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KUPKA-REEB PHENOMENA AND UNIVERSAL UNFOLDINGES OF CERTAIN FOLIATION SINGULARITIES

Dedicated to Professor Yozo Matsushima on his 60th birthday

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If $\tilde{\omega}$ is an integrable 1-form, under certain circumstances, $\tilde{\omega}$ is given as the pull back of a 1-form ω on a lower dimensional space by a submersion, that is, $\tilde{\omega}$ is a trivial unfolding of ω (Kupka-Reeb phenomenon). Especially, if we have an integrable 1-form ω which is a universal unfolding of some other 1-form, then every unfolding of ω is trivial. Thus we obtain "stable" singularities as universal unfoldings.

In this note, we construct universal unfoldings of some complex foliation singularities as an application of the versality theorem proved in [5]. For generalities on unfolding theory of complex analytic foliations, we refer to [4] and [5]. We briefly discuss universal unfoldings in section 1. In section 2, we consider the form $\omega=(\alpha x+\beta y)ydx-(\gamma x+\delta y)xdy$ on $C^2=\{(x,y)\}$ and show that, under some condition on α, β, γ and δ, we can construct a universal unfolding $\tilde{\omega}$ of ω (Theorem 2.1). As a foliation singularity, $\tilde{\omega}$ turns out to be a simple one (Remark 2.2). This fact can be used, for example, to find the solutions of the differential equation $\omega=0$ and its "perturbations". In section 3, we take up the form $\varpi=x_1\cdots x_n\sum_{i=1}^{n} a_i\dfrac{dx_i}{x_i}$ studied by Cerveau and Neto in [1]. They proved, among others, that every unfolding of (a form whose $n-1$ st jet is equal to) ϖ is trivial, provided that $a_i\neq a_j\neq 0$. We give (Theorem 3.2) an alternative proof of this using the versality theorem in [5]. When $n=3$ and two of the a_i's are the same, we show that some unfolding of ϖ is identical with one of the universal unfoldings constructed in section 2. We also indicate how to "stabilize" ϖ in general when two or more of the a_i's are the same (Proposition 3.8).

1. Universal unfoldings. Let $F=(\omega)$ be a codim 1 local foliation at the origin 0 in C^n ([5] section 1).

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Definition 1.1. An unfolding $\mathcal{F}$ of F is universal if it is versal and if the infinitesimal unfolding map of $\mathcal{F}$ ([5] section 1) is injective.

Definition 1.2. An unfolding $\mathcal{F}$ of F is trivial if there is a local holomorphic submersion $\Phi:(\boldsymbol{C}^n\times\boldsymbol{C}^m, 0)\to(\boldsymbol{C}^n, 0)$ such that $\mathcal{F}$ is generated by the pull back $\Phi^*\omega$ of ω by Φ, where $\boldsymbol{C}^m$ is the parameter space of $\mathcal{F}$.

Proposition 1.3. *Let F be a local foliation at 0 in $\boldsymbol{C}^n$. If $\mathcal{F}=(\tilde{\omega})$ is a universal unfolding of F, then every unfolding of $\mathcal{F}$ is trivial.*

Proof. Let $x=(x_1, \cdots, x_n)$ be a coordinate system on $\boldsymbol{C}^n$ and let $\boldsymbol{C}^m=\{t=(t_1, \cdots, t_m)\}$ be the parameter space of $\mathcal{F}$. Thus $\mathcal{F}$ is a local foliation at the origin in $\boldsymbol{C}^n\times\boldsymbol{C}^m$. Let $\mathcal{F}'$ be an arbitrary unfolding of $\mathcal{F}$ with parameter space $\boldsymbol{C}^l=\{s=(s_1,\cdots,s_l)\}$. Then we may think of $\mathcal{F}'$ as an unfolding of F with parameter space $\boldsymbol{C}^m\times\boldsymbol{C}^l$. By the universality of $\mathcal{F}'$ there are map germs Φ and φ such that (i) the diagram

