Miyanishi, M.
Osaka J. Math.
19 (1982), 901-921

REGULAR SUBRINGS OF A POLYNOMIAL RING, II

Dedicated to Professor Yozô Matsushima on his sixtieth birthday

MASAYOSHI MIYANISHI

(Received December 26, 1980)

Introduction. This is a continuation of the previous work of the author's [7] on a finitely generated, two-dimensional, regular subring contained in a polynomial ring. Let k be an algebraically closed field of characteristic zero, which we fix as the ground field throughout this article. Let $X=\mathrm{Spec}(A)$ be a nonsingular affine surface defined over k. *An $\boldsymbol{A}^1$-fibraton on X over a curve Y* is a surjective morphism $\rho: X\rightarrow Y$ from X to a nonsingular curve Y whose general fibers are isomorphic to the affine line $\boldsymbol{A}_k^1$. It is known that every fiber of ρ is supported by a disjoint union of irreducible components, each of which is isomorphic to $\boldsymbol{A}_k^1$ (cf. [7]). Let $F=\rho^*(P)$ be a fiber of ρ lying over a point P of Y, and write $F=\sum_{i=1}^{s} n_iC_i$, where C_i is isomorphic to $\boldsymbol{A}_k^1$ and $n_i>0$ for every i. We say that F is *a singular fiber of the first kind* (resp. *the second kind*) if $s\geqq 2$ and $n_i=1$ for some i (resp. $n_i\geqq 2$ for every i). We also say that F is *a multiple fiber of multiplicity* μ if $\mu:=\mathrm{G.C.D.}(n_1, \cdots, n_s)>1$.

Let $R:=k[u_1, \cdots, u_r]$ be a polynomial ring of dimension r over k, and let A be a finitely generated, two-dimensional, regular k-subalgebra of R. Let $X:=\mathrm{Spec}(A)$, which is a nonsingular affine rational surface. We know that the group A^* of invertible elements of A coincides with $k^*:=k-(0)$, that X has logarithmic Kodaira dimension $\bar{\kappa}(X)=-\infty$, and that A is isomorphic to a polynomial ring of dimension 2 over k provided A is a unique factorization domain (cf. [7]). The condition that $\bar{\kappa}(X)=-\infty$ implies that there exists an $\boldsymbol{A}^1$-fibration $\rho: X\rightarrow Y$ over a nonsingular curve Y (cf. Miyanishi-Sugie [8], Fujita [2]). In the present case, since X is dominated by the affine r-space $\boldsymbol{A}_k^r=\mathrm{Spec}(R)$, Y is isomorphic to $\boldsymbol{A}_k^1$ or the projective line $\boldsymbol{P}_k^1$.

The purpose of this paper is to study the converse: When is a nonsingular affine surface X with an $\boldsymbol{A}^1$-fibration ρ over $\boldsymbol{A}_k^1$ or $\boldsymbol{P}_k^1$ dominated by $\boldsymbol{A}_k^r$ $(r\geqq 2)$? If $X=\mathrm{Spec}(A)$ has an $\boldsymbol{A}^1$-fibration over $\boldsymbol{A}_k^1$, we can give the following criterion (Theorem 3.3):

X is dominated by $\boldsymbol{A}_k^r$, that is, A is contained in R as a k-subalgebra, if and only if ρ has at most one singular fiber of the second kind.

This is done by solving a Diophantine equation in $k[u_1, \cdots, u_r]$ (Theorem 1.2). Meanwhile, if $X=\mathrm{Spec}(A)$ has an $\boldsymbol{A}^1$-fibration over $\boldsymbol{P}_k^1$, the situation becomes very much complicated. Namely, in order to discuss the embeddability of A into $k[u_1, \cdots, u_r]$ in full generality, we have to know what the solutions of the following Diophantine equation in $k[u_1, \cdots, u_r]$ look like:

$$x_1^{a_1} \cdots x_l^{a_l}+y_1^{b_1} \cdots y_m^{b_m}+z_1^{c_1} \cdots z_n^{c_n} = 0\,,$$

where $a_i \geqq 2$, $b_j \geqq 2$, $c_s \geqq 2$ for every index i $(1 \leqq i \leqq l)$, j $(1 \leqq j \leqq m)$, s $(1 \leqq s \leqq n)$. We only give partial answers to the embeddability problem in terms of multiple fibers of ρ, which are stated as follows:

(1) Assume that A is contained in R as a k-subalgebra. Then the fibration ρ has at most three multiple fibers. If ρ has three multiple fibers, their multiplicities $\{\mu_1, \mu_2, \mu_3\}$ are given, up to permutation, by one of the following triplets: $\{2, 2, n\}$ $(n \geqq 2)$, $\{2, 3, 3\}$, $\{2, 3, 4\}$ and $\{2, 3, 5\}$ (cf. Theorem 3.5).

(2) Assume, conversely, that ρ satisfies the following two conditions:

(i) ρ has no singular fibers of the second kind except at most three multiple fibers, each of which is supported by a single irreducible component;

(ii) if ρ has three multiple fibers, the set of multiplicities $\{\mu_1, \mu_2, \mu_3\}$ is, up to permutation, one of the triplets given in the assertion (1).

Then A is contained in a polynomial ring as a k-subalgebra (cf. Theorem 3.7).

In order to obtain these results, we consider an affine hypersurface S_{p_1,p_2,p_3} in $\boldsymbol{A}_k^3=\mathrm{Spec}(k[x_1, x_2, x_3])$ defined by an equation

$$x_1^{p_1}+x_2^{p_2}+x_3^{p_3} = 0 \quad \text{with} \quad p_1, p_2, p_3 \geqq 2\,,$$

and also a complete intersection Σ_{p_1,p_2,p_3,p_4} in $\boldsymbol{A}_k^4=\mathrm{Spec}(k[x_1, x_2, x_3, x_4])$ defined by equations

$$x_1^{p_1}+x_2^{p_2}+x_3^{p_3}=0 \quad \text{and} \quad ax_1^{p_1}+x_2^{p_2}+x_4^{p_4} = 0$$

with $p_1, p_2, p_3, p_4 \geqq 2$ and $a \in k-\{0, 1\}$. Indeed, we have to compute $\bar{\kappa}(S^*_{p_1,p_2,p_3})$, where $S^*_{p_1,p_2,p_3}=S_{p_1,p_2,p_3}-(0)$, and determine when there exists a dominant morphism from $\boldsymbol{A}_k^r$ to $S^*_{p_1,p_2,p_3}$ or $\Sigma^*_{p_1,p_2,p_3,p_4}:=\Sigma_{p_1,p_2,p_3,p_4}-(0)$ (cf. Theorems 2.8 and 2.15).

The terminology and the notations in this article conform to the use in the previous paper [7] and the general current practice. We shall list up the notations in frequent use.

$\boldsymbol{A}_k^r$: the affine space of dimension r defined over k;

$\boldsymbol{P}_k^r$: the projective space of dimension r defined over k;

$\bar{\kappa}(X)$: the logarithmic Kodaira dimension of a nonsingular algebraic variety X;

A^*: the multiplicative group consisting of the invertible elements of A;

$(a_1, \cdots, a_n)$ (or $G.C.D.\,(a_1, \cdots, a_n)$): the greatest common divisor of positive integers $a_1, \cdots, a_n$;

L.C.M. $(a_1, \cdots, a_n)$: the least common multiple of positive integers $a_1, \cdots, a_n$;
$\{a_1, \cdots, a_n\}$: an n-tuple of integers;
$D \sim D'$: a divisor D is linearly equivalent to a divisor D';
For a dominant morphism $\pi: X \to C$ and a point P of C, π^*P denotes the (scheme-theoretic) complete inverse image, and $\pi^{-1}(P)$ denotes the set-theoretic inverse image.

