# LOCAL PROPERTIES OF $p$-BLOCK ALGEBRAS
# OF FINITE GROUPS

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## I. Introduction

The Brauer correspondence is one of the basic methods in the theory of modular representations of finite groups. Moreover, some of the other 1–1 correspondences (of modules or blocks) share good relationships with it. For example, the Green correspondence is one of such ones (Nagao, see (7.7) Feit [8]) and it was noted by Alperin [1] that the Glauberman correspondences of characters for relatively prime operator groups may be proved via the First Main Theorem of Brauer. Here, we pick up what we call the Fong correspondences of blocks and study the relationships of them with the Brauer one. In particular, we get a generalization of one of the results in Okuyama and Wajima [12].

Our second concern is with a certain special $p$-block algebra of a finite group —one which is separable over its center. Such a block has an extreme property on the number of its characters, namely if $d$ is the defect of it, then it has exactly $p^d$ ordinary irreducible characters and one modular irreducible character. According to Brauer [4], this happens if the block has the inertial index one and its defect group is abelian. We shall show that the converse of this fact is true, so that the separability is completely characterized by the local property (except the case of defect zero, of course). The proof will be carried out through the analysis on the annihilator ideal of the radical of a group algebra, especially of its center, which originates in a Brauer's old remark [3] back in 1950's. Some general facts about separable algebras are also helpful.

The notation is standard. $G$ will always denote a finite group and $p$ a prime number. We fix a complete, discrete rank one valuation ring $R$, with quotient field $L$ of characteristic zero and residue field $k$ of characteristic $p$. We assume that $L$ has the $|G|$-th roots of unity. If $a$ is an element of an $R$-module $M$, then $\bar{a}$ denotes its image under the natural map $M \to k \otimes M = \bar{M}$, where $\otimes = \otimes_R$ (throughout this paper). By a $p$-block of $G$, we mean here a block (ideal) of the group ring $R[G]$. If $A$ is a ring, then $J(A)$ and $Z(A)$ denote its Jacobson radical and its center respectively and for a subset $S$ of $A$, $(0: S)$ denotes the set of right annihilators of $S$ in $A$. Finally, $M(n, A)$ denotes the full matrix

ring over $A$ of degree $n$.

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## II. Fong correspondences

In what follows, $B$ will denote a $p$-block of $G$ with a defect group $D$ and the block idempotent $e$; $B=R[G]e$ Let $K$ be a normal $p'$-subgroup of $G$ and $B_K$ one of its $p$-blocks which are covered by $B$. So it is uniquely determined up to $G$-conjugacy and Irr $(B_K)$ consists of a single element, say $\theta$. Let $f$ be the block idempotent of $B_K$. The inertial group $I=I_G(f)$ of $f$ in $G$ (under the action of $G$ by conjugation) is equal to that of $\theta$ and we may assume that $I \supset D$. Furthermore, there exists a unique $p$-block $B_I$ of $I$ such that it covers $B_K$, it has defect group $D$ and $B_I^G=B$. Actually, $B \leftrightarrow B_I$ defines a 1–1 correspondence between the set of $p$-blocks of $G$ which cover $B_K$ and the set of such $p$-blocks of $I$. (Fong [9]). We call $B_I$ the Fong correspondent of $B$ over $K$.

Now, by the Glauberman's Theorem, there exists a unique $\theta' \in$ Irr $(C_K(D))$ such that $(\theta, \theta')_{C_K(D)}$ is prime to $p$. Let $N=N_G(D)$. It is easy to see that $I \cap N=I_N(\theta')$. The following fact is a consequence of Alperin's argument [1] in his proof of the Glauberman's Theorem.

**Theorem 1.** *Let $b$ be the Brauer correspondent of $B$ in $N$. If $b_1$ is the Brauer correspondent of $B_I$ in $N_I(D)=I \cap N$, then it is the Fong correspondent of over $C_K(D)$.*