$$\begin{array}{ccc} (\boldsymbol{C}^n\times\boldsymbol{C}^m\times\boldsymbol{C}^l, 0) & \xrightarrow{\Phi} & (\boldsymbol{C}^n\times\boldsymbol{C}^m, 0) \\ {\scriptstyle\pi'}\downarrow & & \downarrow{\scriptstyle\pi} \\ (\boldsymbol{C}^m\times\boldsymbol{C}^l, 0) & \xrightarrow{\varphi} & (\boldsymbol{C}^m, 0), \end{array}$$

where π' and π are canonical projections, is commutative, (ii) for each (t, s), the restriction of Φ to $\pi'^{-1}(t, s)$ is a biholomorphic map into $\pi^{-1}(\varphi(t, s))$ and (iii) $\mathcal{F}'$ is generated by $\Phi^*\tilde{\omega}$. In order to prove the proposition, it suffices to show that Φ is a submersion. We may write $\Phi(x, t, s)=(\psi(x, t, s), \varphi(t, s))$, where ψ is a local map $(\boldsymbol{C}^n\times\boldsymbol{C}^m\times\boldsymbol{C}^l, 0)\to(\boldsymbol{C}^n, 0)$. By the above property (ii), ψ is a submersion. Consider the diagram

$$\begin{array}{ccc} T_{\boldsymbol{C}^m\times\boldsymbol{C}^l,0} & \underset{\iota}{\overset{d\varphi}{\rightleftarrows}} & T_{\boldsymbol{C}^m,0} \\ & {\scriptstyle\rho'}\searrow \quad \swarrow{\scriptstyle\rho} & \\ & U(F), & \end{array}$$

where ρ and ρ' are the infinitesimal unfolding maps ([4] (2.9), (4.3), [5] section 1) of $\mathcal{F}$ and $\mathcal{F}'$ respectively, $d\varphi$ is the differential of φ and ι is the natural inclusion. Since $\mathcal{F}'$ is an unfolding of $\mathcal{F}$, we have $\rho'\circ\iota=\rho$. Also, by the naturality of infinitesimal unfolding maps, we have $\rho'=\rho\circ d\varphi$. Hence $\rho=\rho\circ d\varphi\circ\iota$. Since $\mathcal{F}$ is a universal unfolding, ρ is injective. Therefore, $d\varphi$ must be surjective and φ is a submersion, Q.E.D.

Remarks 1.4. 1°. For Proposition 1.3, the codimension of F need not be one.
2°. For the universal unfoldings given in this note, we could use [5] (4.1) Corollary instead of Proposition 1.3.

2. Universal unfoldings of some singularities on C^2. We consider the 1-form

$$\omega = (\alpha x+\beta y)ydx-(\gamma x+\delta y)xdy$$

on $C^2=\{(x, y)\}$ with α, β, γ and δ complex numbers. We assume that the set of zeros of ω consists only of the origin 0, that is, we assume that

$$\beta\neq 0,\ \gamma\neq 0 \text{ and } D=\alpha\delta-\beta\gamma\neq 0\,.$$

Let $F=(\omega)$ be the codim 1 local foliation at 0 in C^2 generated by the germ of ω at 0 ([5] section 1). We set

$$A = \gamma\{D+\beta(\alpha-\gamma)\},\ \ B = \beta\{D+\gamma(\delta-\beta)\} \text{ and } C=\alpha\beta-\gamma\delta\,.$$

Then it is not difficult to show that F is a Haefliger foliation ([4] (1.10) Definition), that is, ω admits an integrating factor, if and only if $A=B=0$.

We assume that F is non-Haefliger hereafter. Furthermore, we consider only the following three cases:

(I) $\quad A=0,\ B\neq 0,$

(II) $\quad A\neq 0,\ B=0,$

(III) $\quad A\neq 0,\ B\neq 0,\ C=0.$

In the case (I), we may set $\alpha=2ak$, $\beta=a$, $\gamma=bk$, $\delta=b-a$. Then we have $D=-a(2a-b)k$ and $B=-a(a+b)(2a-b)k$. The constants a, b and k are arbitrary as long as $abk\neq 0$, $a\neq -b$ and $2a\neq b$.
In the case (II), we may set $\alpha=a-b$, $\beta=ak$, $\gamma=b$ and $\delta=2bk$. Then we have $D=-b(2b-a)k$ and $A=-b(b+a)(2b-a)k$. The constants a, b and k are arbitrary as long as $abk\neq 0$, $b\neq -a$ and $2b\neq a$.
In the case (III), we may set $\alpha=ak$, $\beta=b$, $\gamma=a$ and $\delta=bk$. Then we have $D=ab(k^2-1)$, $A=a^2b(k-1)(k+2)$ and $B=ab^2(k-1)(k+2)$. The constants a, b and k are arbitrary as long as $ab\neq 0$, $k^2\neq 1$ and $k\neq -2$.

Theorem 2.1. *If one of the conditions* (I), (II) *and* (III) *is satisfied, then there is a universal unfolding* $\mathcal{F}=(\tilde{\omega})$ *of the foliation* $F=(\omega)$. $\mathcal{F}$ *is a* (*codim* 1, *local*) *foliation at* 0 *in* $C^4=\{(x, y, s, t)\}$. *In each case we may choose the following as a generator* $\tilde{\omega}$ *of* $\mathcal{F}$:

(I) $\quad \tilde{\omega} = (2akx+ay+as)ydx-\{bkx^2+(b-a)xy+bxs+bt\}dy+axyds+aydt\,.$

(II) $\quad \tilde{\omega} = \{(a-b)xy+aky^2-ays-at\}dx-(bx+2bky-bs)xdy+bxyds+bxdt\,.$

(III) $\quad \tilde{\omega} = \{akxy+by^2-ys-(k+1)at\}dx-\{ax^2+bkxy+xs-(k+1)bt\}dy$
$\quad\quad +(k+1)(xy-t)ds+(ax-by+s)dt\,.$

Proof. Let $\mathcal{O}$ be the ring of germs of holomorphic functions at 0 in $\boldsymbol{C}^2$ and let Ω be the $\mathcal{O}$-module of germs of holomorphic 1-forms. We set $\Omega_F = \Omega/F$. If we denote by f and g the germs of the functions $(\alpha x+\beta y)y$ and $-(\gamma x+\delta y)x$, respectively, we have ([4] (4.5), [5] section 1) $\mathrm{Ext}^1_{\mathcal{O}}(\Omega_F, \mathcal{O}) = \mathcal{O}/(f, g)$, where (f, g) is the ideal generated by f and g. For any element h in $\mathcal{O}$, we denote by $[h]$ the class of h in $\mathcal{O}/(f, g)$. In our case, since $\beta \neq 0$ and $\gamma \neq 0$, we have

$$Ext^1_{\mathcal{O}}(\Omega_F, \mathcal{O}) = \boldsymbol{C}^4$$

and we may take $[1]$, $[x]$, $[y]$ and $[xy]$ as basis elements. Next we determine the set $U(F)$ of equivalence classes of first order unfoldings of F, which is given by ([4] (6.1) Theorem, (6.8) Remark)

$$U(F) = \{[h] \in Ext^1_{\mathcal{O}}(\Omega_F, \mathcal{O}) \mid h d\omega = \eta \wedge \omega \text{ for some } \eta \in \Omega\} .$$

First we have

$$d\omega = -\{(\alpha+2\gamma)x + (\delta+2\beta)y\} dx \wedge dy .$$

If $\alpha+2\gamma = \delta+2\beta = 0$, then $A=B=0$. Hence by our assumption, $d\omega \neq 0$. Since the coefficients of ω are homogeneous polynomials of degree 2, if an element of the form $\lambda_1[1]+\lambda_2[x]+\lambda_3[y]+\lambda_4[xy]$ is in $U(F)$, then we must have $\lambda_1 = 0$. Now the element $[xy]$ is in $U(F)$, since if we set $h_1 = Dxy$, we have

$$h_1 d\omega = \eta_1 \wedge \omega$$

with $\eta_1 = \{D+\beta(\alpha-\gamma)\} y dx + \{D+\gamma(\delta-\beta)\} x dy$. We now look for elements of the form $h = \lambda x + \mu y$ such that $[h] \in U(F)$. It is not difficult to see that the equation $h d\omega = \eta \wedge \omega$ has a solution for η if and only if

