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ON THE INDECOMPOSABILITY OF AMALGAMATED SUMS

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Let R be a semiprimary ring with identity element. In this paper, we study when a factor module of a direct sum of local or colocal R-modules of finite length is indecomposable.

Let (E): $0\to K\xrightarrow{f}\bigoplus_{i=1}^{n}L_i\xrightarrow{g}M\to 0$ be a nonsplit exact sequence of right R-modules of finite length and $f_i\colon K\to L_i$ be the i^{th} coordinate map of f for each $i=1,\cdots,n$. As we will see in the paper, if M is indecomposable, then the following condition holds:

(*) For each $j=1,\cdots,n$ and each $h=(h_i)_{i=1}^{n}\colon \bigoplus_{i=1}^{n}L_i\to L_j$, $hf=\sum_{i=1}^{n}h_if_i=0$ implies that h_i is not an isomorphism for each $i=1,\cdots,n$.

The converse is not true in general, but in Tachikawa [3] we see the converse holds under rather strong conditions. Moreover in [1, section 2], we showed that this converse assertion is still true in the case of each of three groups of weaker conditions than those required in [3]. But in [1, Proposition 2.7], the third group of conditions, we assumed a condition on composition lengths of the L_i's which was not assumed in the other two cases. In this paper, we remove this condition on composition lengths and show that the condition (*) implies the indecomposability of M if each L_i is local and colocal, and each f_i is a monomorphism (see (3.3)).

In section 1, we consider the fundamental properties of the map $f=(f_i)_{i=1}^{n}$ in the sequence (E) satisfying the condition (*). Section 2 is a generalization of tools used in [1, section 2] (this generalization is not essential to understanding the main results) and in section 3 we give the main results.

Throughout the paper R is a ring with identity element, J the Jacobson radical of R, every module is a unitary right R-module. We denote by Mod R and by mod R the category of all R-modules and R-modules of finite length, respectively. We call an R-module M completely indecomposable in case the endomorphism ring $\mathrm{End}_R(M)$ is a local ring. For maps $f\colon K\to L$ and $g\colon L\to M$, and for a decomposition $D\colon L=\bigoplus_I L_i$ of L, the notations $(f,D)=(f_i)_I^T$ and

$(D, g)=(g_i)_I$ are matrix expressions of f and g relative to D, respectively. The notation $I_1 \amalg \cdots \amalg I_n = I$ means that the union $I_1 \cup \cdots \cup I_n = I$ is disjoint. Finally, the socle of M is denoted by soc M, the m^{th} socle of M is denoted by $\operatorname{soc}_m M$ for each R-module M. (It is well known that if R is a semiprimary ring, $\operatorname{soc}_m M$ is equal to the left annihilator of J^m in M.)

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1. Infusible maps

DEFINITION. Let $f\colon K \to L$ be a homomorphism of R-modules, $D\colon L = \bigoplus_{i=1}^{n} L_i$ be a decomposition of L and let $(f, D)=(f_i)_{i=1}^{T\ n}$. Then the pair (f, D) is said to be *infusible* in case for each $j=1, \cdots, n$ and each $h=(h_i)_{i=1}^{n}\colon \bigoplus_{i=1}^{n} L_i \to L_j$; $hf = \sum_{n=1}^{n} h_i f_i = 0$ implies that h_i is not an isomorphism for each $i=1, \cdots, n$. Further the pair (f, D) is called *fusible* in case it is not infusible.

Dually, for a homomorphism $g\colon L \to M$ of R-modules and for a decomposition $D\colon L = \bigoplus_{i=1}^{n} L_i$ of L, the pair (D, g) is said to be *coinfusible* in case for each $j=1, \cdots, n$ and each $h=(h_i)_{i=1}^{T\ n}\colon L_j \to \bigoplus_{i=1}^{n} L_i$; $gh=0$ implies that each h_i is not an isomorphism for each $i=1, \cdots, n$. Further the pair (D, g) is called *cofusible* if it is not coinfusible.

Finally, assume that the R-module L above can be written as a finite direct sum of completely indecomposable R-modules. Then we simply say that the map f (resp. g) is infusible (resp. coinfusible) in case the pair (f, D) (resp. (D, g)) is infusible (resp. coinfusible) for every decomposition D of L, and the map f (resp. g) is said to be fusible (resp. cofusible) if it is not infusible (resp. coinfusible).