1. A Diophantine equation, I

1.1. Let $R := k[u_1, \cdots, u_r]$ be a polynomial ring of dimension r over k. Let us consider a Diophantine equation in $(m+n)$-variables,

$$x_1^{a_1} \cdots x_m^{a_m} - y_1^{b_1} \cdots y_n^{b_n} = 1 , \qquad \cdots\cdots\cdots\cdots\cdots\cdots (1)$$

where $m, n \geqq 1$ and a_i's and b_j's are integers larger than 1, and look for its solutions in R. A solution $\{x_i = f_i, y_j = g_j; 1 \leqq i \leqq m, 1 \leqq j \leqq n\}$ is called *a constant solution* if $f_i \in k$ and $g_j \in k$ for every i and every j. Otherwise, it is called *a non-constant solution.*

1.2. We shall prove the following

Theorem. *A non-constant solution of the equation* (1) *in R has one of the following forms*:

(1) $x_i = 0$ *for some* $1 \leqq i \leqq m$, $y_j = c_j \in k$ *for every* $1 \leqq j \leqq n$, *where* $c_1^{b_1} \cdots c_n^{b_n} = -1$;

(2) $y_j = 0$ *for some* $1 \leqq j \leqq n$, *and* $x_i = c_i \in k$ *for every* $1 \leqq i \leqq m$, *where* $c_1^{a} \cdots c_m^{a_m} = 1$.

The proof will be given in the paragraph 1.3.

1.3. Let $\{x_i = f_i, y_j = g_j\}$ be a non-constant solution such that $f_i \notin k$ and $g_j \notin k$ for some i and j. By reducing the number of variables in the equation (1) if necessary, we may assume that $f_i \notin k$ and $g_j \notin k$ for every $1 \leqq i \leqq m$ and every $1 \leqq j \leqq n$.

On the other hand, we may assume that R is a polynomial ring in one variable u. In effect, let $\gamma_1(u), \cdots, \gamma_r(u)$ be sufficiently general polynomials in $k[u]$, and let $\varphi_i := f_i(\gamma_1(u), \cdots, \gamma_r(u))$ and $\psi_j := g_j(\gamma_1(u), \cdots, \gamma_r(u))$. Then $\{x_i = \varphi_i, y_j = \psi_j\}$ is a non-constant solution of the equation (1) in $k[u]$ such that $\varphi_i \notin k$ and $\psi_j \notin k$ for every $1 \leqq i \leqq m$ and every $1 \leqq j \leqq n$. Such polynomials $\gamma_1(u), \cdots, \gamma_r(u)$ exist because k is an infinite field. If we can show the non-existence of such a solution in $k[u]$, it implies the non-existence of a non-constant solution of (1) in R such that $f_i \notin k$ and $g_j \notin k$ for some i and j. Thus, we may assume that $R = k[u]$.

By replacing again the equation (1) by an equation of the same kind in more unknowns if necessary, we may assume that $f_i = c_i(u - \alpha_i)$ and $g_j = d_j(u_j - \beta_j)$, where $c_i, \alpha_i, d_j, \beta_j \in k$, and $\alpha_i \neq \alpha'_i$, $\beta_j \neq \beta'_j$ whenever $i \neq i'$ and $j \neq j'$. Finally, we obtain a relation in a variable u,

$$c(u-\alpha_1)^{a_1}\cdots(u-\alpha_m)^{a_m}-d(u-\beta_1)^{b_1}\cdots(u-\beta_n)^{b_n}=1\,, \quad \cdots\cdots\cdots\cdots(2)$$

where $c, d \in k^*$. We shall show that such identity in u is impossible.

Note that every α_i is distinct from $\beta_1, \cdots, \beta_n$ and every β_j is distinct from $\alpha_1, \cdots, \alpha_m$. By differentiating both hand sides of the equation (2) in u, we obtain a relation,

$$c\prod_{i=1}^{m}(u-\alpha_i)^{a_i}\cdot\left\{\sum_{i=1}^{m}\frac{a_i}{u-\alpha_i}\right\}=d\prod_{j=1}^{n}(u-\beta_j)^{b_j}\cdot\left\{\sum_{j=1}^{n}\frac{b_j}{u-\beta_j}\right\}. \quad \cdots\cdots\cdots(3)$$

Note that we have

$$\deg\left(\prod_{i=1}^{m}(u-\alpha_i)\cdot\left\{\sum_{i=1}^{m}\frac{a_i}{u-\alpha_i}\right\}\right)\leqq m-1, \text{ and}$$

$$\deg\left(\prod_{j=1}^{n}(u-\beta_j)\cdot\left\{\sum_{j=1}^{n}\frac{b_j}{u-\beta_j}\right\}\right)\leqq n-1\,.$$

Since $a_i \geqq 2$ and $b_j \geqq 2$ by assumption, the relation (3) implies that

$$\prod_{j=1}^{n}(u-\beta_j)\cdot\left\{\sum_{j=1}^{n}\frac{b_j}{u-\beta_j}\right\}$$

is divisible by $\prod_{i=1}^{m}(u-\alpha_i)$. Hence we obtain $m \leqq n-1$. Similarly, we have $n \leqq m-1$. This is a contradiction. Therefore, we have shown that if $\{x_i=f_i, y_j=g_j\}$ is a non-constant solution of the equation (1), then either $f_i \in k$ for every $1 \leqq i \leqq m$ or $g_j \in k$ for every $1 \leqq j \leqq n$.

Suppose that the first case takes place, i.e., $f_i=c_i \in k$ for every $1 \leqq i \leqq m$. Then $g_j \notin k$ for some j. If $\prod_{i=1}^{m} c_i^{a_i} \neq 1$, then g_j would be a unit in R; this is a contradiction. Hence $\prod_{i=1}^{m} c_i^{a_i}=1$, and $g_j=0$ for some j. The other case can be treated in a similar way. Q.E.D.

2. A Diophantine equation, II

2.1. In this section, we shall consider a Diophantine equation

$$x_1^{p_1}+x_2^{p_2}+x_3^{p_3}=0\,, \quad \cdots\cdots\cdots\cdots\cdots\cdots(4)$$

where p_1, p_2 and p_3 are integers larger than 1, and look for non-constant solutions in $R:=k[u_1, \cdots, u_r]$. Let S_{p_1,p_2,p_3} be the affine hypersurface in $A_k^3:=\mathrm{Spec}(k[x_1, x_2, x_3])$ defined by the equation (4), and let $S^*_{p_1,p_2,p_3}:=S_{p_1,p_2,p_3}-(0)$, where (0) is the point (0, 0, 0). When there is no fear of confusion, we denote S_{p_1,p_2,p_3} and $S^*_{p_1,p_2,p_3}$ simply by S and S^*, respectively. It is easy to see that S

is a normal surface with the unique singular point (0). The resolution of singularity of S at the point (0) is completely understood (cf. Orlik-Wagreich [10]). We recall some of the results which we need in our subsequent arguments.