$$\begin{vmatrix} \gamma & 0 & (\alpha+2\gamma)\lambda \\ \delta & \alpha & (\delta+2\beta)\lambda + (\alpha+2\gamma)\mu \\ 0 & \beta & (\delta+2\beta)\mu \end{vmatrix} = 0$$

or equivalently $A\mu + B\lambda = 0$. Thus if we set $h_2 = Ax - By$, then

$$h_2 d\omega = \eta_2 \wedge \omega$$

with $\eta_2 = \left(2+\frac{\alpha}{\gamma}\right) A dx - \left(2+\frac{\delta}{\beta}\right) B dy$. Hence we see that $U(F) = \boldsymbol{C}^2$ and we may take $[h_1] = [Dxy]$ and $[h_2] = [Ax-By]$ as basis elements. The element h_1 determines a first order unfolding

$$\tilde{\omega}_1 = \omega + \omega_1^{(1)} s + h_1 ds$$

with $\omega_1^{(1)} = dh_1 - \eta_1 = \beta(\gamma-\alpha) y dx + \gamma(\beta-\delta) x dy$, where s is a parameter. The second order (in s) term in $d\tilde{\omega}_1 \wedge \tilde{\omega}_1$ is

$$d\omega_1^{(1)} \wedge \omega_1^{(1)} s^2 + (h_1 d\omega_1^{(1)} - dh_1 \wedge \omega_1^{(1)}) s ds .$$

We have $d\omega_1^{(1)} \wedge \omega_1^{(1)} = 0$, since $\omega_1^{(1)}$ is a 1-form on $\boldsymbol{C}^2$. Using $d\omega_1^{(1)} = C dx \wedge dy$, we have $h_1 d\omega_1^{(1)} - dh_1 \wedge \omega_1^{(1)} = (DC - DC) xy dx \wedge dy = 0$. Hence $\tilde{\omega}_1$ satisfies the integrability condition $d\tilde{\omega}_1 \wedge \tilde{\omega}_1 = 0$. Thus $\tilde{\omega}_1$ is actually an unfolding of ω. The element h_2 determines a first order unfolding

$$\tilde{\omega}_2 = \omega + \omega_2^{(1)} t + h_2 dt$$

with $\omega_2^{(1)} = dh_2 - \eta_2 = -\left(1 + \dfrac{\alpha}{\gamma}\right) A dx + \left(1 + \dfrac{\delta}{\beta}\right) B dy$,

where t is a parameter. Since $d\omega_2^{(1)} = 0$, the second order term in $d\tilde{\omega}_2 \wedge \tilde{\omega}_2$ is $-dh_2 \wedge \omega_2^{(1)} \wedge t dt = \dfrac{1}{\beta\gamma} ABC dx \wedge dy \wedge t dt$. Hence under the condition (I), (II) or (III), this vanishes. Thus $\tilde{\omega}_2$ is actually an unfolding of ω.

(I) $A = 0$, $B \neq 0$. In this case, we have $h_1 = Dxy = -a(2a-b)kxy$, $\omega_1^{(1)} = -(2a-b)k(aydx - bxdy)$, $h_2 = -By = aCy$ and $\omega_2^{(1)} = -bCdy$. We set $h_1' = axy$, $\omega_1^{(1)\prime} = aydx - bxdy$, $h_2' = ay$ and $\omega_2^{(1)\prime} = -bdy$. Then clearly

$$\tilde{\omega}_1' = \omega + \omega_1^{(1)\prime} s + h_1' ds \quad \text{and} \quad \tilde{\omega}_2' = \omega + \omega_2^{(1)\prime} t + h_2' dt$$

are unfoldings of ω. We combine $\tilde{\omega}_1'$ and $\tilde{\omega}_2'$ to obtain a form on $\boldsymbol{C}^4 = \{(x, y, s, t)\}$:

$$\tilde{\omega} = \omega + \omega_1^{(1)\prime} s + \omega_2^{(1)\prime} t + h_1' ds + h_2' dt .$$

