In HNP-rings these modules have a comparatively simple structure. They consist of the projective modules, and those quasi-projectives that are direct sums of cyclic uniserial modules [3], [17]. Therefore, it is of interest to determine those rings whose quasi-projectives are projective modulo their annihilator.

It is known that for left perfect rings every left quasi-projective module is projective modulo its annihilator [6]. Rangaswamy and Vanaja [13] showed that a commutative Dedekind domain is a complete local Dedekind domain if and only if its ring of quotients is quasi-projective. This construction gives a class of Dedekind domains which have quasi-projectives not projective modulo their annihilator. Implicit in their results is the following proposition: A commutative Dedekind domain has every quasi-projective module projective modulo its annihilator if and only if it is semi-local and not a complete local Dedekind domain.

In this paper we set out to determine the structure of the HNP-rings whose quasi-projective modules are projective modulo their annihilator. In so doing, we actually determine the structure of the quasi-projective modules over semi-local HNP-rings. The following theorem is proved: An HNP-ring has all quasi-projectives projective modulo their annihilator if and only if it is semi-local and is not Morita equivalent to a complete, local, Dedekind domain. As a consequence of this result, it is shown that hereditary, Noetherian rings with all quasi-projectives projective modulo their annihilator are ring direct sums of the above rings and Artinian, hereditary rings.

All rings considered are associative with an identity, and all modules are unital. All conditions will be assumed to hold on both sides unless otherwise stated. For example, an HNP-ring is a prime ring which is both left and right hereditary and left and right Noetherian. However, our rings are not necessarily commutative so that in our terminology Dedekind domains and principal ideal domains may be non-commutative.

As an HNP-ring satisfies Goldie's conditions on the right as well as the left, it has a classical quotient ring which is simple Artinian. For such a ring any one sided ideal is essential if and only if it contains a regular element [7]. We will use $Q(R)$ to denote the quotient ring, or simply Q when the ring R is understood. The following notation will be used: For a ring R, $J(R)$ or simply J when R is understood, will denote the Jacobson radical of R. Given a module M, $M^{(A)}$ will be used to denote a direct sum of A copies of M. For the basic definitions and properties of torsion and torsion-free modules, regular elements, and rings of quotients, we refer to [16] and [19]. For the basic properties of uniserial modules and serial rings, we refer to [3].

This paper is inspired to a large degree by [10], [13], [17], including the proofs of some of the results.

1. Quasi-projective modules over bounded HNP-rings

An HNP-ring is said to be *bounded* in case every essential one sided ideal contains a non-zero two sided ideal. By the results of Lenagan (9), the bounded HNP-rings are exactly the simple, Artinian rings and those HNP-rings not primitive. The bounded HNP-rings satisfy the following property which will be used in some of the proofs of our results: Any essential one sided ideal contains a product of non-zero prime ideals ([9], Theorem 3.3).

The main object of this section is to show that a quasi-projective module over a bounded HNP-ring is either torsion or torsion-free. To do this we will need the concept of purity. A sumbodule K of a module M over a ring R will be said to be *pure* if $r\cdot K=r\cdot M\cap K$ for all $r\in R$. The following properties hold for pure modules:

(1) If the quotient module by a pure submodule is a direct sum of cyclically presented modules then the pure submodule is a direct summand.

(2) Any direct summand of a module is pure.

(3) A pure submodule of a pure submodule is pure.

The following result is due to Eisenbud and Robson ([4], Section 3):

Proposition 1. *Let R be a boundecd HNP-ring, and M a finitely generated* *module. Then M is a direct sum of torsion cyclic modules and a projective module.*

Using a variation of the argument given in ([10], Theorem 3.12) or ([8], Theorem 5) yields the following proposition.

Proposition 2. *Let R be a bounded HNP-ring, and M an R-module and S a pure submodule of bounded order (That is, there exists $0 \neq r \in R$ such that $r \cdot S = 0$). Then S is a direct summand of M.*

Lemma 1. *Let R be a bounded HNP-ring, and M an R-module. If M is not torsion-free, then any cyclic direct summand of the torsion submodule M_T of M is a direct summand of M.*

Proof. Let C be a cyclic direct summand of M_T. We will first show that for any submodule N between C and M such that N/C is a direct sum of torsion cyclics and a projective module implies that C is a direct summand of N. So $N/C = N' \oplus P$, P projective and N' torsion. Let π be the canonical homomorphism from N to P. Then $N = P_1 \oplus L$, $P_1 \cong P$, $L = \ker(\pi)$, $C \subseteq L$. An easy exercise shows that L is torsion. As C is a direct summand of M_T, $C \subseteq L$ implies that C is a direct summand of L. Therefore, C is a direct summand of N.