2.2. Let G_m be the multiplicative group scheme defined over k. We need the following:

Lemma. *Let X be a nonsingular quasi-projective surface with an effective separated G_m-action. Assume that X has no fixed points. Let $Y:=X/G_m$ be the quotient variety and let $\pi\colon X\to Y$ be the canonical projection. Then we have:*

(1) *Y is a nonsingular curve;*

(2) *$\pi^{-1}(y)\cong A^1_*$ for every point $y\in Y$, where A^1_* is the affine line A^1_k with one point deleted off;*

(3) *π^*y is a multiple fiber with multiplicity μ if and only if the stabilizer group σ_x is a cyclic group of order μ for a point x in $\pi^{-1}(y)$.*

Proof. Let x be a point of X. By virtue of Sumihiro [11; Cor. 2], there exists a G_m-stable affine open neighborhood $U:=\mathrm{Spec}(A)$ of x. Let B be the subalgebra of G_m-invariants in A. Then $\bar{U}:=\mathrm{Spec}(B)$ is an affine open neiphborhood of $y:=\pi(x)$. Since A is regular, B is normal. Hence Y is a nonsingular curve. It is known by the theory of quotient varieties with respect to reductive group actions (e.g., Mumford [9; Chap. 1]) that $\pi^{-1}(y)$ consists of a single orbit under the stated assumption. Hence the assertion (2) holds.

Consider a G_m-equivariant completion $X\to Z$, where we may assume that Z is a nonsingular projective surface (cf. Sumihiro [11]). Let $O(x)$ be the orbit through x, and let C be the closure of $O(x)$ in Z. Then C contains a fixed point z. We can find a system of local coordinates (u, v) at z such that $u=0$ defines a branch of C through z and the induced G_m-action on the tangent space $T_{z,Z}$ is normalized as $t(\xi, \eta)=(t^\alpha\xi, t^\beta\eta)$, where $t\in k^*$, α and β are integers and $\xi=\partial/\partial u$ and $\eta=\partial/\partial v$. Replacing the G_m-action $(t, z)\mapsto {}^tz$ on Z by a G_m-action $(t, z)\mapsto {}^{t^{-1}}z$ and interchanging the roles of u and v if necessary, we may assume that $\beta>0$. Since $\hat{\mathcal{O}}_{z,Z}\cong k[[u, v]]$, α and β are prime to each other; if $\alpha=0$ then $\beta=1$. Let $y:=\pi(x)$. Then $\hat{\mathcal{O}}_{y,Y}\cong k[[u^\beta v^{-\alpha}]]$, and the orbit $O(x)$ is defined by $u=0$ in a neighborhood of z. Hence the multiplicity of π^*y is β, and the stabilizer group of a point (hence of the point x) of the orbit $O(x)$ is $\boldsymbol{Z}/\beta\boldsymbol{Z}$. Hence the assertion (3) holds true. Q.E.D.

2.3. Let p_1, p_2 and p_3 be the same as for the equation (4). Let $d:=L.C.M.(p_1, p_2, p_3)$ and define the integers q_i $(1\leqq i\leqq 3)$ by $d=p_iq_i$. The group scheme G_m acts effectively on $S^*_{p_1,p_2,p_3}$ by

$$t(x_1, x_2, x_3) = (t^{q_1}x_1, t^{q_2}x_2, t^{q_3}x_3)\ .$$

Then $S^*_{p_1,p_2,p_3}$ has no fixed points. Let $C:=S^*/G_m$ and let $\pi\colon S^*\to C$ be the canonical projection. Then we have:

Lemma. (1) *The genus g of C is given by*

$$g=\frac{d^2}{2q_1q_2q_3}-\frac{d}{2}\left\{\frac{(q_1,q_2)}{q_1q_2}+\frac{(q_2,q_3)}{q_2q_3}+\frac{(q_3,q_1)}{q_3q_1}\right\}+1\,.$$

(2) *π has no multiple fibers but possibly $\frac{d(q_1,q_2)}{q_1q_2}$ fibers with multiplicity (q_1,q_2), $\frac{d(q_2,q_3)}{q_2q_3}$ fibers with multiplicity (q_2,q_3) and $\frac{d(q_3,q_1)}{q_3q_1}$ fibers with multiplicity (q_3,q_1).*

Proof. (1) Let T be the hypersurface in $A_k^3:=\mathrm{Spec}(k[y_1,y_2,y_3])$ defined by $y_1^d+y_2^d+y_3^d=0$, and let $T^*:=T-(0)$. Let Φ: $T^*\to S^*$ be the morphism defined by $(x_1,x_2,x_3)\mapsto(y_1^{q_1},y_2^{q_2},y_3^{q_3})$. Let G_m act on T^* via $t(y_1,y_2,y_3)=(ty_1,ty_2,ty_3)$. Then Φ is a G_m-equivariant morphism. Let $D:=T^*/G_m$. Then Φ induces a surjective morphism φ: $D\to C$ such that $\pi\cdot\Phi=\varphi\cdot\pi'$, where π': $T^*\to D$ is the canonical quotient morphism. Then it is easy to show that $\deg\varphi=q_1q_2q_3$ and the morphism φ ramifies at d points (on D) with ramification index $q_3(q_1,q_2)$, at d points with ramification index $q_2(q_3,q_1)$ and at d points with ramification index $q_1(q_2,q_3)$. Since D has genus $\frac{1}{2}(d-1)(d-2)$, the genus g of C is obtained by the Riemann-Hurwitz formula applied to φ: $D\to C$. The assertion (2) can be verified by means of Lemma 2.2. Q.E.D.

2.4. Let p_i $(1\leqq i\leqq 4)$ be integers larger than 1. Let Σ_{p_1,p_2,p_3,p_4} be the surface in $A_k^4:=\mathrm{Spec}(k[x_1,x_2,x_3,x_4])$ defined by equations,

$$x_1^{p_1}+x_2^{p_2}+x_3^{p_3}=0 \quad\text{and}\quad ax_1^{p_1}+x_2^{p_2}+x_4^{p_4}=0\,,$$

where $a\in k-\{0,1\}$. Let $\Sigma^*_{p_1,p_2,p_3,p_4}:=\Sigma_{p_1,p_2,p_3,p_4}-(0)$; we denote these objects by Σ and Σ^* if there is no fear of confusion. Then Σ^* is a nonsingular surface with an effective action of the group scheme G_m defined by

$$t(x_1,x_2,x_3,x_4)=(t^{q_1}x_1,t^{q_2}x_2,t^{q_3}x_3,t^{q_4}x_4)\,,$$

where the integers q_i $(1\leqq i\leqq 4)$ are defined by

$$d=p_iq_i\ (1\leqq i\leqq 4)\quad\text{and}\quad d=L.C.M.\,(p_1,p_2,p_3,p_4)\,.$$

The G_m-action on Σ^* given above has no fixed points. Let $C:=\Sigma^*/G_m$ and let π: $\Sigma^*\to C$ be the canonical quotient morphism. We have the following:

Lemma. (1) *The genus g of C is given by the formula*:

$$g=\frac{d^3}{q_1q_2q_3q_4}-\frac{d^2}{2}\left\{\frac{(q_1,q_2,q_3)}{q_1q_2q_3}+\frac{(q_1,q_2,q_4)}{q_1q_2q_4}+\frac{(q_1,q_3,q_4)}{q_1q_3q_4}+\frac{(q_2,q_3,q_4)}{q_2q_3q_4}\right\}+1\,.$$

(2) *π has no multiple fibers but possibly* $\frac{d^2(q_1, q_2, q_3)}{q_1q_2q_3}$ *fibers with multiplicity* (q_1, q_2, q_3), $\frac{d^2(q_1, q_2, q_4)}{q_1q_2q_4}$ *fibers with multiplicity* (q_1, q_2, q_4), $\frac{d^2(q_1, q_3, q_4)}{q_1q_3q_4}$ *fibers with multiplicity* (q_1, q_3, q_4) *and* $\frac{d^2(q_2, q_3, q_4)}{q_2q_3q_4}$ *fibers with multiplicity* (q_2, q_3, q_4).