We now show that $\tilde{\omega}$ satisfies the integrability condition and is thus an unfolding of ω. Noting that $d\omega_2^{(1)\prime} = 0$, we have

$$d\tilde{\omega} = d\omega + d\omega_1^{(1)\prime} s - (\omega_1^{(1)\prime} - dh_1') \wedge ds - (\omega_2^{(1)\prime} - dh_2') \wedge dt .$$

For our purpose, it suffices to show that the terms in $d\tilde{\omega} \wedge \tilde{\omega}$ involving sdt, tds or $ds \wedge dt$ vanish. First, the coefficient of sdt is

$$(\omega_2^{(1)\prime} - dh_2') \wedge \omega_1^{(1)\prime} + h_2' d\omega_1^{(1)\prime} = a(a+b) y dx \wedge dy - a(a+b) y dx \wedge dy = 0 .$$

The coefficient of tds is

$$(\omega_1^{(1)\prime} - dh_1') \wedge \omega_2^{(1)\prime} = b(a+b) x dy \wedge dy = 0 .$$

Finally the coefficient of $ds \wedge dt$ is

$$h_1'(\omega_2^{(1)\prime} - dh_2') - h_2'(\omega_1^{(1)\prime} - dh_1') = -a(a+b) xy dy + a(a+b) xy dy = 0 .$$

Therefore, $\tilde{\omega}$ is integrable. Let $\mathscr{F} = (\tilde{\omega})$ be the unfolding of $F = (\omega)$ generated by $\tilde{\omega}$. The infinitesimal unfolding map of $\mathcal{F}$ sends the tangent vectors $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$ of the parameter space $C^2=\{(s, t)\}$ to the classes $[h_1']$ and $[h_2']$ in $U(F)$. Since $[h_1']$ and $[h_2']$ form a basis of $U(F)$, by the versality theorem in [5], $\mathcal{F}$ is a universal unfolding of F.

(II) $A \neq 0$, $B=0$. Similar to the case (I).

(III) $A\neq 0$, $B\neq 0$, $C=0$. In this case, we have $h_1=Dxy=ab(k^2-1)xy$, $\omega_1^{(1)}=-ab(k-1)d(xy)$, $h_2=ab(k-1)(k+2)(ax-by)$ and $\omega_2^{(1)}=-ab(k^2-1)(k+2)(adx-bdy)$. We set $h_1''=(k+1)xy$, $\omega_1^{(1)''}=-d(xy)$, $h_2''=ax-by$ and $\omega_2^{(1)''}=-(k+1)(adx-bdy)$. Then clearly

$$\tilde{\omega}_1'' = \omega+\omega_1^{(1)''}s+h_1''ds \quad\text{and}\quad \tilde{\omega}_2'' = \omega+\omega_2^{(1)''}t+h_2''dt$$

are unfoldings of ω. We construct a universal unfolding of ω by combining $\tilde{\omega}_1''$ and $\tilde{\omega}_2''$. If we simply add them as in the case (I), we do not get an integrable form. However, again by a straightforward computation, it can be shown that the form

$$\tilde{\omega} = \omega+\omega_1^{(1)''}s+\omega_2^{(1)''}t+h_1''ds+h_2''dt-(k+1)tds+sdt$$

on $C^4=\{(x, y, s, t)\}$ is integrable (see also Remark 2.2 below). Thus $\mathcal{F}=(\tilde{\omega})$ is an unfolding of F. Moreover, since $[h_1'']$ and $[h_2'']$ form a basis of $U(F)$, by the versality theorem in [5], $\mathcal{F}$ is a universal unfolding of F, Q.E.D.

REMARK 2.2. For each $\tilde{\omega}$ in Theorem 2.1, we have $d\tilde{\omega}(0)\neq 0$. Hence there must be a coordinate system on C^4 in terms of which the form $\tilde{\omega}$ involves only two variables (Kupka-Reeb phenomenon [3], [1] p. 2). In fact, in each case, such a coordinate system (x', y', s', t') is given as follows:

(I) If $x' = x, y' = y, s' = s$ and $t' = kx^2+xy+xs+t$, then
$\tilde{\omega} = ay'dt'-bt'dy'$.