To show that C is pure, it is sufficient to show that for any system $\sum_{j=1}^{t} r_{ij} x_j = c_i$, $(i \in I)$, $(c_i \in C)$, $(r_{ij} \in R)$ which is solvable in M also has a solution in C ([2], Theorem 2.4). Suppose m_j $(1 \leq j \leq t)$ is a solution in M. Consider the submodule N generated by C and the m_j. Then N/C is finitely generated, so by Proposition 1, N/C is a direct sum of a projective module and a torsion module. Therefore $N = C \oplus K$. So $m_j = c'_j + k_j$ $(c'_j \in C)$, $(k_j \in K)$, $(1 \leq j \leq t)$. Clearly $\{c'_j\}$ is a solution in C. Thus C is pure in M. Applying Proposition 2, C is a direct summand of M.

A module M is said to be *projective relative* to a module P if for every factor module N of P the natural map $\mathrm{Hom}_R(M, P) \to \mathrm{Hom}_R(M, N)$ is epic. The class of modules to which a given module M is projective is closed under submodules, quotient mdoules, and finite direct sums [14]. An easy consequence of this fact is the following condition: A module $M_1 \oplus M_2$ is quasi-projective if and only if M_i is projective relative to M_j $(i, j = 1, 2)$.

In the proof of the next theorem and some of the subsequent results, we make use of the following observation: Applying ([17], Corollary 4) a module over a bounded HNP-ring is injective if and only if it is divisible. Thus, for a bounded HNP-ring a module is reduced (that is, it has no divisible submodules) if and only if it has no injective submodules.

Theorem 1. *Let R be a bounded HNP-ring and M a quasi-projective module. Then M is torsion or torsion-free.*

Proof. Suppose M is not torsion. Then by ([17], Lemma 14), M is projective relative to R. If M is not torsion-free, the torsion submodule M_T is non-zero. We first show that M_T is not injective. Suppose M_T is injective. Then M_T would be a direct summand of M and hence quasi-projective. But this would contradict ([18], Theorem 2.12). Thus we may apply ([17], Theorem 10) to obtain a torsion cyclic direct summand C of M_T. By Lemma 1, C is also a direct summand of M. Therefore, C is also projective relative to R, a contradiction. This completes the proof.

2. Torsion-free quasi-projective modules

The HNP-rings which are maximal orders in their rings of quotients are called *Dedekind prime rings*. Suppose R is a bounded Dedekind prime ring and P a non-zero prime ideal in R. Let R_P be the localization of R with respect to P in the sense of Marubayashi ([10], pages 96–97), and set $J(R_P)=P_1$. Observing that $R/P^n \cong R_P/P_1^n$ ([10], page 97) and letting $R(P^\infty)=\varinjlim R/P^n$ where $\varinjlim R/P^n$ is the inductive limit of R/P^n, the following lemma gives a decomposition of the torsion divisible module Q/R due originally to Marubayashi ([11], Lemma 5.1).

Lemma 2. *Let R be a bounded Dedekind prime ring. Then $Q/R \cong \oplus \sum R(P^\infty)$ where P varies over all non-zero prime ideals of R.*

For the following propositions we will need some definitions: Two rings R and S which are orders in a given quotient ring Q are said to be *left equivalent* if there exist units $a, b \in Q$ such that $Ra \subseteq S$, $Sb \subseteq R$. A module is said to be *uniform* if every non-zero submodule is essential. The injective hull of such a module is clearly indecomposable. Let R be a Noetherian ring and M a module over R. Since the injective hull of M is a direct sum of uniform modules, M has an essential submodule N, which is a direct sum of uniform submodules, say $N=\oplus \sum N_i$ $(i \in I)$. The cardinality $|I|$ is called the *rank* of M. Since $E(M)=E(N)=\oplus \sum E(N_i)$ $(i \in I)$ by the Krull-Schmidt-Azumaya Theorem, $|I|$ is uniquely determined by $E(M)$, and hence by M.