Proof. Similar to the proof of Lemma 2.3.

2.5. As an application of Lemma 2.4, we have the following examples:

$\{p_1, p_2, p_3, p_4\}$	$g(\Sigma^*/G_m)$	multiple fibers of π: $\Sigma^* \to C := \Sigma^*/G_m$
$\{2, 2, 2, 2s\}$	1	4 fibers with multiplicity s
$\{2, 2, 2, 2s+1\}$	0	4 fibers with multiplicity $2s+1$
$\{2, 2, 3, 3\}$	2	no multiple fibers
$\{2, 2, 3, 4\}$	0	2 fibers with multiplicity 2 4 fibers with multiplicity 3
$\{2, 2, 3, 5\}$	0	2 fibers with multiplicity 5 2 fibers with multiplicity 3

2.6. From this paragraph on up to 2.14, we shall retain the notations of 2.1. Let $p_1' := p_1/(q_2, q_3)$, $p_2' := p_2/(q_1, q_3)$ and $p_3' := p_3/(q_1, q_2)$; p_i' $(1 \leqq i \leqq 3)$ are integers because, for example, $d = p_1q_1$ and $(q_1, (q_2, q_3)) = 1$ imply that p_1 is divisible by (q_2, q_3). As an easy application of Lemma 2.3, we know that $g=0$ (resp. $g=1$, resp. $g>1$) if and only if $\frac{1}{p_1'}+\frac{1}{p_2'}+\frac{1}{p_3'}>1$ (resp. $=1$, resp. <1).

2.7. We have the following:

Lemma. *Assume that* $p_1 \leqq p_2 \leqq p_3$ *and* $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1$. *Then we have*:

(1) $\{p_1, p_2, p_3\} = \{2, 3, 6\}$, $\{2, 4, 4\}$ *or* $\{3, 3, 3\}$.

(2) *$C := S^*/G_m$ is a nonsingular elliptic curve, and π: $S^* \to C$ has no multiple fibers, i.e., S^* is an A^1_*-bundle over C.*

(3) *Let $b := d/q_1q_2q_3$. Then $b = 1, 2, 3$ for $\{p_1, p_2, p_3\} = \{2, 3, 6\}$, $\{2, 4, 4\}$ and $\{3, 3, 3\}$, respectively. There exists an invertible sheaf $\mathcal{L}$ of degree b over C such that the ruled surface $V := Proj(\mathcal{O}_C \oplus \mathcal{L})$ over C with the zero section M_0 and the infinity section M_∞ deleted off is isomorphic to S^*.*

(4) $\bar{\kappa}(S^*) = 0$.

Proof. (1) follows from a well-known straightforward computation. (2) follows from Lemma 2.3. Since S^* is an A^1_*-bundle over C, S^* is obtained from a ruled surface in the way as specified in the assertion (3). Then $(M_0^2) = -b$, $(M_\infty^2) = b$ and $(M_0 \cdot M_\infty) = 0$. The number $b := \deg \mathcal{L}$ is equal to $d/q_1q_2q_3$, because M_0 is the unique exceptional curve which arises from the minimal resolution of singularity of the point (0, 0, 0) of S (cf. Orlik-Wagreich [10]). Note that the canonical divisor K_V of V is linearly equivalent to $-M_0-M_\infty$. The boundary divisor of S^* in V is $D:=M_0+M_\infty$. Hence $D+K_V\sim 0$. Therefore, we have $\bar{\kappa}(S^*)=0$. Q.E.D.

2.8. We shall prove

Theorem (cf. Iitaka [4]). *$S^*_{p_1,p_2,p_3}$ has the logarithmic Kodaira dimension $\bar{\kappa}(S^*_{p_1,p_2,p_3})=-\infty$, 0, 1 according as $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}>1$, $=1$, <1, respectively.*

The proof will be given in the paragraphs 2.9~2.11.

2.9. Let V be a nonsingular projective surface with a surjective morphism φ: $V\to C:=S^*/G_m$ satisfying the following conditions:

(i) V contains $S^*_{p_1,p_2,p_3}$ as a dense open set, and $\varphi|_{S^*}=\pi$: $S^*\to C$;

(ii) $V-S^*$ contains no exceptional curves of the first kind which are contained in fibers of φ.

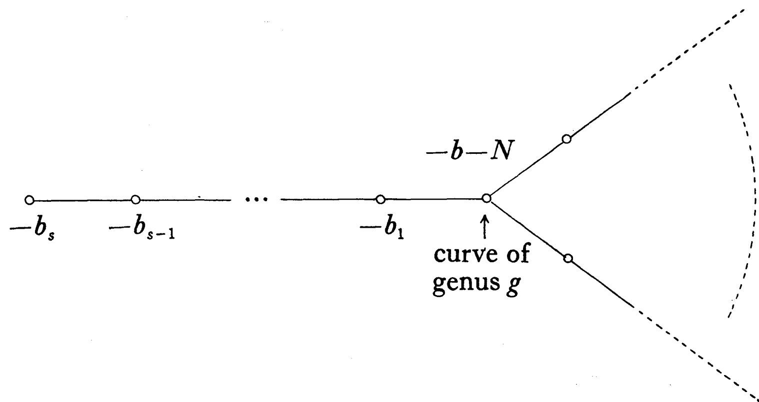
It is clear that general fibers of φ are isomorphic to $\boldsymbol{P}^1_k$. The resolution of singularity of S_{p_1,p_2,p_3} at the unique singular point (0)=(0, 0, 0) is described in detail in Orlik-Wagreich [10]. We recall some of the necessary results. The morphism π: $S^*\to C$ has multiple fibers if one of (q_1, q_2), (q_2, q_3) and (q_3, q_1) is larger than 1. If $(q_1, q_2)>1$, there are $d(q_1, q_2)/q_1q_2$ fibers of multiplicity (q_1, q_2) (cf. Lemma 2.3). For a multiple fiber F of multiplicity (q_1, q_2), set $\alpha:=(q_1, q_2)$ and determine an integer β uniquely by the condition that $q_3\beta\equiv 1 \pmod{\alpha}$ and $0<\beta<\alpha$. Define positive integers $b_1, \cdots, b_s\geqq 2$ by writing $\alpha/(\alpha-\beta)$ in the form of a continued fraction

$$\frac{\alpha}{\alpha-\beta}=b_1-\cfrac{1}{b_2-\cfrac{1}{\ddots-\cfrac{1}{b_s}}},$$

which we write in the form $\alpha/(\alpha-\beta)=[b_1, \cdots, b_s]$. For multiple fibers of multiplicity (q_2, q_3) or (q_1, q_3), we determine the corresponding integers α, β, $b_1, \cdots, b_s$ etc. Let N be the number of the multiple fibers of π. Let

$$b:=\frac{d}{q_1q_2q_3}-\sum_{i=1}^{N}\frac{\beta_i}{\alpha_i},$$

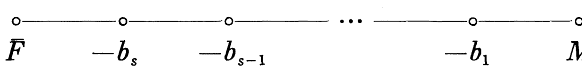
where $\{\alpha_i, \beta_i\}$ ranges over all pairs of integers which are determined for all multiple fibers of π in the above-mentioned fashion. Let g be the genus of C. Then the dual graph of the exceptional curves which arise from the resolution of singularity of the point (0) of S_{p_1,p_2,p_3} has a vertex with weight $-b-N$ (corresponding to a nonsingular curve of genus g) and has N branches, each of which is a linear chain of nonsingular rational curves as exhibited in the following figure:



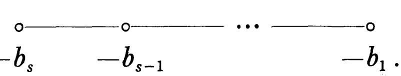
2.10. The fibration $\varphi: V\to C$ has two cross-sections M_0' and M_∞' and N singular fibers $\Phi_1, \cdots, \Phi_N$ such that:

(i) M_0' and M_∞' are nonsingular curves of genus g; $(M_0'^2)=-b-N$ and $(M_\infty'^2)=b$;

(ii) Let Φ be a singular fiber of φ; then $\Phi\cap S^*=\alpha F$ with $F\cong A_*^1$, i.e., a multiple fiber of multiplicity $\alpha>1$; the component $\bar{F}$ of Φ (=the closure of F in V) is connected to the cross-section M_0' by s components as exhibited in

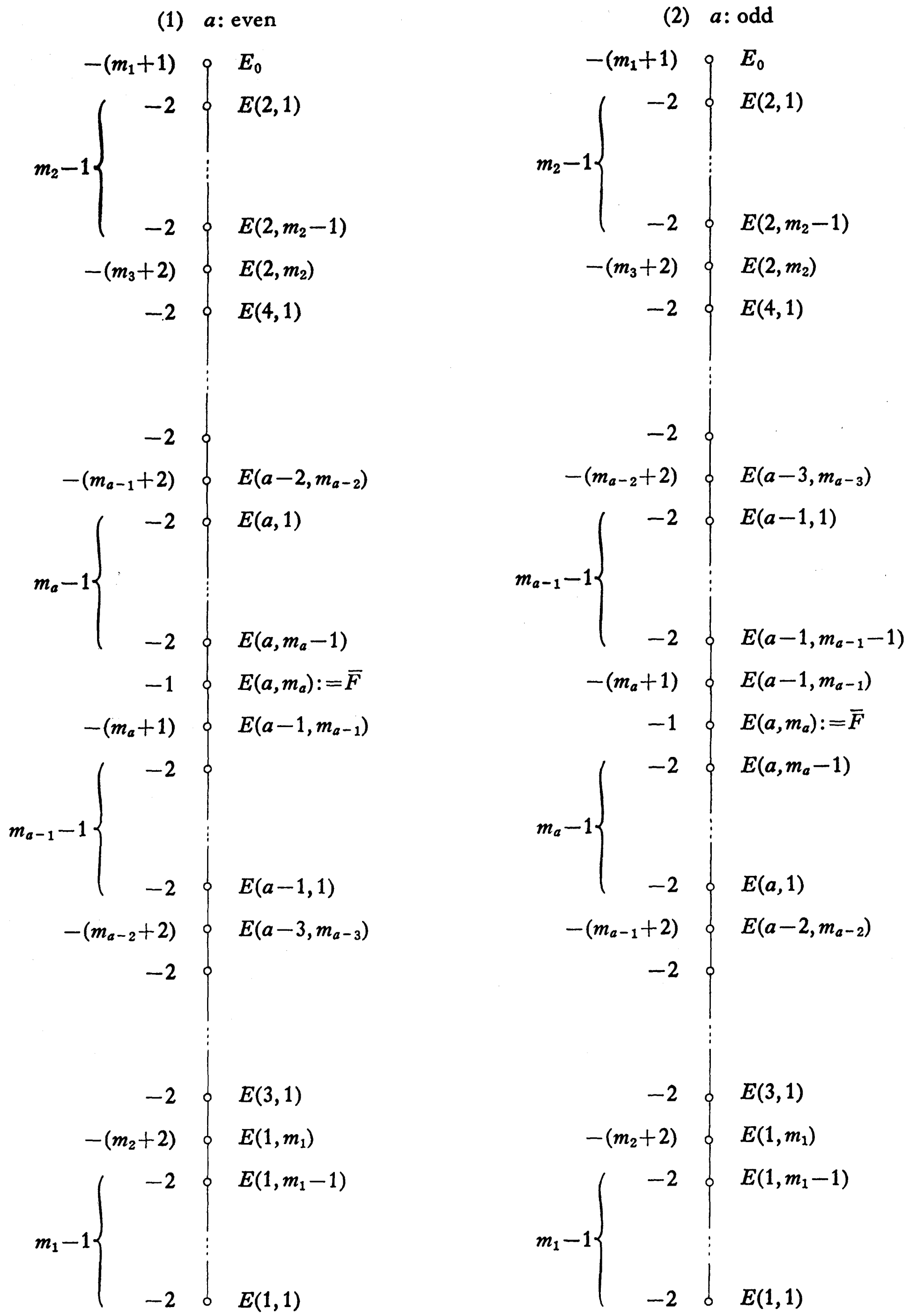


By assumption, $\Phi-F$ contains no exceptional curves of the first kind. Hence $\bar{F}$ is the unique exceptional curve of the first kind contained in the singular fiber Φ. Then it is easily ascertained that the dual graph of the fiber Φ is a linear chain. It looks like the one given in Miyanishi [6; p. 95]. To fix the notations, we represent it in the next page. The upper half of the chain between E_0 and $E(a, m_a)$ (with $E(a, m_a)$ excluded) corresponds to the chain



Hence we have $\dfrac{\alpha}{\alpha-\beta}=[b_1, \cdots, b_s]$

$$=\begin{cases}[m_1+1, \underbrace{2, \cdots, 2}_{m_2-1}, m_3+2, 2, \cdots, 2, m_{a-1}+2, \underbrace{2, \cdots, 2}_{m_a-1}] & \text{if a is even} \\ [m_1+1, \underbrace{2, \cdots, 2}_{m_2-1}, m_3+2, 2, \cdots, 2, m_{a-2}+2, \underbrace{2, \cdots, 2}_{m_{a-1}-1}, m_a+1] & \text{if a is odd .}\end{cases}$$



Note that α is the multiplicity of $\bar{F}$ in the fiber Φ. This is clear because $\Phi\cap S^*=\alpha F$. We can check this fact as follows. The multiplicity $\mu(i,j)$ $(1\leqq i\leqq a;\ 1\leqq j\leqq m_i)$ of the component $E(i,j)$ in Φ is given by the function

$\mu(i, j)$ defined inductively by:

$$\mu(0, m_0) := 1,\ \mu(1, j) = j \qquad \text{for } 1 \leqq j \leqq m_1\,,$$
$$\mu(i, 1) = \mu(i-1, m_{i-1}) + \mu(i-2, m_{i-2}) \qquad \text{for } 1 < i \leqq a\,,$$
$$\mu(i, j) = \mu(i, j-1) + \mu(i-1, m_{i-1}) \qquad \text{for } 1 < j \leqq m_i\,.$$

On the other hand, the integer α is regained by the method as indicated in the appendix of [10; p. 76] from the above development of $\alpha/(\alpha-\beta)$ into a continued fraction.

2.11. Note that $V-S^*$ consists of nonsingular components crossing normally. It is also easy to see that there exists a unique contraction σ: $V \to V_0$, where

(i) φ_0: $V_0 \to C$ is a relatively minimal ruled surface;

(ii) Let $M_0 := \sigma_* M_0'$ and $M_\infty := \sigma_* M_\infty'$; Then $(M_0^2) = -(b+N)$ and $(M_\infty^2) = b+N$.