Proof. Note that $f$ is central in $R[I]$ and is, among others, a block idempotent of a $p$-block of $DK$ with defect group $D$. Let $\rho$ be the block idempotent of $B_I$. By the assumption, we have $\bar{f}=\bar{\rho}+z$ for some $z \in Z(k[I])$. By the Brauer homomorphism from $Z(k[I])$ into $Z(k[N_I(D)])$, $\bar{\rho}$ is mapped onto the block idempotent of $\bar{b}_1$ by definition, while $\bar{f}$ is mapped onto a block idempotent, say $i$, of $k[D \times C_K(D)]$ (note that the intersection of $C_I(D)$ with the supports of $f$ is actually the intersection of $C_{DK}(D)$ with them). We see from Alperin's argument [1] that Irr $(R[C_K(D)]i)= \{\theta'\}$. In particular, $b_1$ covers $R[C_K(D)]i$. Therefore there exists a $p$-block of $N$ to which $b_1$ corresponds by the Fong correspondence, which is necessary $b_1^N$ and has defect group $D$. We now have, from $b_1^G=(b_1^N)^G=(b_1^I)^G=B_I^G=B$, that $b_1^N=b$. This completes the proof.

We next consider the case $G=I$. So $f$ is central in $R[G]$ and $B$ is a component of $A=R[G]f$. The centralizer ring $C=C_A(B_K)$ is a twisted group ring of $G/K$ over $R$; $C=\oplus \sum_{\bar{g} \in G/K} Rc_{\bar{g}}$ and $c_{\bar{g}}c_{\bar{h}}=\alpha(\bar{g}, \bar{h})c_{\bar{g}\bar{h}}$ for some $\alpha(\bar{g}, \bar{h}) \in R$. And there is an $R$-algebra isomorphism $\phi: A \simeq C \otimes B_K$ such that if $g \in G$, then

$\phi(gf) = c_{\bar{g}} \otimes n_g$ for some $n_g \in B_K$. This is a usual argument in the theory of central separable algebras (note that $B_K$ is isomorphic to the full matrix ring over $R$ of degree $\theta(1)$), except the structure of $C$, which was observed by Morita [13] and also by Dade in his theory of Clifford systems [7]. So we omit the detail.

The order, say $m$, of the 2-cocycle $\alpha$ in the cohomology group $H^2(G/K, L^* - \{0\})$, $L^*$ being an algebraic closure of $L$, is a divisor of $|K|^2$ (Fong [9]). After taking a suitable finite extension of $L$ if necessary, we may assume that $\alpha(\bar{g}, \bar{h})^m = 1$ for all $g, h \in G$. With such a 2-cocycle, we construct a central extension

$$1 \longrightarrow M \longrightarrow \hat{G} \overset{\rho}{\longrightarrow} G/K \longrightarrow 1$$

where
$$M = \{x \in R; x^m = 1\} .$$

We may assume that $\{c_{\bar{g}}\}_{\bar{g} \in G/K} \subset \hat{G},$ so that it constitutes a coset representative of $M$ in $\hat{G}$. As $|M| = m$ is prime to $p$, $a = 1/|M| \sum_{x \in M} i(x^{-1})x$ is a central idempotent of $R[G]$, where $i(x)$ indicates the identification $i(x) = x$ as the element of the coefficient ring $R$ and $C$ is isomorphic to the component $R[\hat{G}]a$ of $R[\hat{G}]$ by the map $c_{\bar{g}} \to c_{\bar{g}}a$. We identify $C$ with $R[\hat{G}]a$ for the convenience of later arguments. We conclude from the above that there is a unique $p$-block $\hat{B}$ of $\hat{G}$ such that $B \simeq \hat{B} \otimes B_K$ as $R$-algebras. Actually, $B \leftrightarrow \hat{B}$ defines a 1–1 correspondence between the set of $p$-blocks of $G$ which cover $B_K$ and the set of $p$-blocks of $\hat{G}$ which cover $R[M]a$. We call $\hat{B}$ again the Fong correspondent of $B$ in $\hat{G}$.

Now, *let $H \supset K$ be a subgroup of $G$* and let $\hat{H} = \rho^{-1}(H/K)$. Let $b$ be a $p$-block of $H$ which covers $B_K$ and $\hat{b}$ the Fong correspondent of $b$ in $\hat{H}$. The next result is a generalization of the Theorem 1 of Okuyama and Wajima [12].

**Theorem 2.** *$b^G$ is defined and equal to $B$ if and only if $\hat{b}^{\hat{G}}$ is defined and equal to $\hat{B}$.*

Proof. Everything in the above arguments goes on with $A_1 = R[H]f$ and $C_1 = C_{A_1}(B_K) = R[\hat{H}]a$. Moreover the isomorphism $\phi$ is compatible with the inclusion $i: H \to G$, i.e., the diagram

$$
\begin{array}{ccc}
\phi: A & \simeq & R[\hat{G}]a \otimes B_K \\
\downarrow i & & \downarrow i \otimes 1 \\
\phi_1: A_1 & \simeq & R[\hat{H}]a \otimes B_K
\end{array}
$$

is commutative, where 1 denotes the identity map of $B_K$.