Proposition 1.1. (a) *For a homomorphism $f\colon K \to L$ of R-modules and for a decomposition $D\colon L = \bigoplus_{i=1}^{n} L_i$ of L, putting $g\colon L \to \operatorname{Coker} f\,(=M)$ the canonical epimorphism and $(D, g)=(g_i)_{i=1}^{n}$, the following statements are equivalent*:

(1) *(f, D) is fusible.*

(2) *There is a homomorphism $h\colon \bigoplus_{i \neq j} L_i \to L_j$ for some $j=1, \cdots, n$ such that $f_j = h(f_i)_{i \neq j}^{T}$.*

(3) *g_j is a split monomorphism for some $j=1, \cdots, n$.*

(4) *There is a split epimorphism $p\colon M \to M'\ (\neq 0)$ and a nonempty subset $I \subseteq \{1, \cdots, n\}$ such that $p(g_i)_I\colon \bigoplus_I L_i \to M'$ is an isomorphism.*

Further if each L_i is completely indecomposable, then the above conditions are equivalent to

(5) *$\{h \in \operatorname{End}_R(L) \mid hf=0\} \not\subseteq J(\operatorname{End}_R(L))$ where $J(-)$ denotes the Jacobson*

radical of $(-)$.

(a)′ *For a homomorphism* $g: L \to M$ *of R-modules and for a decomposition* D: $L=\bigoplus_{i=1}^{n} L_i$ *of* L, *putting* f: $(K=)\operatorname{Ker} g \to L$ *the inclusion map and* $(f, D)=(f_i)_{i=1}^{T\ n}$, *the following statements are equivalent*:

(1)′ (D, g) *is cofusible.*

(2)′ *There is a homomorphism* $h: L_j \to \bigoplus_{i \neq j} L_i$ *for some* $j=1, \cdots, n$ *such that* $g_j=(g_i)_{i\neq j}h$.

(3)′ f_j *is a split epimorphism for some* $j=1, \cdots, n$.

(4)′ *There is a split monomorphism* q: $(0\neq) K' \to K$ *and a nonempty subset* $I \subseteq \{1, \cdots, n\}$ *such that* $(f_i)_I^T q$: $K' \to \bigoplus_I L_i$ *is an isomorphism.*

Further if each L_i *is completely indecomposable, then the above conditions are equivalent to*

(5)′ $\{h \in \operatorname{End}_R(L) \mid gh=0\} \not\subseteq J(\operatorname{End}_R(L))$.

Proof. We prove only (a). (1)⇒(2). If the pair (f, D) is fusible, then there exist $i_0=1, \cdots, n$ and h: $L \to L_{i_0}$, $h=(h_i)_{i=1}^{n}$ such that $hf=0$ and h_j is an isomorphism for some $j=1, \cdots, n$. Since h_j is an isomorphism, we may assume $i_0=j$ and $h_j=1_{L_j}$. Then $hf=0$ implies that $f_j=(-h_i)_{i\neq j}(f_i)_{i\neq j}^T$.

(2)⇒(3). Suppose that $f_j=h(f_i)_{i\neq j}^T$ and $h=(h_i)_{i\neq j}$. Taking $h'=(-h_1, \cdots, -h_{j-1}, 1_{L_j}, -h_{j+1}, \cdots, -h_n)$, we have $h'f=0$. Therefore there is a homomorphism p: $M \to L_j$ such that $h'=pg$. Let k_j: $L_j \to L$ be the inclusion map. Then $pg_j=pgk_j=h'k_j=1_{L_j}$. Thus g_i is a split monomorphism.

(3)⇒(4). Trivial.

(4)⇒(1). Suppose (4) holds. Taking $h=p_i(p(g_i)_I)^{-1}pg$ where p_i: $\bigoplus_I L_i \to L_i$ $(i \in I)$ is the canonical projection, we have $hf=0$ and $h_i=1_{L_i}$ thus (f, D) is fusible.

If each L_i is completely indecomposable, then $J(\operatorname{End}_R(L))=\{(f_{ij}) \in \operatorname{End}_R(\bigoplus_{i=1}^{n} L_i) \mid f_{ij}$ is not an isomorphism for each $i, j=1, \cdots, n\}$. From this fact, equivalence of (1) and (5) is immediate. //

REMARK 1. (a) There is a homomorphism f: $K \to L$ and decompositions D, D' of L such that (f, D) is fusible but (f, D') is infusible.

(a)′ There is a homomorphism g: $L \to M$ and decompositions D, D' of L such that (D, g) is cofusible but (D, g) is coinfusible.