(II) If $x' = x, y' = y, s' = s$ and $t' = -xy-ky^2+ys+t$, then
$\tilde{\omega} = bx'dt'-at'dx'$.

(III) If $x' = x, y' = y, s' = ax+by+s$ and $t' = -xy+t$, then
$\tilde{\omega} = s'dt'-(k+1)t'ds'$.

From this we can readily find the singular set and the leaves of the foliation $\mathcal{F}=(\tilde{\omega})$. The leaves of $\mathcal{F}$ are given, in terms of the old system (x, y, s, t), by

$$(2.3)\qquad \begin{cases} (\text{ I }) & cy^b = (kx^2+xy+xs+t)^a, \\ (\text{II}) & cx^a = (-xy-ky^2+ys+t)^b, \\ (\text{III}) & c(xy-t) = (ax-by+s)^{k+1}, \end{cases}$$

where c is an arbitrary constant. Also, if we consider, for each fixed (s, t), the foliation $F_{s,t}=(\omega_{s,t})$ on $C^2=\{(x, y)\}$ generated by

(I) $\omega_{s,t}=(2akx+ay+as)ydx-\{bkx^2+(b-a)xy+bxs+bt\}dy$,

(II) $\omega_{s,t}=\{(a-b)xy+aky^2-ays-at\}dx-(bx+2bky-bs)xdy$,

(III) $\omega_{s,t}=\{akxy+by^2-ys-(k+1)at\}dx-\{ax^2+bkxy+xs-(k+1)bt\}dy$,

then (2.3) also gives the leaves of $F_{s,t}$, or solutions of the differential equation $\omega_{s,t}=0$.

EXAMPLES 2.4. 1°. If $\alpha=0$, $\beta=1$, $\gamma=1$ and $\delta=0$, then $D=-1$, $A=B=-2$ and $C=0$. Hence (III) is satisfied and $a=b=1$, $k=0$. Thus

$$\tilde{\omega}=(y^2-ys-t)dx-(x^2+xs-t)dy+(xy-t)ds+(x-y+s)dt$$

is a universal unfolding of $\omega=y^2dx-x^2dy$.
2°. If $\alpha=2$, $\beta=1$, $\gamma=1$ and $\delta=2$, then $D=3$, $A=B=4$ and $C=0$. Hence (III) is satisfied and $a=b=1$, $k=2$. Thus

$$\tilde{\omega}=(2xy+y^2-ys-3t)dx-(x^2+2xy+xs-3t)dy+3(xy-t)ds+(x-y+s)dt$$

is a universal unfolding of $\omega=(2x+y)ydx-(x+2y)xdy$.
3°. If $\alpha=2$, $\beta=-1$, $\gamma=1$ and $\delta=0$, then $D=1$, $A=0$, $B=-2$ and $C=-2$. Hence (I) is satisfied and $a=b=k=-1$. Thus

$$\tilde{\omega}=(2x-y-s)ydx-(x^2-xs-t)dy-xyds-ydt$$

is a universal unfolding of $\omega=(2x-y)ydx-x^2dy$.

Examples 2° and 3° give universal unfoldings of singularities of Dumortier [2] p. 95.

3. Singularity of Cerveau and Neto.

Consider the integrable 1-form

$$\varpi=x_1\cdots x_n\sum_{i=1}^{n}a_i\frac{dx_i}{x_i} \tag{3.1}$$

on $C^n=\{(x_1,\cdots,x_n)\}$. First we give an alternative proof, which uses the versality theorem in [5], of the following result of Cerveau and Neto [1].