Let $d_H$ be the $R$-homomorphism from $R[G]$ into $R[H]$ which extends the characteristic function of $G$ on $H$. From the definition, we see easily that $(d_{\hat{H}} \otimes 1)\phi = \phi_1 d_H$. On the other hand, since $Z(B_K) = Rf \simeq R$, we may identify $Z(R[\hat{G}]a \otimes B_K)$ with $Z(R[\hat{G}]a)$ and with the convention we have $d_{\hat{H}}\phi = \phi_1 d_H$ on

$Z(A)$.  This means that the diagram

$$
\begin{array}{ccccc}
Z(\bar{A}) & \xrightarrow{\ \bar{d}_H\ } & Z(\bar{A}_1) & \xrightarrow{\ \bar{\omega}\ } & k \\
\downarrow{\scriptstyle \bar{\phi}} & & \downarrow{\scriptstyle \bar{\phi}} & & \| \\
Z(k[\hat{G}])a & \xrightarrow{\ \bar{d}_{\hat{H}}\ } & Z(k[\hat{H}]a) & \xrightarrow{\ \bar{\omega}_1\ } & k,
\end{array}
$$

is commutative, where $\omega$ ($\omega_1$ resp.) denotes the central character of $b$ ($\hat{b}$ resp.). Our assertion follows immediately from this.

## III.  Separable blocks

We begin with the following Proposition which may be known partly.

**Proposition 1.**  *Let $B$ be a $p$-block of $G$ with a defect group $D$.  Then the following are equivalent.*
  (1)  $\bar{B}$ *is separable over* $Z(\bar{B})=\overline{Z(B)}$.
  (2)  $J(\bar{B})=J(Z(\bar{B}))\bar{B}$.
  (3)  $\bar{B}\simeq M(n, Z(\bar{B}))$, *for some* $n\geq 1$.
  (4)  $B$ *is separable over* $Z(B)$.
  (5)  $J(B)=J(Z(B))B$.
  (6)  $B\simeq M(n, Z(B))$, *for some* $n\geq 1$.
  (7)  *Every ordinary irreducible character of $B$ has equal degree.*
  (8)  *Every ordinary irreducible character of $B$ is modularly irreducible,*
  (9)  $|\mathrm{Irr}\,(B)|=|D|$  *and*  $|\mathrm{IBr}\,(B)|=1$

Proof.  The equivalence of the first six statements follows easily from some general facts about separable algebras (see Auslander and Goldman [2]). For example, (1) and (2) are equivalent by the Theorem 4.7 in it. Therefore, it follows from the lifting idempotent theorem that (1) (and (2)) imply $\bar{B}/J(\bar{B})\simeq M(n, k)$ for some $n\geq 1$. In particular, $\bar{B}$ has a unique (non-isomorphic) indecomposable projective module $\bar{B}i$ (with $i^2=i$) and then $\bar{B}\simeq M(n, E)$, where $E=i\bar{B}i$. From this we have $E=k+J(E)=Z(\bar{B})+J(Z(\bar{B}))E$, as $J(E)=iJ(\bar{B})i=J(Z(\bar{B}))E$ and thus $E=Z(\bar{B})$ by Nakayama's Lemma, which implies (3). Similarly, we can show that (3)→(6) and (4)→(6).  The rest is well known.
  (6)→(7):  trivial
  (7)→(8):  As is well known, it holds that
G.C.D. $\{\chi(1);\ \chi\in\mathrm{Irr}(B)\}=$ G.C.D. $\{\psi(1);\ \psi\in\mathrm{IBr}\,(B)\}$ in general.  So it follows from the assumption that $\chi(1)=\psi(1)$ for every $\chi\in\mathrm{Irr}(B)$ and every $\psi\in\mathrm{IBr}(B)$ and our assertion is obvious.
  (8)→(9):  It is obvious that $|\mathrm{IBr}(B)|=1$ and then we have $|\mathrm{Irr}(B)|=|D|$ since the single Cartan invariant of $B$ is necessary $|D|$.
  (9)→(1)  (Külshammer [11]):  Let $i$ be a primitive idempotent of $\bar{B}$.