For example: (a). Let L_1, L_2, L_3 be nonzero R-modules of finite length, $L=L_1 \oplus L_2 \oplus L_3$ and $M=L_1$ and let g_i: $L_i \to M$ be a homomorphism for each $i=1, 2, 3$ such that g_1 is the identity, g_3 is not a split monomorphism; and put $g=(g_1, g_2, g_3)$: $L \to M$. Consider the following exact sequence: $0 \to \operatorname{Ker} g \xrightarrow{f} L \xrightarrow{g} M \to 0$. Let D: $L=L_1 \oplus (L_2 \oplus L_3)$, D': $L=(L_1 \oplus L_2) \oplus L_3$ be two decompositions of L. Then (f, D) is fusible since g_1 is a split monomorphism but (f, D') is infusible since (g_1, g_2) and g_3 are not split monomorphisms (1, 1; 3).

(a)′: Dual.

REMARK 2. Let D: $L=\bigoplus_{i=1}^{n} L_i$ and D': $L=\bigoplus_{j=1}^{m} N_j$ be two decompositions of an R-module L where $N_j=\bigoplus_{I_j} L_i$ for each $j=1, \cdots, m$ and $\{1, \cdots, n\}=I_1 \amalg \cdots \amalg I_m$. Then from (1.1; 3 and 3′) it holds that

(a) For a map f: $K \to L$ in Mod R, if (f, D') is fusible, then (f, D) is fusible; and

(a)′ For a map g: $L \to M$ in Mod R, if (D', g) is cofusible, then (D, g) is cofusible.

From the above remark and (1.1; 5 and 5′), we obtain the following

Corollary 1.2. *Suppose that an R-module L has a finite decomposition* D: $L=\bigoplus_{i=1}^{n} L_i$ *with each* L_i *completely indecomposable. Then*

(a) *A homomorphism* f: $K \to L$ *in* Mod R *is infusible if and only if the pair* (f, D) *is infusible*; *and*

(a)′ *A homomorphism* g: $L \to M$ *in* Mod R *is coinfusible if and only if the pair* (D, g) *is coinfusible.*

REMARK. Let (E): $0 \to K \xrightarrow{f} L \xrightarrow{g} M \to 0$ be a nonsplit exact sequence in mod R. In [1] we called (E) a (*)-sequence iff the map f is infusible.

Corollary 1.3. *Let* (E): $0 \to K \xrightarrow{f} L \xrightarrow{g} M \to 0$ *be a nonsplit exact sequence in* mod R. *Then*

(a) *If M is indecomposable, f is infusible*; *and*

(a)′ *If K is indecomposable, g is coinfusible.*

Proof. We only show that (a) holds. If f is fusible, thus (f, D) is fusible for some decomposition D: $L=\bigoplus_{i=1}^{n} L_i$, then from (1.1; 3) and the fact that the sequence (E) does not split, L_i is a proper nonzero direct summand of M for some $i=1, \cdots, n$. //

REMARK. (a) There is a nonsplit monomorphism f in mod R such that f is infusible but Coker f is decomposable.

(a)′ There is a nonsplit epimorphism g in mod R such that g is coinfusible but Ker g is decomposable.

For example: (a) Let the exact sequence $0 \to K \xrightarrow{f} L \xrightarrow{g} M \to 0$ be the projective cover of $M=M_1 \oplus M_2$ where M_1 and M_2 are nonprojective indecomposables in mod R. Then f does not split and by (1.1; 3), f is infusible but Coker $f=M$ is decomposable.

(a)′: Dual. //

From (1.2; 2 and 2′) we have

Corollary 1.4. (a) *Let* $K_i \lneqq L_i$ *for each* $i=1, 2$ *and* $h: K_1 \to K_2$ *be an isomorphism. Define* $f_1 = k_1$, $f_2 = k_2 h$ *where* $k_i: K_i \to L_i$ *is the inclusion map for each* $i=1, 2$. *Then* h *or* h^{-1} *is extendable to a homomorphism* $L_1 \to L_2$ *or* $L_2 \to L_1$, *respectively iff* $(f, D) = (f_i)_{i=1,2}^T: K_1 \to L_1 \oplus L_2$ *is fusible.*

(a)′ *Let* $K_i \lneqq L_i$ *for each* $i=1, 2$ *and* $h: L_1/K_1 \to L_2/K_2$ *be an isomorphism. Define* $g_1 = hp_1$, $g_2 = p_2$ *where* $p_i: L_i \to L_i/K_i$ *is the projection for each* $i=1, 2$. *Then* h *or* h^{-1} *is liftable to a homomorphism* $L_1 \to L_2$ *or* $L_2 \to L_1$, *respectively iff* $(D, g) = (g_i)_{i=1,2}: L_1 \oplus L_2 \to L_2/K_2$ *is cofusible.* //

The forms of (1.1; 4 and 4′) are mainly used to show that (f, D) is fusible and (D, g) is cofusible below.