Proposition 3. *Let R be an HNP-ring which is left equivalent to a bounded Dedekind prime ring S. Let σ denote the number of distinct prime ideals of S, and c_0 the first infinite cardinal. Then any torsion-free quasi-projective module over R of rank $m \geq \sigma \cdot c_0$ is projective.*

Proof. Let M be a torsion-free quasi-projective module with $m=$ rank $(M) \geq \sigma \cdot c_0$, and E the injective hull of M. Thus M contains an essential submodule $\oplus \sum Rx_\alpha$ $(\alpha \in A)$ with each Rx_α uniform and $|A|=m$. By hypothesis R is left equivalent to a bounded Dedekind prime ring S. Since the

$S(P^{\infty})$ are countably generated over S, Lemma 2 implies that $Q(S)$ has $\sigma \cdot c_0$ generators. An easy application of the definition of left equivalence shows that $Q(R)$ has at most $\sigma \cdot c_0$ generators. Therefore, the number of generators of M is also m.

Suppose the rank of ${}_R R$ is n. Consider, $\oplus \sum_{\alpha \in A} (Rx_\alpha)^{(n)} \subseteq M^{(n)}$. As each Rx_α is torsion-free, $(Rx_\alpha)^{(n)}$ contains a copy of R ([17], Lemma 14). Thus $M^{(n)}$ contains a copy of $(R)^{(A)}$. Since $M^{(n)}$ is quasi-projective, $M^{(n)}$ and hence M, is projective relative to $R^{(A)}$. As M is an epimorphic image of $R^{(A)}$, M is a direct summand of $R^{(A)}$, and so is projective.

Proposition 4. *Let R be a semi-local HNP-ring. Then any torsion-free quasi-projective module of infinite rank is projective.*

Proof. If $J=0$, R is semi-simple and there is nothing to prove. When $J \neq 0$, by ([15], Theorem 6.4), R is left equivalent to a Dedekind prime ring S with $J(S) \neq 0$. Thus S is bounded ([9], Proposition 3.6). Likewise, using ([15], Theorem 6.4) again, it is clear that S is also semi-local and has only a finite number of maximal ideals. Thus the proposition follows from Proposition 3.

A Dedekind domain R which is complete with respect to the J-adic topology is said to be a *complete* Dedekind domain. The following result due to Singh will be needed ([16], Theorem 8):

Proposition 5 (Singh). *Let R be a bounded HNP-ring. Then Q is quasi-projective if and only if R is Morita equivalent to a complete local Dedekind domain.*

We now turn our attention to those torsion-free quasi-projectives of finite rank. The structure of these modules is virtually known [17]. Using the results of [16], [17] we obtain,

Proposition 6 (Singh). *Let R be a bounded HNP-ring, and M a torsion-free quasi-projective module of finite rank. Then,*
(1) *If R is not Morita equivalent to a complete local Dedekind domain, then M is finitely generated projective.*
(2) *If R is Morita equivalent to a complete local Dedekind domain, then $M=P \oplus E$, P finitely generated projective and E a direct summand of a finite number of copies of Q.*

Proof. Case 1 follows from ([17], Corollary 4), ([17[, Theorem 18), ([16], Lemma 1) and Proposition 5. For (2) observe that any injective submodule of M is a direct summand of M, and therefore is a diıect summand of a finite number of copies of Q ([16], Theorem 5). This implies that $M=P \oplus E$, E a direct summand of a finite number of coipes of Q, and P a submodule of M containing no injective submodules. Another application of ([17], Theorem 18) shows that

P is finitely generated projective.

We now investigate some of the properties of torsion-free quasi-projective modules over bounded HNP-rings. The following propositions generalize 5.2 and 5.3 of Rangaswamy and Vanaja [13]. But first some definitions and lemmas.

An element $x \in M$ is said to be *uniform* in case Rx is a uniform submodule of M. Let M be a torsion module and $x \in M$ a uniform element. The *height* of x is defined to be the supremum of the lengths of the composition series of all uniform submodules of M containing x. A module is said to be *uniserial* in case all of its submodules are linearly ordered by inclusion. The socle of a module M, denoted by Socle (M), is the largest semi-simple submodule of M.