Theorem 3.2. *Let $\bar{F}$ be the foliation generated by ϖ in (3.1). If $a_i+a_j\neq 0$, then the set $U(\bar{F})$ of first order unfoldings of $\bar{F}$ is zero. Thus $\bar{F}$ is a universal unfolding of $\bar{F}$ itself and every unfolding of $\bar{F}$ is trivial.*

Proof. Let $\mathcal{O}$ be the ring of germs of holomorphic functions at 0 in C^n and let Ω be the $\mathcal{O}$-module of germs of holomorphic 1-forms. We set $\Omega_{\bar{F}}=\Omega/\bar{F}$. Also we denote by f_i the germ of the function $x_1\cdots\hat{x}_i\cdots x_n$ (omit x_i)

at 0. Then we have

$$(3.3)\qquad Ext^1_{\mathcal{O}}(\Omega_{\bar{F}}, \mathcal{O}) = \mathcal{O}/(f_1, \cdots, f_n),$$

where $(f_1, \cdots, f_n)$ is the ideal generated by $f_1, \cdots, f_n$. Now we find the set $U(\bar{F})$, which is given by

$$U(\bar{F}) = \{[h] \in Ext^1_{\mathcal{O}}(\Omega_{\bar{F}}, \mathcal{O}) \mid h d\bar{\omega} = \eta \wedge \bar{\omega} \quad \text{for some } \eta \in \Omega\}.$$

We have $d\bar{\omega} = x_1 \cdots x_n \sum_{i<j} (a_j - a_i) \frac{dx_i \wedge dx_j}{x_i x_j}$ and, if we write $\eta = \sum_{i=1}^n g_i dx_i$, $\eta \wedge \bar{\omega} = x_1 \cdots x_n \sum_{i<j} \left(\frac{a_j g_i}{x_j} - \frac{a_i g_j}{x_i}\right) dx_i \wedge dx_j$. Hence $h d\bar{\omega} = \eta \wedge \bar{\omega}$ is equivalent to

$$(3.4)\qquad (a_j - a_i)h = a_j x_i g_i - a_i x_j g_j \quad \text{for all } i, j \text{ with } 1 \leq i < j \leq n.$$

By (3.3), we may assume that each monomial in h involves at most $n-2$ different x_i's. Thus, if $a_i \neq a_j \neq 0$, (3.4) is satisfied only when $h=0$. Therefore $U(\bar{F}) = 0$. By the versality theorem in [5], $\bar{F}$ is a universal unfolding of $\bar{F}$ itself and every unfolding of $\bar{F}$ is trivial ([5] (4.1) Corollary), Q.E.D.

Now we consider the case where two or more of the a_i's are the same. First we assume that $n=3$ and $a_1 \neq a_2 = a_3$. We may set $a_2 = a_3 = 1$. Also we set $a_1 = \lambda$. Thus

$$(3.5)\qquad \bar{\omega} = \lambda x_2 x_3 dx_1 + x_1 x_3 dx_2 + x_1 x_2 dx_3,$$

where $\lambda \neq 1$. We also impose a technical condition $\lambda \neq -2$. The following proposition, which is a direct consequence of Theorem 2.1, shows that we can "stabilize" $\bar{\omega}$ in (3.5) by unfolding it suitably.

Proposition 3.6. *Let $\tilde{\omega}$ be the 1-form on $\mathbf{C}^4 = \{(x_1, x_2, x_3, t)\}$ given by*

$$\tilde{\omega} = \lambda(x_2 x_3 + t)dx_1 + x_1 x_3 dx_2 + x_1 x_2 dx_3 + x_1 dt, \quad \lambda \neq 0, 1, -2.$$

Then $\tilde{\omega}$ is integrable; $d\tilde{\omega} \wedge \tilde{\omega} = 0$, and the foliation $\mathcal{F} = (\tilde{\omega})$ can be viewed as a universal unfolding of a certain foliation on $\mathbf{C}^2$. Thus every unfolding of $\mathcal{F}$ is trivial.

Proof. Consider the 1-form

$$\omega = (-2x + y)y dx - \{\lambda x - (\lambda + 1)y\} x dy$$

on $\mathbf{C}^2 = \{(x, y)\}$. Then we have $D = \lambda + 2$, $A = 0$, $B = -(\lambda - 1)(\lambda + 2)$. Thus by Theorem 2.1 (I),

$$\tilde{\omega} = (-2x + y + s)y dx - \{\lambda x^2 - (\lambda + 1)xy - \lambda xs - \lambda t\} dy + xy ds + y dt$$

is a universal unfolding of ω. By the coordinate transformation $x_1 = y$, $x_2 = x$, $x_3 = -x + y + s$, $t = t$ of $\mathbf{C}^4$, $\tilde{\omega}$ becomes the one in the statement, Q.E.D.