2. Covering property

Recall that a functor r of Mod R to itself is called a preradical in case it is a subfunctor of the identity functor of Mod R, that is, for any $M \in \text{Mod}\,R$; $rM \leqslant M$ and for any map $f: M \to N$ in Mod R, $rf: rM \to rN$ is the restriction map of f. A preradical r is called a radical (resp. an idempotent preradical) in case for any $M \in \text{Mod}\,R$, $r(M/rM)=0$ (resp. $r(rM)=rM$). For any $N \leqslant M$ in Mod R the notations $N \ll M$ and $N \leqslant_e M$ mean "N is small in M" and "N is essential in M", respectively. An R-module M is called local (resp. colocal) in case M has the unique maximal (resp. minimal) submodule. We denote the composition length of M by $|M|$.

Lemma 2.1. *Let* $L = \bigoplus_{i=1}^{n} L_i$ *and* $M = M_1 \oplus M_2$ *be decompositions of* R-*modules such that each* L_i *is completely indecomposable and let* $p_j: M \to M_j$ *denote the projections. If* $f: L \to M$ *is an isomorphism, then there exists a partition* $\{1, \cdots, n\} = I_1 \amalg I_2$ *such that the restriction map* $p_j f: \oplus_{I_j} L_i \to M_j$ *is an isomorphism for each* $j=1, 2$.

Proof. See [1, Lemma 1.4].

REMARK. By induction, this Lemma holds more generally. Let $M = \bigoplus_{i=1}^{n} L_i = \bigoplus_{j=1}^{r} M_j$ be direct decompositions of an R-module M with completely indecomposable modules L_i. Then there exists a partition $\{1, \cdots, n\} = I_1 \amalg \cdots \amalg I_r$ such that the induced map $p_j: N_j \to M_j$ is an isomorphism for each $j=1, \cdots, r$ where $N_j = \oplus_{I_j} L_i$ and $p_j: M \to M_j$ is the projection.

DEFINITION. Let $D: L = \bigoplus_{i=1}^{n} L_i$ be a decomposition of an R-module L and $g: L \to M$, $f: M \to L$ be homomorphisms. Then g is said to have the *covering* (resp. *cocovered*) *property* and f is said to have the *covered* (resp. *cocovering*) *property* in case for any decomposition $M = M_1 \oplus M_2$ there is a decomposition

$L=N_1\oplus N_2$ such that the restriction maps $p_jg\colon N_j\to M_j$ and g itself are epimorphisms (resp. monomorphisms) and the restriction maps $q_jf\colon M_j\to N_j$ and f itself are epimorphisms (resp. monomorphisms), respectively where $p_j\colon M\to M_j$ and $q_j\colon L\to N_j$ are the projections. In addition, if in the definitions above, N_j is always given by the form $N_j=\oplus_{I_j}L_i$ for each $j=1, 2$ where $I_1\amalg I_2=\{1, \cdots, n\}$, then we say that g has the *D-compatible* covering (resp. cocovered) property and so on.

Proposition 2.2. *Let* $0\to K\xrightarrow{f}L\xrightarrow{g}M\to 0$ *be a nonsplit exact sequence in* $\operatorname{mod}R$, $D\colon L=\bigoplus_{i=1}^{n}L_i$ *a decomposition of* L *and* $(f, D)=(f_i)_{i=1}^{T\,n}$, $(D, g)=(g_i)_{i=1}^{n}$. *Then*

(a) *If* K *is simple, then* M *is indecomposable iff* (f, D) *is infusible and* g *has the D-compatible covering property; and*

(b) *If* f_1 *is monic and* $\operatorname{Coker} f_1$ *is simple, then* M *is indecomposable iff* (f, D) *is infusible and* $(g_i)_{i\neq 1}$ *has the* $\oplus_{i\neq 1}L_i$*-compatible cocovered property.*

The dual statements also hold.

Proof. We prove only (a) and (b). (a). ($\Rightarrow$). By (1.3), (f, D) is infusible and it is trivial that g has the D-compatible covering property. ($\Leftarrow$). Suppose $M=M_1\oplus M_2$; $M_1, M_2\neq 0$. Then there is a decomposition $L=N_1\oplus N_2$ such that the induced maps $N_j\to M_j$ are epimorphisms where N_j's are direct sums of some L_i's. But since K is simple, $|N_j|=|M_j|$ for some $j=1, 2$ which means that the map $N_j\to M_j$ is an isomorphism. Hence (f, D) is fusible by (1.1).