Lemma 3. *Let M be a torsion module and R a bounded HNP-ring. If a uniform element $0 \neq x \in Socle(M)$ has infinite height then there exists an injective submodule of M containing x.*

Proof. Consider the set $\{U_\alpha\}_{\alpha \in A}$ of uniform submodules of M containing x. It suffices to show that at least one of them has infinite length since it will be injective ([17], Lemma 2). Let $E(M)$ be the injective hull of M and $E(U_\alpha) \subseteq E(M)$ the injective hull of each U_α $(\alpha \in A)$. It is clear that $E(M)$ is also torsion. Since R is hereditary, $E(U_\alpha)+E(U_\beta)$ is injective for each pair $\alpha, \beta \in A$, so applying ([17], Theorem 2), $E(U_\alpha)+E(U_\beta)$ is a direct sum of uniserial modules each with a non-zero essential socle. But the socle of $E(U_\alpha)+E(U_\beta)$ is just Rx. Therefore, $E(Rx)=E(U_\alpha)=E(U_\beta)$ for all $\alpha, \beta \in A$. This fact and the fact that x has infinite height imply that there exists a proper ascending sequence of uniserial submodules of M containing x. The union of these uniserial submodules is a uniform submodule of M containing x of infinite length which is injective.

Lemma 4. *Let R be a bounded HNP-ring, M a torsion-free R-module, and E the injective hull of M. Suppose that N is a direct summand of E. Consider the module $K=M \cap N$ of E. Then for all regular elements $r \in R$, $r \cdot M \cap K = r \cdot K$.*

Proof. Suppose $rx \in K$, r regular $(x \in M)$. Then $x=y+z$, $y \in N$, z in the complement of N. Thus $rx=ry+rz$ so that $rx=ry$. This implies that $x \in N$.

The following properties hold for the module K as defined above, and for modules $L \subseteq K$, $K \subseteq S \subseteq M$ and r a regular element of R:

(1) $r \cdot (K/L) = r \cdot (M/L) \cap K/L$.

(2) $r \cdot K = r \cdot S \cap K$.

A module M is said to be *torsionless* in case there is a monomorphism of M into a direct product of A copies of R for some indexing set A. This is clearly equivalent to the following condition: For each $0 \neq x \in M$, there exists a homomorphism $f\colon M \to R$ with $f(x) \neq 0$.

Proposition 7. *Let R be a bounded HNP-ring which is not Morita equivalent to a complete local Dedekind domain. Then any torsion-free quasi-projective module is torsionless.*

Proof. Let M be a torsion-free quasi-projective module, and $0 \neq x \in M$. In order to show that M is torsionless, we will construct an $h \in \mathrm{Hom}_R(M, R)$ with $h(x) \neq 0$.

Let E be the injective hull of M. Then there exists a direct summand L of E such that Rx is essential in L. Consider $K = M \cap L$. Now M is reduced (That is M has no divisible submodules), otherwise M would have an injective, quasi-projective direct summand which would imply that the quotient ring Q is quasi-projective, clearly in violation of our hypothesis and Proposition 5. Therefore, there exists a regular element $r \in R$ such that $r \cdot K = r \cdot RK \neq K$. Since R is bounded, $r \cdot R$ contains a product $P_1 \cdots P_n$ of non-zero prime ideals. Thus for some i, there exists a P_i, say P, such that $P \cdot K \neq K$.

Let T be the torsion submodule of M/PK. Thus $0 \neq K/PK \subseteq T$, since P necessarily contains a regular element. We will show that there exists a uniform element in the socle of K/PK with finite height in T. Suppose no such element exists. Then by Lemma 3, each uniform element in the socle of K/PK is contained in a torsion injective submodule of T. Thus there exists an injective module N/PK such that $K/PK \subseteq N/PK \subseteq T \subseteq M/PK$. By Lemma 4 and the subsequent remarks $d(K/PK) = d(N/PK) \cap K/PK$ for each regular element $d \in R$. But N/PK is injective, hence is divisible, so that $d(N/PK) = N/PK$. Thus $d(K/PK) = N/PK \cap K/PK = K/PK$ for each regular element $d \in R$. This is a contradiction, since P contains a regular element.

Since there exist uniform elements in Socle (K/PK) of finite height, we may apply ([17], Theorem 10) and Lemma 1 to find a non-zero cyclic direct summand C of M/PK such that $\mathrm{Socle}(K/PK) \cap C \neq 0$. The module C may be expressed as $Ry/(Ry \cap PK)$, for some $y \in M$.