From the second condition, we have $\bar{B}\simeq M(n, E)$ for some $n\geq 1$, where $E=i\bar{B}i=\mathrm{End}_{\bar{B}}(\bar{B}i)$. It is sufficient to show that $E$ is commutative. We know that $\dim_k E$ (=the single Cartan invariant)$=|D|$. On the other hand, we have that $|D|=|\mathrm{Irr}(B)|=\dim_k Z(\bar{B})=\dim_k Z(E)$. Therefore we have $E=Z(E)$, completing the proof.

As was mentioned in the introduction, the separability is completely characterized by the local property, namely,

**Theorem 3.** *Let $B$ be a $p$-block of $G$ with a defect group $D$ and $b$ the $p$-block of $N_G(D)$ with $b^G=B$. Then the following are equivalent.*

(1) *$\bar{B}$ is separable over its center.*

(2) *$\bar{b}$ is separable over its center.*

(3) *$D$ is abelian and the inertial index of $B$ is one.*

*Furthermore, if $B$ is separable over its center, then it is isomorphic to a full matrix ring over $R[D]$ (of some degree).*

Before proceeding with the proof, we introduce some additional notations. Let $C_1, C_2, \cdots, C_r$ be the $p$-regular classes of $G$ and let $S_i$ be the $p'$-section containing $C_i$ $(1\leq i\leq r)$. Let furthermore $\hat{C}_i(\hat{S}_i$ resp.$)$ denote the class sum of $C_i(S_i$ resp.$)$ in the group algebra $k[G]$ and $Z'_p=\sum_{1\leq i\leq r} k\hat{S}_i$. For a $p$-subgroup $P$ of $G$, $Z_P$ denotes the $k$-subspace of $Z=Z(k[G])$ spanned by the set $\{\hat{C}; C$ is a conjugate class with a defect group contained in $P\}$ and we put $Z_P^0=\sum_{Q<P} Z_Q$. As is well known, $Z_P$ is an ideal of $Z$. We define a linear function $\lambda$ of $k[G]$ by $\lambda(\sum_{g\in G} a_g g)=a_1$, where $1$ denotes the identity element of $G$ and $a_g\in k$. Remember that $\lambda$ is non-singular in the sense that its kernel contains no non-zero right or left ideals.

For the convenience of the reader, we list up here some preliminary results with brief proofs. The first one is due to Brauer [3]. Several proofs are available now. A relatively simple proof (by Iizuka) is given below. We put $J=J(k[G])$ for brevity.

**Lemma 1.** $J=\{x\in k[G]; xZ'_p=0\}$

Proof. Let $\{\bar{\psi}_1, \bar{\psi}_2, \cdots, \bar{\psi}_r\}$ be the set of irreducible $k$-characters of $G$. We see easily that $J=\{x\in k[G]; \psi_i(k[G]x)=0$ for all $i$, $1\leq i\leq r\}$, in fact, the latter ideal contains no idempotent. On the other hand, using that the matrix $[\bar{\psi}_i(x_j)]$ is non-singular $(x_j\in C_j)$, we get by a direct computation that for $a=\sum_{h\in G} a_g g\in k[G]$, $\bar{\psi}_i(a)=0$ for all $i$ if and only if $\sum_{g\in S_j} a_g=\lambda(a\hat{S}_{j*})=0$ for all $j$ $(1\leq i,j\leq r)$, where $S_{j*}=\{g^{-1}; g\in S_j\}$. Our assertion follows from this and the equality $a\hat{S}_i=\sum_{g\in G}\lambda(g^{-1}a\hat{S}_i)g$.

The second Lemma is known more generally to symmetric algebras (Nakayama).

**Lemma 2.** $(0:J)=k[G]c$, *for some* $c \in Z$.

Proof. Let $c = \sum_{g \in G} \mu(g^{-1})g$, where $\mu = \sum_{1 \leq i \leq r} \bar{\psi}_i$.

We see easily that $\lambda(cx) = \mu(x)$ for all $x \in k[G]$. On the other hand the linear function $\mu: k[G] \rightarrow k$ is essentially the sum of the trace functions of the simple components of the split $k$-algebra $k[G]/J$ and so it is non-singular as a linear function of $k[G]/J$. Thus it follows from the above equality that $(0:c)=J$, equivalently $(0:J)=k[G]c$, as $k[G]$ is Frobeniusean.