(b). ($\Rightarrow$). Trivial. ($\Leftarrow$). Note that $(g_i)_{i\neq 1}$ is monic, since f_1 is monic and $\operatorname{Ker} g\leqslant \operatorname{Im} f$. We put $g_0=(g_i)_{i\neq 1}$ and $L_0=\bigoplus_{i\neq 1}L_i$. Suppose $M=M_1\oplus M_2$; $M_1, M_2\neq 0$. Then there is a decomposition $L_0=N_1\oplus N_2$ such that the maps $N_j\to M_j$ induced by g_0 are monomorphisms where N_j's are direct sums of some L_i's. But since $\operatorname{Coker} f_1$ is simple and $\operatorname{Coker} g_0=(g_1L_1+g_0L_0)/g_0L_0\simeq g_1L_1/(g_1L_1\cap g_0L_0)=g_1L_1/g_1f_1K$; we have $|\operatorname{Coker} g_0|=1$ or 0. Hence $|N_j|=|M_j|$ for some $j=1, 2$ thus (f, D) is fusible. //

The following proposition is a generalization of [1, Lemma 2.4].

Proposition 2.3. *Let* $D\colon L=\bigoplus_{i=1}^{n}L_i$ *be a decomposition of* L *in* $\operatorname{Mod}R$. *Then we have*

(a) *For an epimorphism* $g\colon L\to M$ *in* $\operatorname{Mod}R$, *if there is a radical* r *such that* L_i/rL_i *is completely indecomposable for each* $i=1, \cdots, n$ *and* $\operatorname{Ker} g\leqslant rL\ll L$, *then* g *has the D-compatible covering property;*

(a)′ *For a monomorphism* $f\colon M\to L$ *in* $\operatorname{Mod}R$, *if there is an idempotent preradical* r *such that* rL_i *is completely indecomposable for each* $i=1, \cdots, n$ *and* $rL\leqslant \operatorname{Im} f$, $rL\leqslant_e L$, *then* f *has the D-compatible cocovering property;*

(b) *For a monomorphism* $g\colon L\to M$ *in* $\operatorname{Mod}R$, *if there is an idempotent* *preradical r such that rL_i is completely indecomposable for each $i=1, \cdots, n$ and $rM \leqslant \operatorname{Im} g$, $rM \leqslant_e M$, then g has the D-compatible cocovered property*; *and*

(b)′ *For an epimorphism $f: M \to L$ in* Mod R, *if there is a radical r such that L_i/rL_i is completely indecomposable for each $i=1, \cdots, n$ and* $\operatorname{Ker} f \leqslant rM \ll M$, *then f has the D-compatible covered property.*

Proof. (a). Since $\operatorname{Ker} g \leqslant rL$ and r is a radical, $r(L/\operatorname{Ker} g) = rL/\operatorname{Ker} g$. Accordingly, we get the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & r(L/\operatorname{Ker} g) & \to & L/\operatorname{Ker} g & \to & L/rL & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & rM & \longrightarrow & M & \longrightarrow & M/rM & \to & 0 \end{array}$$

with exact rows where all vertical maps are isomorphisms, in particular the induced map $\bar{g}: L/rL \to M/rM$ is an isomorphism. Now let $M = M_1 \oplus M_2$, p_j: $M \to M_j$ be the canonical projections and $s: L \to L/rL$, $t: M \to M/rM$ the canonical epimorphisms. Then we have $L/rL = \bigoplus_{i=1}^{n} s(L_i)$ and $M/rM = t(M_1) \oplus t(M_2)$ where $s(L_i) \cong L_i/rL_i$ is completely indecomposable for each $i=1, \cdots, n$. From (2.1), we obtain that there is a partition $\{1, \cdots, n\} = I_1 \amalg I_2$ such that the restriction maps $\bar{p}_j \bar{g}: \oplus_{I_j} s(L_i) \to t(M_j)$ are isomorphisms. But since $\operatorname{Ker} g \leqslant rL \ll L$, we have $rM \ll M$ thus t is a small epimorphism. Hence the restriction maps $p_j g: \oplus_{I_j} L_i \to M_j$ are epimorphisms. The rest of the proof is similar to (a). //

The following corollary is just [1, Lemma 2.4] and its dual. But for completeness, we shall rewrite it below.