As in the proof of ([13], Lemma 5.2), let $\pi_c\colon M/PK \to Ry/(Ry \cap PK)$ be the canonical projection of M/PK to $Ry/(Ry \cap PK)$, and $\pi_1\colon M \to M/PK$, $\pi_2\colon M \to M/(Ry \cap PK)$ be the natural maps. Consider the following diagram:

$$\begin{tikzcd} & M \arrow[dl, dashed] \arrow[d, "\pi_c \circ \pi_1"] & \\ M \arrow[r, "\pi_2"'] & M/(Ry \cap PK) \arrow[r] & 0 \end{tikzcd}$$

By the quasi-projectivity of M, there exists an $h\colon M \to M$ making the diagram commute. As $\pi_2 h(M) \subseteq Ry/(Ry \cap PK)$ so $h(M) \subseteq Ry$. Since Ry is projective, it suffices to show that $h(x) \neq 0$. Suppose $Rx \subseteq \ker(h) \cap K$. Let $0 \neq a \in K$. As Rx

is essential in K, so is $\ker(h)\cap K$. Thus $Ra\cap(\ker(h)\cap K)$ is essential in Ra. Since R is prime Goldie and Ra projective, there exists a regular element $b\in R$ such that $ba\in\ker(h)\cap K$. Thus $0=h(ba)=b\cdot h(a)$. But b regular implies that $h(a)=0$. Thus $K\subseteq\ker(h)$, a contradiction. This completes the proof.

Proposition 8. *Let R be a bounded HNP-ring which is not Morita equivalent to a complete local Dedekind domain. Then any torsion-free quasi-projective module of at most countable rank is projective.*

Proof. Applying ([17], Theorem 18), it suffices to consider the countable infinite case. Let M be a torsion-free, quasi-projective module of countable infinite rank. Applying Proposition 7 and using an argument similar to the one given in ([13], Corollary 5.3), one can show that any submodule of finite rank of M is projective. To show that M is projective, we apply a modification of the argument given in ([12], Lemma 8.3). The injective hull of M has the form $E(M)=\oplus\sum_{i=1}^{\infty}E_i$, with each E_i indecomposable.

Consider the ascending sequence of finite rank submodules $K_n=(\oplus\sum^n E_i)\cap M$. We show that the sequence K_n has the following properties:

(1) $K_1\subseteq K_2\subseteq\cdots$.

(2) Each K_j is a projective module.

(3) K_j is a direct summand of K_{j+1}, $(j\geq 1)$.

(4) $M=\cup K_j$.

These four conditions imply that M is projective.

It is clear that conditions (1) and (2) are satisfied. The module K_{j+1}/K_j is finitely generated. Using Lemma 4, a simple argument shows that K_{j+1}/K_j is torsion-free. By Proposition 1, K_{j+1}/K_j is projective. Thus K_j is a direct summand of K_{j+1}.

Let $a\in M$. Then $a\in\oplus\sum^n E_i$ for some n. Thus $a\in K_n$. Therefore, $M=\cup K_n$.

3. Modules projective modulo their annihilator

The following theorem serves to characterize those HNP-rings whose quasi-projective modules are all projective modulo their annihilator. However, before giving the proof we make the following observation: A ring is said to be *serial* in case it is a direct sum of uniserial modules both as a left and right module over itself. By a well known theorem of Nakayama ([3], Theorem 1.2), any module over an Artinian, serial ring is a direct sum of cyclic uniserial modules. Furthermore, any proper factor ring of an HNP-ring is an Artinian, serial ring ([3], Corollary 3.2). These results along with ([17], Lemma 9) and ([18], Theorem 2.13) show that any torsion quasi-projective module over a bounded HNP-ring is a direct sum of cyclic, uniserial modules.

Theorem 2. *Let R be an HNP-ring. Then R has every quasi-projective module projective modulo its annihilator if and only if*

(1) *R is semi-local.*

(2) *R is not Morita equivalent to a complete local Dedekind domain.*

Proof. To prove the necessity of (1), consider $\{S_\alpha\}_{\alpha\in A}$ a complete set of pairwise non-isomorphic simple modules over R. Then $T=\oplus\sum S_\alpha$ is projective over R/J. The necessity of (1) now follows from ([5], Corollary 2.2). The necessity of (2) is a consequence of Proposition 5 and Q not being projective over R.

Suppose R satisfies (1) and (2). By Theorem 1 any quasi-projective module is torsion or torsion-free. The torsion-free quasi-projectives are all projective so we need only examine the torsion case.