The canonical divisor K_{V_0} is given by

$$K_{V_0} \sim -M_0 - M_\infty + \varphi_0^*(K_C) \quad \text{and} \quad M_\infty \sim M_0 + \varphi_0^*(\delta)\,,$$

where K_C is the canonical divisor of C and δ is a divisor on C with $\deg(\delta) = b+N$. In effect, $V_0 \cong \mathrm{Proj}(\mathcal{O}_C \oplus \mathcal{O}_C(\delta))$, and M_0 and M_∞ correspond to the zero section and the infinite section of V_0, respectively.

Each irreducible component $E(i, j)$ of the singular fiber has the contribution $k(i, j)$ in the canonical divisor K_V determined inductively as follows:

$$k(0, m_0) := 0,\ k(1, j) = j \qquad \text{for } 1 \leqq j \leqq m_1\,,$$
$$k(i, 1) = k(i-1, m_{i-1}) + k(i-2, m_{i-2}) + 1 \qquad \text{for } 1 < i \leqq a\,,$$
$$k(i, j) = k(i, j-1) + k(i-1, m_{i-1}) + 1 \qquad \text{for } 1 < j \leqq m_i\,.$$

On the other hand, $E(i, j)$ has multiplicity $n(i, j)$ in $\sigma^*(M_\infty)$, which is determined by

$$n(0, m_0) := 0,\ n(1, j) = 1 \qquad \text{for } 1 \leqq j \leqq m_1\,,$$
$$n(i, 1) = n(i-1, m_{i-1}) + n(i-2, m_{i-2}) \qquad \text{for } 1 < i \leqq a\,,$$
$$n(i, j) = n(i, j-1) + n(i-1, m_{i-1}) \qquad \text{for } 1 < j \leqq m_i\,.$$

Let D be the reduced effective divisor such that $\mathrm{Supp}(D) = V-S^*$. Then it is straightforward to show that the coefficient $\nu(i, j)$ of $E(i, j)$ in $D+K_V-\Phi$ is given by,

$$\nu(i, j) = \begin{cases} 0 & \text{if } (i, j) \neq (a, m_a) \\ -1 & \text{if } (i, j) = (a, m_a)\,. \end{cases}$$

Therefore we have:

$$D+K_V \sim \sum_{i=1}^{N} \Phi_i - \sum_{i=1}^{N} \bar{F}_i + \varphi^*(K_C)$$
$$\geqq \sum_{i=1}^{N} \left(1-\frac{1}{\alpha_i}\right)\Phi_i + \varphi^*(K_C)\,,$$

where α_i is the multiplicity of $\bar{F}_i$ in Φ_i. Let

$$A := \left(\sum_{i=1}^{N}\left(1-\frac{1}{\alpha_i}\right)\Phi_i + \varphi^*(K_C)\cdot M_0'\right).$$

Note that α_i has one of the values (q_1, q_2), (q_2, q_3) and (q_3, q_1) (cf. 2.9) and that A is, in effect, equal to

$$\left(\sum_{P\in C}\left(1-\frac{1}{\alpha_P}\right)\varphi^*(P) + \varphi^*(K_C)\cdot M_0'\right),$$

where $\pi^*(P)=\alpha_P F_P$ with $F_P \cong A_*^1$. Then we can calculate A as follows:

$$\begin{aligned} A &= \frac{d(q_1, q_2)}{q_1q_2} + \frac{d(q_2, q_3)}{q_2q_3} + \frac{d(q_3, q_1)}{q_3q_1} - \frac{d(q_1, q_2)}{q_1q_2}\cdot\frac{1}{(q_1, q_2)} \\ &\quad - \frac{d(q_2, q_3)}{q_2q_3}\cdot\frac{1}{(q_2, q_3)} - \frac{d(q_3, q_1)}{q_3q_1}\cdot\frac{1}{(q_3, q_1)} + 2g - 2 \\ &= \frac{d^2}{q_1q_2q_3}\left(1-\frac{1}{p_1}-\frac{1}{p_2}-\frac{1}{p_3}\right). \end{aligned}$$

We have clearly $\bar{\kappa}(S^*)=1$ if $A>0$, because $D+K_V$ is linearly equivalent to a divisor supported by fibers and the components contained in fibers of φ. If $A=0$ we have $\bar{\kappa}(S^*)=0$ (cf. 2.7). If $A<0$, i.e., $1<\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}$, we have the following under an additional assumption $2\leqq p_1 \leqq p_2 \leqq p_3$: $\{p_1, p_2, p_3\}=\{2, 2, n\}$ $(n\geqq 2)$, $\{2, 3, 3\}$, $\{2, 3, 4\}$ or $\{2, 3, 5\}$. In each of the above four cases for $A<0$, the foregoing arguments of evaluating $D+K_V$ shows that $\bar{\kappa}(S^*)=-\infty$; note that if $A<0$ then $g=0$. This completes the proof of Theorem 2.8.

2.12. If $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}>1$, the surface S_{p_1,p_2,p_3} is the quotient variety of A_k^2 with respect to a linear action of a Kleinian subgroup G of $GL(2, k)$ (cf. Brieskorn [1]). In effect, G acts freely on $A_k^2-(0)$. Hence there exists an étale finite morphism ρ: $A_k^2-(0)\to S^*$, and $A_k^2-(0)$ is algebraically simply connected.

Suppose that the ground field k is the field C of complex numbers. Let U be the universal covering space of $S^*_{p_1,p_2,p_3}$. Then it is known(*) that:

(*) This was communicated by Dr. A. Fujiki.

$$U\cong \boldsymbol{C}^2-(0) \Leftrightarrow 1<\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}$$

$$U\cong \boldsymbol{C}^2 \Leftrightarrow 1=\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}$$

$$U\cong \boldsymbol{C}\times D \Leftrightarrow 1>\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3},$$

where D is a unit disc.

2.13. For later use, we shall prove:

Lemma. *Suppose that $\bar{\kappa}(S^*_{p_1,p_2,p_3})>0$ and $C\cong \boldsymbol{P}^1_k$. Then $\pi\colon S^*\to C$ has three or more multiple fibers.*

Proof. We have the inequalities,

$$\frac{(q_2, q_3)}{p_1}+\frac{(q_3, q_1)}{p_2}+\frac{(q_1, q_2)}{p_3}>1>\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3},$$

(cf. 2.6 and 2.8). Hence it is impossible that $(q_2, q_3)=(q_3, q_1)=(q_1, q_2)=1$. If $(q_2, q_3)>1$, $(q_3, q_1)>1$ and $(q_1, q_2)>1$, π has three or more multiple fibers. We shall consider the cases where one or two of (q_2, q_3), (q_3, q_1) and (q_1, q_2) equal 1. Assume first that $(q_2, q_3)=1$, $(q_3, q_1)>1$ and $(q_1, q_2)>1$. Suppose that $d(q_3, q_1)/q_3q_1 = d(q_1, q_2)/q_1q_2=1$. Then $q_3=p_1(q_1, q_3)$ and $q_2=p_1(q_1, q_2)$. Hence (q_2, q_3) is divisible by p_1. Since $p_1>1$, this contradicts the assumption that $(q_2, q_3)=1$. Hence $\frac{d(q_3, q_1)}{q_3q_1}>1$ or $\frac{d(q_1, q_2)}{q_1q_2}>1$. Thus π has three or more multiple fibers. Consider next the case where $(q_2, q_3)=(q_3, q_1)=1$ and $(q_1, q_2)>1$. Then the above inequalities imply that $(q_1, q_2)>p_3$. Hence $q_3(q_1, q_2)>d$, and

$$1\geqq\frac{(q_1, q_2)}{q_2}>\frac{d}{q_2q_3}.$$

However, since $(q_2, q_3)=1$, d is divisible by q_2q_3. This is a contradiction. Thus this case does not occur. The other cases can be treated in a similar fashion. Q.E.D.