We now have

**Lemma 3.** $Z'_p = (0:J) \cap Z$.

Proof. Using the notation in the above Lemma, the map $k[G] \ni x \rightarrow xc \in (0:J)$ induces an isomorphism $k[G]/J \simeq (0:J)$, which commutes with the action of $G$ by conjugation. Therefore, we have $\dim_k(0:J) \cap Z = \dim_k Z(k[G]/J) = |\mathrm{IBr}(G)|$ $= r = \dim_k Z'_p$. Since we have $(0:J) \cap Z \supset Z'_p$ by Lemma 1, we get the equality as desired.

As a corollary of the above Lemma and its proof, we have

**Lemma 4.** *Let $e$ be a block idempotent of a $p$-block $B$. Then* $\dim_k \bar{e}Z'_p = |\mathrm{IBr}(B)|$.

The final Lemma is a beautiful formula remarked by Iizuka and Watanable [10] (For the proof, use the fact that $\chi(\hat{S}_i)/|G| \in R$ for any $\chi \in \mathrm{Irr}(G)$ (Frobenius))

**Lemma 5.** *Let $S_1$ be the $p'$-section consisting of the $p$-elements of $G$. If $\bar{e}$ is a block idempotent of a block of $k[G]$ and $\bar{e} = \sum_{1 \leq i \leq r} a_i \hat{C}_i$, $a_i \in k$, then we have* $\hat{S}_1 \bar{e} = \sum_{1 \leq i \leq r} a_i \hat{S}_i$.

Proof (of the Theorem 3). We need only to show that (1) implies (3). Let $\bar{e}$ be the block idempotent of $\bar{B}$. We have that $\bar{e}Z_D = \bar{e}Z$, as is well known. Let $\sigma$ be the Brauer homomorphism from $Z = Z(k[G])$ into $Z(k[N])$, where $N = N_G(D)$. It induces a $k$-algebra isomorphism $Z_D/Z_D^0 \simeq Z(k[N])_D/Z(k[N])_D^0$ and then

$$\bar{e}Z/\bar{e}Z_D^0 \simeq \sigma(\bar{e})Z(k[N])/\sigma(\bar{e})Z(k[N])_D^0 .$$

Now, by Proposition 1 (2) and Lemma 3, we have $\bar{e}Z'_p = (0: \bar{e}J(Z))$, which means that $\bar{e}Z'_p$ is the socle of $\bar{e}Z$. Furthermore, since $\dim_k \bar{e}Z'_p = 1$ by Proposition 1 (9) and Lemma 4, it must be contained in every non-zero ideal of $\bar{e}Z$. In particular, we have $\bar{e}Z_D^0 = 0$ since $\bar{e}Z_D^0 \not\supset \bar{e}Z'_p$ by Lemma 5. So the above isomorphism may be written as

$(1)° \quad \bar{e}Z \simeq \sigma(\bar{e})Z(k[N])/\sigma(\bar{e})Z(k[N])_D^0$

Let $\bar{e}_1$ be a primitive idempotent of $Z(k[DC_G(D)])$ with $\bar{e}_1 = \sigma(\bar{e})\bar{e}_1$ and $T = I_N(\bar{e}_1)$ its inertial group in $N$. As is proved easily, the relative trace map $Tr_T^N$:

$$Z(k[T]) \ni x \to \sum_{u \in N/T} u^{-1}xu \in Z(k[N]) \quad \text{induces}$$

a $k$-algebra isomorphism $\bar{e}_1 Z(k[T]) \simeq \sigma(\bar{e})Z(k[N])$ (see Broué [5]) and so we have

$(2)° \quad \bar{e}_1 Z(k[T])/\bar{e}_1 Z(k[T])_D^0 \simeq \sigma(\bar{e})Z(k[N])/\sigma(\bar{e})Z(k[N])_D^0$

On the other hand, the relative trace map $Tr_{DC_G(D)}^T$ induces an epimorphism (remember that $[T : DC_G((D)]$ is prime to $p$),

$(3)° \quad \bar{e}_1 Z(k[DC_G(D)]) \to \bar{e}_1 Z(k[T])/\bar{e}_1 Z(k[T])_D^0 \to 0$

Furthermore, we know the following (Broué [5])

$(4)° \quad \bar{e}_1 Z(k[DC_G(D)]) = \bar{e}_1 Z(k[D]) \quad (\simeq Z(k[D]))$

Since we have $\dim_k \bar{e}Z = |D|$ by Proposition 1 (9), we get from the above that $\dim_k Z(k[D]) \geq |D|$, which of course means simply the equality and then the relative trace map $Tr_{DC_G(D)}^T$ is a monomorphism. Therefore we have that $D$ is abelian and $T = C_G(D)$. This proves that (1) implies (3). The latter statement follows from (3) and a remark by Broué and Puig [6] (see the section one of it).