Suppose M is a quasi-projective torsion module. Then there exists a set $\{C_\alpha\}_{\alpha\in A}$ of cyclic uniserial modules such that $M=\oplus\sum C_\alpha$. As each C_α is uniserial, Artinian, C_α/JC_α is simple. Suppose C_γ and C_β appear in the decomposition of M with $C_\gamma/JC_\gamma\cong C_\beta/JC_\beta$. Consider the following diagram with π_γ and π_β the natural epimorphisms:

$$\begin{tikzcd} & C_\beta \arrow[d] & \\ C_\gamma \arrow[r] & C_\gamma/JC_\gamma \arrow[r] & 0 \end{tikzcd}$$

Since C_β is projective relative to C_γ, there exists a $\varphi\colon C_\beta\to C_\gamma$ which is necessarily epic. Therefore, C_γ is an epimorphic image of C_β, and since C_γ is projective relative to C_β, C_γ is isomorphic to a direct summand of C_β. But C_β is indecomposable. Therefore $C_\beta\cong C_\gamma$. As R is semi-local, R has only a finite number of non-isomorphic simple modules. Combining this fact with the previous observation, we see that $M=\oplus\sum_{i=1}^{k} C_i^{(A_i)}$ with k a positive integer representing the number of non-isomorphic simples occuring in the decomposition of M/JM. As $K=C_1\oplus\cdots\oplus C_k$ is finitely generated, applying ([16], Lemma 2) $l_R(K)\neq 0$. But $l_R(K)=l_R(M)$ so $l_R(M)\neq 0$. This means that $R/l_R(M)$ is Artinian ([3], Corollary 3.2). By ([6], Theorem 2.3), M is projective over $R/l_R(M)$.

Combining the proof of Theorem 2 with Proposition 4 and Theorem 1, we have actually determined the quasi-projective modules over semi-local HNP-rings.

Proposition 9. *Let R be a semi-local HNP-ring. If a module M is quasi-projective, it has a decomposition of one of the following types:*

(1) *M is projective.*

(2) $M\cong\oplus\sum^{n} C_i^{(A_i)}$ *where each C_i is cyclic, Artinian, uniserial and* $M/JM\cong\oplus\sum^{k}(C_i/JC_i)^{(A_i)}$, *$k$ a positive integer.*

(3) $M \cong P \oplus E$ *P a finitely generated projective module, E a direct summand of a finite number of copies of Q. In which case R is Morita equivalent to a complete local Dedekind domian.*

We make the following observation which is true in any setting: Suppose R is a ring direct sum of rings R_i ($i=1, \cdots, n$) and M a quasi-projective module over R. Then $M=M_1 \oplus \cdots \oplus M_n$ with each M_i quasi-projective over R_i. In fact, since each $R_i = e_i R e_i$, with the $\{e_i\}$ being a set of central, orthogonal idempotents, we may take $M_i = e_i M$. Now we are ready to prove the following result:

Theorem 3. *Let R be a hereditary Noetherian ring. Then R has every quasi-projective module projective modulo its annihilator if and only if R is a ring direct sum of Artinian hereditary rings, and semi-local HNP-rings which are not Morita equivalent to complete local Dedekind domains.*

Proof. By [1] every hereditary Noetherian ring is a ring direct sum of a finite number of hereditary Artinian rings and HNP-rings. So let $R = \oplus \sum R_i$ where each R_i is either a hereditary Artinian ring or an HNP-ring. Let M_i be a module over R_i. Since the R action on M_i corresponds to the R_i action, M_i is quasi-projective over R_i if and only if it is quasi-projective over R. It then follows that M_i is projective modulo its annihilator as an R_i module if and only if it is projective modulo its annihilator as an R-module. This observation and Theorem 2 yield the necessity of the conditions.

For sufficiency, let M be a quasi-projective module over R. Thus $M = M_1 \oplus \cdots \oplus M_n$ with each M_i quasi-projective over R_i. If R_i is a hereditary, Artinian ring, M_i is projective modulo its annihilator ([6], Proposition 2.3). If R_i is an HNP-ring, M_i is projective modulo its annihilator by Theorem 2. Let $I_i = l_R(M_i)$. Then $l_R(M_i) = R_1 \oplus \cdots \oplus I_i \oplus \cdots \oplus R_n$. Therefore, $l_R(M) = \cap\, l_R(M_i) = I_1 \oplus \cdots \oplus I_n$. So $R/l_R(M) \cong R_1/I_1 \oplus \cdots \oplus R_n/I_n$. As each M_i is projective modulo its annihilator, each M_i is a direct summand of $(R_i/I_i)^{(A_i)}$ $(1 \leq i \leq n)$. This shows that M is a direct summand of a free $R/l_R(M)$ module. That is, M is projective over $R/l_R(M)$.

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Instituto de Matematica
Universidade Federal da Bahia
Salvador, Bahia, Brasil