2.14. We shall prove the following:

Theorem. (1) *If $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}\leqq 1$, then there are no non-constant morphisms from $\boldsymbol{A}^r_k$ to $S^*_{p_1,p_2,p_3}$.*

(2) *If $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}>1$, then there are dominant morphisms from $\boldsymbol{A}^2_k$ to $S^*_{p_1,p_2,p_3}$.*

Proof. (1) If $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1$, S^* is an $\boldsymbol{A}^1_*$-bundle over a nonsingular elliptic curve C. Thus, if f: $A_k^r \to S^*$ is a non-constant morphism, $f(A_k^r)$ is contained in a fiber of π, which is isomorphic to A_*^1. This is impossible. So, we may assume that $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}<1$, i.e., $\bar{\kappa}(S^*)>0$. Let f: $A_k^r \to S^*$ be a non-constant morphism if such a morphism exists at all. If f is dominant, we may assume without loss of generality that $r=2$. Then we have

$$-\infty = \bar{\kappa}(A_k^2) \geqq \bar{\kappa}(S^*) = 1\,,$$

which is impossible. Hence $f(A_k^r)$ is a rational curve with at most one place at infinity, and $f(A_k^r)$ is not contained in any fiber of π. Thus we have a dominant morphism

$$\psi := \pi \cdot f\colon A_k^r \to S^* \to C\,.$$

Hence C is isomorphic to P_k^1, and $\psi(A_k^r)$ is isomorphic to A_k^1 or P_k^1. Consider first the case where $\psi(A_k^r) \cong A_k^1$. By 2.13, there exist points P, Q of C such that P, $Q \in \psi(A_k^r)$ and that π^*P and π^*Q are multiple fibers of multiplicity μ and ν, respectively. Choose an inhomogeneous coordinate t of A_k^1 such that P and Q are defined by $t=0$ and $t=1$, respectively. Then there exist non-constant polynomials g and h in $R := k[u_1, \cdots, u_r]$ such that $\psi^*(t)=g^\mu$ and $\psi^*(t-1)=h^\nu$. This implies that $\{x=g, y=h\}$ is a solution of the Diophantine equation

$$x^\mu - y^\nu = 1\,.$$

This contradicts Theorem 1.2. Consider next the case where $\psi(A_k^r) \cong P_k^1$. In order to prove, by *reductio ad absurdum*, the non-existence of such a non-constant morphism as ψ, we may assume, by embedding the ground field k into the field C of complex numbers in a suitable way, that $k=C$. Restricting ψ onto a suitable line A_C^1 in A_C^r, we may assume that $r=1$. Then the Nevanlinna theory (cf. Hayman [3]) implies that

$$\sum_{i=1}^{N}\left(1-\frac{1}{\alpha_i}\right)-2 \leqq 0\,,$$

where N is the number of multiple fibers of π and α_i's are multiplicities. The left-hand side of the above inequality is, in effect, equal to A in 2.11. Hence we have $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3} \geqq 1$. This is a contradiction. Thus there are no non-constant morphisms f: $A_k^r \to S^*$ provided $\bar{\kappa}(S^*) \geqq 0$.

(2) We may assume that $p_1 \leqq p_2 \leqq p_3$. Then $\{p_1, p_2, p_3\}$ is one of the following triplets: $\{2, 2, n\}$ $(n \geqq 2)$, $\{2, 3, 3\}$, $\{2, 3, 4\}$, $\{2, 3, 5\}$. Except in the case where $\{p_1, p_2, p_3\} = \{2, 3, 5\}$, one can easily find a solution $\{x_1=f_1, x_2=f_2, x_3=f_3\}$ of the equation

$$x_1^{p_1}+x_2^{p_2}+x_3^{p_3} = 0$$

in a polynomial ring $R:=k[u_1, \cdots, u_r]$ such that the subvarieties $\{f_i=0\}$ $(1\leqq i\leqq 3)$ have no common points in A_k^r and that trans. $\deg_k k(f_1, f_2, f_3)=2$. Then the assignment $x_i\mapsto f_i$ $(1\leqq i\leqq 3)$ gives rise to a dominant morphism f: $A_k^r\rightarrow S^*$. For example, if $\{p_1, p_2, p_3\}=\{2, 2, 2\}$, such a solution is given by

$$x_1=\frac{\xi^2+\eta^2}{2},\ x_2=\frac{\xi^2-\eta^2}{2\sqrt{-1}},\ x_3=\sqrt{-1}\cdot\xi\eta\ ,$$

where ξ, η are polynomials in R such that $\{\xi=0\}$ and $\{\eta=0\}$ have no common points in A_k^r and that trans. $\deg_k k(\xi, \eta)=2$. The case where $\{p_1, p_2, p_3\}=\{2, 3, 5\}$ seems more subtle.(*) We look for a dominant morphism f: $A_k^2\rightarrow S^*$. Since A_k^2 is algebraically simply connected, such a morphism f (if it exists at all) is factored by a dominant morphism $\tilde{f}$: $A_k^2\rightarrow A_k^2-(0)$ such that $f=\rho\cdot\tilde{f}$ (cf. 2.12). Conversely, if a dominant morphism $\tilde{f}$ is given, $f:=\rho\cdot\tilde{f}$ is a required dominant morphism. Hence we have only to find a dominant morphism $\tilde{f}$: $A_k^2\rightarrow A_k^2-(0)$. Such a morphism $\tilde{f}$ exists because a dominant morphism f: $A_k^2\rightarrow S_{2,2,2}^*$ provides one. Note that this argument works also for the other cases. Q.E.D.

2.15. We shall prove:

Theorem. *Let $\Sigma_{p_1,p_2,p_3,p_4}^*$ be the nonsingular surface defined in* 2.4. *Assume that $\{p_1, p_2, p_3, p_4\}$ is one of the following quadruplets*: $\{2, 2, 2, 2s+1\}$ $(s\geqq 1)$, $\{2, 2, 3, 4\}$, $\{2, 2, 3, 5\}$, *i.e., those in the examples in* 2.5 *with* $g(\Sigma^*/G_m)=0$. *Then there are no non-constant morphisms from A_k^r to $\Sigma_{p_1,p_2,p_3,p_4}^*$.*

Proof. We only consider the case where $\{p_1, p_2, p_3, p_4\}=\{2, 2, 2, 3\}$. The other cases can be treated in a similar fashion. Suppose that f: $A_k^r\rightarrow\Sigma^*$ is a non-constant morphism. With the notations of 2.4, C is then isomorphic to P_k^1. Let $\psi:=\pi\cdot f$. Then $\psi(A_k^r)$ is isomorphic to A_k^1 or P_k^1. The case where $\psi(A_k^r)\cong A_k^1$ is impossible because π has four multiple fibers of multiplicity 3 (cf. 2.5 and the proof of Theorem 2.14). Hence $\psi(A_k^r)\cong P_k^1$. Let $3F_i$ $(1\leqq i\leqq 4)$ be the multiple fibers of π. Then $f^*(F_i)$ is defined by $f_i=0$ with $f_i\in R:=k[u_1, \cdots, u_r]$. Since $3F_1\sim 3F_2\sim 3F_3$, for example, we have a relation

$$f_3^3=f_2^3+bf_1^3,\ \text{where}\ b\in k^*\ .$$

Since $f^*(F_1)\cap f^*(F_2)\cap f^*(F_3)=\phi$, we can define a non-constant morphism

$$g\colon A_k^r\rightarrow S_{3,3,3}^*\subset \operatorname{Spec}(k[x_1, x_2, x_3]/(x_1^3+x_2^3+x_3^3))$$

by $g^*(x_1)=b^{1/3}f_1$, $g^*(x_2)=f_2$ and $g^*(x_3)=-f_3$. This is impossible because $S_{3,3,3}^*/G_m$ is an elliptic curve. Q.E.D.