Corollary 2.4. *Let R be a semiprimary ring, $D: L = \bigoplus_{i=1}^{n} L_i$ a decomposition of L in* Mod R. *Then it holds that*

(a) *For an epimorphism $g: L \to M$ in* Mod R, *if all L_i's are local then g has the D-compatible covering property*;

(a)′ *For a monomorphism $f: M \to L$ in* Mod R, *if all L_i's are colocal then f has the D-compatible cocovering property*;

(b) *For a monomorphism $g: L \to M$ in* Mod R, *if all L_i's are colocal then g has the D-compatible cocovered property*; *and*

(b)′ *For an epimorphism $f: M \to L$ in* Mod R, *if all L_i's are local then f has the D-compatible covered property.*

Proof. (a). In the proof of (2.3), put $r = J$ (Jacobson radical) and note that $MJ \ll M$ and there is a subset $I \subseteq \{1, \cdots, n\}$ such that the induced map $\oplus_I s(L_i) \to M/MJ$ is an isomorphism since L/LJ is semisimple. The rest of the proof is similar. //

The following proposition is a generalization of [1, Theorem 2.5 and Proposition 2.6].

Proposition 2.5. *Let* $0 \to K \xrightarrow{f} L \xrightarrow{g} M \to 0$ *be a nonsplit exact sequence in* mod R *and* $D: L = \bigoplus_{i=1}^{n} L_i$ *a decomposition of* L. *Then it holds that*

(a) *If* K *is simple and there is a radical* r *such that* L_i/rL_i *is indecomposable for each* $i=1, \cdots, n$ *and* $\operatorname{Ker} g \leqslant rL \ll L$, *then* M *is indecomposable iff* (f, D) *is infusible*;

(a)′ *If* M *is simple and there is an idempotent preradical* r *such that* rL_i *is indecomposable for each* $i=1, \cdots, n$ *and* $rL \leqslant \operatorname{Im} f$, $rL \leqslant_e L$, *then* K *is indecomposable iff* (D, g) *is coinfusible*;

(b) *If* f_1 *is monic,* $\operatorname{Coker} f_1$ *is simple and there is an idempotent preradical* r *such that* rL_i *is indecomposable for each* $i=2, \cdots, n$ *and* $rM \leqslant \operatorname{Im}(g_i)_{i \neq 1}$, $rM \leqslant_e M$, *then* M *is indecomposable iff* (f, D) *is infusible*; *and*

(b)′ *If* g_1 *is epic,* $\operatorname{Ker} g_1$ *is simple and there is a radical* r *such that* L_i/rL_i *is indecomposable for each* $i=2, \cdots, n$ *and* $\operatorname{Ker}(f_i)_{i \neq 1}^{T} \leqslant rM \ll M$, *then* K *is indecomposable iff* (D, g) *is coinfusible, where* $(f, D) = (f_i)_{i=1}^{T\,n}$ *and* $(D, g) = (g_i)_{i=1}^{n}$.

Proof. Clear from (2.2), (2.3) and the fact that every indecomposable module of finite length is completely indecomposable. //

The following corollary is just [1, Theorem 2.5 and Proposition 2.6] and their duals.

Corollary 2.6. *Let* R *be a semiprimary ring and* $0 \to K \xrightarrow{f} L \xrightarrow{g} M \to 0$ *be a nonsplit exact sequence in* mod R, *and let* $D: L = \bigoplus_{i=1}^{n} L_i$ *be a decomposition of* L. *Then it holds that*

(a) *If* K *is simple and all* L_i*'s are local, then* M *is indecomposable iff* (f, D) *is infusible*;

(a)′ *If* M *is simple and all* L_i*'s are colocal, then* K *is indecomposable iff* (D, g) *is coinfusible*;

(b) *If* f_1 *is monic,* $\operatorname{Coker} f_1$ *is simple and* L_i *is colocal for each* $i=2, \cdots, n$, *then* M *is indecomposable iff* (f, D) *is infusible*; *and*

(b)′ *If* g_1 *is epic,* $\operatorname{Ker} g_1$ *is simple and* L_i *is local for each* $i=2, \cdots, n$, *then* K *is indecomposable iff* (D, g) *is coinfusible, where* $(f, D) = (f_i)_{i=1}^{T\,n}$ *and* $(D, g) = (g_i)_{i=1}^{n}$. //

REMARK. (2.6.a) is also a generalization of [2, Theorem 3.7] in the semiprimary case.