REMARK 3.7. Let $\bar{F}$ be the foliation in $C^3=\{(x_1, x_2, x_3)\}$ generated by ϖ in (3.5). If we set, for each integer $p\geq 1$, $h_p=x_1^p$, then $[h_p]$ is in $U(\bar{F})$. In fact, h_p determines an actual unfolding

$$\tilde{\omega}_p=\{\lambda x_2x_3+(\lambda+p-1)x_1^{p-1}t\}dx_1+x_1x_3dx_2+x_1x_2dx_3+x_1^p dt$$

of ϖ. Moreover, if $p\neq q$, then $[h_p]\neq[h_q]$ in $U(\bar{F})$. Hence the unfoldings $\tilde{\omega}_p$ and $\tilde{\omega}_q$ are not equivalent. However, the above proposition shows that essentially it suffices if we unfold ϖ to $\tilde{\omega}_1$ which is the unfolding determined by $h_1=x_1$. Thus if we consider the form

$$\tilde{\tilde{\omega}}=\{\lambda x_2x_3+\lambda t+(\lambda+1)x_1s\}dx_1+x_1x_3dx_2+x_1x_2dx_3+x_1dt+x_1^2ds$$

on $C^5=\{(x_1, x_2, x_3, t, s)\}$, which is readily checked to be integrable, as an unfolding of ϖ, it contains the two independent unfoldings determined by $h_1=x_1$ and $h_2=x_1^2$. However, as an unfolding of $\tilde{\omega}$ in Proposition 3.6, it is trivial.

More generally consider the form ϖ in (3.1) and assume that $a_1=\lambda_1, \cdots, a_m=\lambda_m, a_{m+1}=\cdots=a_n=1, m\geq 1, \lambda_i\neq\lambda_j\neq 0, 1$. Thus

$$\varpi=x_1\cdots x_n\sum_{i=1}^{m}\lambda_i\frac{dx_i}{x_i}+x_1\cdots x_n\sum_{i=m+1}^{n}\frac{dx_i}{x_i}.$$

Let $\bar{F}$ be the foliation generated by ϖ. If $h=x_1\cdots x_m$, then it is not difficult to show that $[h]$ is in $U(\bar{F})$ and is not obstructed. In fact, the following proposition shows that we can "stabilize" ϖ if we unfold ϖ to the unfolding $\tilde{\omega}$ determined by $h=x_1\cdots x_m$.

Proposition 3.8. *Let $\tilde{\omega}$ be the 1-form on $C^{n+1}=\{(x_1, \cdots, x_n, t)\}$ given by*

$$\tilde{\omega}=x_1\cdots x_m(x_{m+1}\cdots x_n+t)\sum_{i=1}^{m}\lambda_i\frac{dx_i}{x_i}+x_1\cdots x_n\sum_{i=m+1}^{n}\frac{dx_i}{x_i}+x_i\cdots x_m dt\,,$$

$$\lambda_i\neq\lambda_j\neq 0, 1\,.$$

Then $\tilde{\omega}$ is integrable. If $\mathcal{F}=(\tilde{\omega})$ is the foliation generated by $\tilde{\omega}$, then $U(\mathcal{F})=0$. Thus $\mathcal{F}$ is a universal unfolding of $\mathcal{F}$ itself and every unfolding of $\mathcal{F}$ is trivial.

Proof. We introduce a new coordinate system $(y_1, \cdots, y_{n+1})$ on C^{n+1} by $y_1=x_1, \cdots, y_m=x_m, y_{m+1}=x_{m+1}\cdots x_n+t, y_{m+2}=x_{m+1}, \cdots, y_{n+1}=x_n$. Then $\tilde{\omega}$ becomes

$$\tilde{\omega}=y_1\cdots y_{m+1}\sum_{i=1}^{m+1}\lambda_i\frac{dy_i}{y_i}\,,$$

where we set $\lambda_{m+1}=1$. The proposition is then proved by a similar argument as in the proof of Theorem 3.2.

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