The third condition in the above Theorem may be written in terms of the canonical character of $R[DC_G(D)]e_1$, namely,

**Corollary 1.** *There exists a central separable $p$-block with an (abelian) defect group $D$ if and only if there exists a $\chi \in \mathrm{Irr}(C_G(D)/D)$ of $p$-defect zero with $I_{N_G(D)/D}(\chi) = C_G(D)/D$.*

In the rest we shall show some applications of the above Theorem. First we prove,

**Theorem 4.** *Let $B$ be a $p$-block of $G$ with block idempotent $e$ and let $e = \sum_{i=1}^t f$ be the decomposition of $e$ into the sum of central primitive idempotents $f_i$'s of $L[G]$. Then $R[G]f_i$ is a maximal $R$-order in $L[G]f_i$ for all $f_i$ if and only if $B$ is central separable.*

Proof. Suppose that $B$ is central separable. By Theorem 3, $B$ is isomorphic to the full matrix ring $M(n, R[D])$ for some $n$, where $D$ is a defect group of $B$. Extend the isomorphism naturally to $\varphi: L[G]e \simeq M(n, L[D])$. If $\{g_1, g_2, \cdots, g_m\}$ is the set of the central primitive idempotents of $L[D]$, then $m$ is equal to $t$ and $M(n, L[D]) = \bigoplus_{t=1}^m M(n, L[D]g_i) = \bigoplus_{i=1}^m M(n, L[D])g_i I_n$, where $I_n$ denotes the identity

matrix of degree $n$. We may assume that $\varphi(f_i)=g_i I_n$ after a suitable arrangmente of indices if necessary. Then $\varphi(R[G]f_i) = M(n, R[D])g_i I_n = M(n, R[D]g_i) \subset M(n, L[D]g_i)$. Using now that $D$ is abelian, we have that $L[D]g_i \simeq L$ and $R[D]g_i \simeq R$. Therefore $R[G]f_i$ is isomorphic to $M(n, R)$, which is maximal in $M(n, L) \simeq L[G]f_i$. The converse is an easy deducement of Proposition 1 and the following Lemma.

**Lemma 6.** *Let* $\chi \in \mathrm{Irr}(G)$ *and* $f$ *the central primitive idempotent of* $L[G]$ *corresponding to* $\chi$. *Then* $R[G]f$ *is a maximal* $R$-*order in* $L[G]f$ *if and only if* $\chi$ *is modularly irreducible.*

Proof. Let $T$ be an integral representation of degree $s$ which affords the character $\chi$, so it gives an isomorphism $L[G]f \simeq M(s, L)$, which sends $R[G]f$ into $M(s, R)$. If $R[G]f$ is maximal, $T(R[G]f)$ must be $M(s, R)$. In particular, by reducing modulo the maximal ideal of $R$, we get that $\{\bar{T}(g); g \in G\}$ generate the full matrix ring $M(s, k)$ over $k$, which implies $\bar{T}$ is irreducible. Suppose conversely that $\bar{T}$ is irreducible. Then $\bar{T}: R[G]f/(\pi)R[G]f \to M(s, k)$ is an epimorphism, where $(\pi)$ denotes the maximal ideal of $R$. So $T: R[G]f \to M(s, R)$ must be an epimorphism by Nakayama's Lemma and hence an isomorphism. Therefore $R[G]f$ is maximal, completing the proof of Lemma 6 (and that of Theorem 4).

So far we have considered various conditions on blocks, which eventually lead to the conclusion of the central separability of them. In this connection, we mention the following, which refines a result of Müller [14].

**Theorem 5** (Müller [14]). *Let* $B$ *be a* $p$-*block with defect group* $D$. *If* $Z(\bar{B})$ *is quasi-Frobeniussean, then* $B$ *is central separable. In particular,* $D$ *is abelian and* $Z(\bar{B}) \simeq k[D]$.

Proof. By Lemmas 1 and 2 of Müller [14], $B$ is central separable. So our assertion is clear by Proposition 1 and Theorem 3.

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