3. Main results

Throughout this section, we assume that R is a semiprimary ring and every module is of finite length. For a map u: $M\to N$ in mod R, the notation u: $M\rightarrowtail N$ (resp. $u: M\twoheadrightarrow N$) means that u is a monomorphism (resp. an epimorphism). We denote by $h(M)$ the height (= Loewy length) of M for each M in mod R, namely $h(M)=\min\{n\in \boldsymbol{N}\cup\{0\}\mid MJ^n=0\}=\min\{m\in\boldsymbol{N}\cup\{0\}\mid \mathrm{soc}_m M=M\}$ where we put $MJ^0=M$, $\mathrm{soc}_0 M=0$.

Lemma 3.1. *Consider the maps*

$$D \overset{u}{\rightarrowtail} C_1 \overset{v}{\twoheadleftarrow} E\oplus F$$
$$E \overset{u'}{\rightarrowtail} C_2 \overset{v'}{\twoheadleftarrow} D$$

in mod R *and put* $h=h(F)$. *Then it holds that*

(a) *If F is colocal, $DJ^{h-1}\leqslant_e D$ and $EJ^{h-1}\leqslant_e E$, then v or v' is an isomorphism*; *and*

(b) *If $DJ^h\leqslant_e D$, then v' is an isomorphism.*

Proof. (a). From the maps above we get

$$DJ^{h-1} \overset{u}{\rightarrowtail} C_1J^{h-1} \overset{v}{\twoheadleftarrow} EJ^{h-1}\oplus FJ^{h-1}$$
$$EJ^{h-1} \overset{u'}{\rightarrowtail} C_2J^{h-1} \overset{v'}{\twoheadleftarrow} DJ^{h-1}.$$

Since F is colocal, $FJ^{h-1}=\mathrm{soc}\, F$ is simple and hence $|EJ^{h-1}|\leqslant|DJ^{h-1}|\leqslant|EJ^{h-1}|+1$. Therefore $|DJ^{h-1}|=|EJ^{h-1}|$ or $|DJ^{h-1}|=|EJ^{h-1}|+1$. Thus $v'|DJ^{h-1}$ or $v|(EJ^{h-1}\oplus FJ^{h-1})$ is a monomorphism. But since $DJ^{h-1}\leqslant_e D$ and $EJ^{h-1}\oplus FJ^{h-1}\leqslant_e E\oplus F$, we see *that* v' or v is a monomorphism hence an isomorphism.

(b) Similar to (a). //

Dually we have

Lemma 3.1′. *Consider the maps*

$$D \overset{u}{\twoheadleftarrow} C_1 \overset{v}{\rightarrowtail} E\oplus F$$
$$E \overset{u'}{\twoheadleftarrow} C_2 \overset{v'}{\rightarrowtail} D$$

in mod R *and put* $h=h(F)$. *Then it holds that*

(a) *If F is local,* $\mathrm{soc}_{h-1}D\ll D$ *and* $\mathrm{soc}_{h-1}E\ll E$, *then v or v' is an isomorphism*; *and*

(b) *If* $\mathrm{soc}_h D\ll D$, *then v' is an isomorphism.* //

Now we state our main results.

Theorem 3.2. *Let* $0\to K\overset{f}{\to} L\overset{g}{\to} M\to 0$ *be a nonsplit exact sequence in* mod R *where* $L=\bigoplus_{i=1}^{n} L_i\ (n\geqslant 2)$, f_1, *the first coordinate map of* f, *is a monomorphism and* $L_1, \cdots, L_n$ *are local and colocal. If it holds either*

(a) $h(L_1)=\min_{i=1}^{n} h(L_i)$, L_1 *is local and colocal*; *or*

(b) $h(L_1)<\min_{i\neq 1} h(L_i)$, L_1 *is local,*

then M *is indecomposable iff* f *is infusible.*

Proof. ($\Rightarrow$). By (1.3). ($\Leftarrow$). Let D be the decomposition: $L=\bigoplus_{i=1}^{n} L_i$ and $M=M_1\oplus M_2$; M_1, $M_2\neq 0$, p_i: $M\to M_i$ be the canonical projections. Put $(D, g)=(g_i)_{i=1}^{n}$. Since all L_i's are local, g has the D-compatible covering property. On the other hand noting that $(g_i)_{i\neq 1}$ is monic since f_1 is, we see that $(g_i)_{i\neq 1}$ has the $\bigoplus_{i\neq 1}L_i$-compatible cocovered property since all L_i's$(i\neq 1)$ are colocal. Hence there are partitions $\{1, \cdots, n\}=I_1\amalg I_2$ and $\{2, \cdots, n\}=J_1\amalg J_2$ such that

$$\bigoplus_{J_1} L_i \overset{p_1 g}{\rightarrowtail} M_1 \overset{p_1 g}{\twoheadleftarrow} \bigoplus_{I_1} L_i$$

$$\bigoplus_{J_2} L_i \overset{p_2 g}{\rightarrowtail} M_2 \overset{p_2 g}{\twoheadleftarrow} \bigoplus_{I_2} L_i\,.$$

Here we may assume $1\in I_1$. Put $G_j=I_j\cap J_j$ and $H_j=J_j-G_j$ for each $j=1, 2$. Then noting that $I_1-G_1=H_2\amalg\{1\}$ and $I_2-G_2=H_1$, the following diagram is induced:

$$\bigoplus_{H_1} L_i \rightarrowtail M_1' \twoheadleftarrow (\bigoplus_{H_2} L_i)\oplus L_1$$

$$\bigoplus_{H_2} L_i \rightarrowtail M_2' \twoheadleftarrow \bigoplus_{H_1} L_i$$

where $M_j'=M_j/p_j g(\bigoplus_{G_j} L_i)$, since for any $G\subseteq I\subseteq\{1, \cdots, n\}$ and any u: $\bigoplus_I L_i\to C$, we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_G L_i & \longrightarrow & \bigoplus_I L_i & \longrightarrow & \bigoplus_{I-G} L_i & \longrightarrow & 0\\
 & & \| & & \| & & \downarrow\cong & & \\
0 & \longrightarrow & \bigoplus_G L_i & \longrightarrow & \bigoplus_I L_i & \longrightarrow & \bigoplus_I L_i/\bigoplus_G L_i & \longrightarrow & 0\\
 & & \downarrow & & u\downarrow & & \downarrow & & \\
0 & \longrightarrow & u(\bigoplus_G L_i) & \longrightarrow & C & \longrightarrow & C/u(\bigoplus_G L_i) & \longrightarrow & 0
\end{array}$$

with exact rows from which we see that the induced map v: $\bigoplus_{I-G}L_i\to C/u(\bigoplus_G L_i)$ is monic (resp. epic) if u is monic (resp. epic).

(i) In case (a) is satisfied. Put $h=h(L_1)$. Then for any subset $I\subseteq\{1, \cdots, n\}$, we have that $\operatorname{soc}(\bigoplus_I L_i)\leqslant(\bigoplus_I L_i)J^{h-1}$ since $\operatorname{soc} L_i=L_i J^{h(L_i)-1}\leqslant L_i J^{h-1}$ for all $i\in I$. Hence from (3.1.a), we obtain that $|M_1'|=|(\bigoplus_{H_2} L_i)\oplus L_1|$ or $|M_2'|=|\bigoplus_{H_1} L_i|$. Therefore $|M_1|=|\bigoplus_{I_1} L_i|$ or $|M_2|=|\bigoplus_{I_2} L_i|$ since $p_j(g_i)_{G_j}$ are monomorphisms. Thus f is fusible.

(ii) In case (b) holds. Similarly we can use (3.1.b) to get $|M_2'|=|\oplus_{H_1} L_i|$ thus $|M_2|=|\oplus_{I_2} L_i|$ and f is fusible. //

Corollary 3.3. *Let* $0\to K\xrightarrow{f} L\xrightarrow{g} M\to 0$ *be a nonsplit exact sequence in* mod R, $D: L=\bigoplus_{i=1}^{n} L_i$ $(n\geqslant 2)$ *a decomposition of L and* $(f, D)=(f_i)^T{}_{i=1}^{n}$. *Suppose that each* L_i *is local and colocal, and each* f_i *is a monomorphism. Then M is indecomposable iff f is infusible* //

Dually we obtain

Theorem 3.2′. *Let* $0\to K\xrightarrow{f} L\xrightarrow{g} M\to 0$ *be a nonsplit exact sequence in* mod R, $D: L=\bigoplus_{i=1}^{n} L_i$ $(n\geqslant 2)$ *a decomposition of L and* $(D, g)=(g_i)^T{}_{i=1}^{n}$. *Suppose that* g_1 *is an epimorphism and* $L_2, \cdots, L_n$ *are local and colocal. If it holds either*

(a) $h(L_1)=\min_{i=1}^{n} h(L_i)$, L_1 *is local and colocal; or*

(b) $h(L_1)<\min_{i\neq 1} h(L_i)$, L_1 *is colocal,*

then K is indecomposable iff g is coinfusible. //

Corollary 3.3′. *In the same situation as above. Suppose that all* L_i*'s are local and colocal, and all* g_i*'s are epimorphisms. Then K is indecomposable iff g is coinfusible.* //

REMARK. (3.2) is also a generalization of [2, Theorem 4.2 and Proposition 4.3] in the semiprimary case.

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