(*) For the following argument, the author owes Dr. A. Fujiki.

3. Regular subrings in a polynomial ring

3.1. Let A be a finitely generated, two-dimensional, regular k-algebra contained in a polynomial ring $R:=k[u_1, \cdots, u_r]$ of dimension r. Let $X:=\mathrm{Spec}(A)$ and let $\boldsymbol{A}_k^r:=\mathrm{Spec}(R)$. Then the inclusion $A \hookrightarrow R$ gives rise to a dominant morphism $f\colon \boldsymbol{A}_k^r \to X$. By restricting f onto a linear plane L in $\boldsymbol{A}_k^r$ which meets general fibers of f in finitely many points, we have a dominant morphism $f_L\colon L \cong \boldsymbol{A}_k^2 \to X$. This implies that A is a k-subalgebra of the two-dimensional polynomial ring. Thus we may assume without loss of generality that $r=2$.

Since $f\colon \boldsymbol{A}_k^2 \to X$ is generically finite, we have $\bar{\kappa}(X)=-\infty$, which follows from the inequality of logarithmic Kodaira dimensions,

$$\bar{\kappa}(X) \leqq \bar{\kappa}(\boldsymbol{A}_k^2) = -\infty .$$

This implies that X contains a cylinderlike open set $U \cong U_0 \times \boldsymbol{A}_k^1$, where U_0 is an affine curve (cf. Miyanishi-Sugie [8]; Fujita [2]). The projection $p\colon U \to U_0$ is induced from a dominant morphism $\rho\colon X \to \boldsymbol{P}_k^1$, where U_0 is an open set of $\boldsymbol{P}_k^1$. Then $\rho(X) \cong \boldsymbol{A}_k^1$ or $\rho(X)=\boldsymbol{P}_k^1$. Indeed, if $\boldsymbol{P}_k^1-\rho(X)$ consists of more than one point, we may write $\rho(X)=\mathrm{Spec}(k[t, h(t)^{-1}])$, where t is an inhomogeneous coordinate of $\boldsymbol{P}_k^1$ and $h(t) \in k[t]-k$; then $k[t, h(t)^{-1}]$ is a k-subalgebra of A (and, hence, of $k[u_1, u_2]$); this contradicts the fact that $A^*=k^*$.

Summing up, we have the following:

Lemma. *Let $X:=\mathrm{Spec}(A)$ be a nonsingular affine surface. Then A is contained in a polynomial ring as a k-subalgebra if and only if there exists a dominant morphism $f\colon \boldsymbol{A}_k^2 \to X$. In this case, we have:*

(1) $A^*=k^*$;

(2) *There exists an $\boldsymbol{A}^1$-fibration $\rho\colon X \to Y$, where $Y \cong \boldsymbol{A}_k^1$ or $\boldsymbol{P}_k^1$;*

(3) *Every fiber of ρ is supported by a disjoint union of irreducible curves, each of which is isomorphic to $\boldsymbol{A}_k^1$.*

For the last assertion, see Miyanishi [7].

3.2. A fiber $\rho^*(P)$ of ρ is *a singular fiber* if either $\rho^{-1}(P)$ is reducible or $\rho^*(P)$ is irreducible and non-reduced. Write $\rho^*(P)=\sum_{i=1}^{s} n_i C_i$, where $C_i \cong \boldsymbol{A}_k^1$ and $n_i>0$. $\rho^*(P)$ is called *a singular fiber of the first kind* if $s \geqq 2$ and $n_i=1$ for some i; $\rho^*(P)$ is called *a singular fiber of the second kind* if $n_i \geqq 2$ for every i. Let $\mu:=G.C.D.\,(n_1, \cdots, n_s)$. If $\mu>1$, the fiber $\rho^*(P)$ is called *a multiple fiber* and μ is called *the multiplicity*.

3.3. We shall prove:

Theorem. *Let $X:=\mathrm{Spec}(A)$ be a nonsingular surface with an $\boldsymbol{A}^1$-fibration $\rho\colon X \to Y$, where $Y \cong \boldsymbol{A}_k^1$. Then A is contained in a polynomial ring as a k-sub-*

algebra if and only if ρ has at most one singular fiber of the second kind.

Proof. (I) Let $f\colon A_k^2 \to X$ be a dominant morphism. Then note that $\rho \cdot f(A_k^2) = Y$. Suppose that ρ has two singular fibers of the second kind $\rho^*(P)$ and $\rho^*(Q)$. Then $f^*\rho^*(P)$ and $f^*\rho^*(Q)$ are defined by the equations

$$g_1^{a_1} \cdots g_m^{a_m} = 0 \quad \text{and} \quad h_1^{b_1} \cdots h_n^{b_n} = 0$$

respectively, where $g_1, \cdots, g_m$ and $h_1, \cdots, h_n$ are non-constant polynomials in $k[u_1, u_2]$ and where $a_i \geqq 2$ $(1 \leqq i \leqq m)$ and $b_j \geqq 2$ $(1 \leqq j \leqq n)$. We may choose an inhomogeneous coordinate t of $Y := \mathrm{Spec}(k[t])$ in such a way that the points P and Q are defined by $t=0$ and $t=1$, respectively. Then we have a relation

$$g_1^{a_1} \cdots g_m^{a_m} - h_1^{b_1} \cdots h_n^{b_n} = 1\,.$$

This is impossible by virtue of Theorem 1.2. Therefore ρ has at most one singular fiber of the second kind provided A is contained in a polynomial ring as a k-subalgebra.

(II) We shall prove the "if" part of the theorem. Let $\rho^*(P) = \sum_{i=1}^{s} n_i C_i$ be a singular fiber of the first kind. We shall show that after replacing X by a suitable affine open set with an A^1-fibration similar to that for X, $\rho^*(P)$ can be assumed to be an irreducible and reduced fiber. For this purpose, embed X into a nonsingular projective surface V as a dense open set. Then $V - X$ consists only of components of codimension 1. Since X is affine, there exists an effective ample divisor D on V such that $\mathrm{Supp}(D) = V - X$. For $\rho^*(P) = \sum_{i=1}^{s} n_i C_i$, suppose that $n_1 = 1$. Then there exists an ample divisor D' on V such that $\mathrm{Supp}(D') = (V - X) \cup \bigcup_{i=2}^{s} \bar{C}_i$, where $\bar{C}_i$ is the closure of C_i in V. Replace X by $X' := X - \mathrm{Supp}(D')$. Then X' is an affine open set of X and $\rho' := \rho|_{X'}\colon X' \to Y$ is an A^1-fibration over Y for which the fiber $\rho^{*\prime}(P)$ is irreducible and reduced.

Performing this operation to all singular fibers of the first kind of ρ, we may assume that ρ has no singular fibers of the first kind. Let $\rho^*(P)$ denote anew a singular fiber of the second kind if such a fiber exists at all. If $\rho^*(P)$ is reducible, we may delete all irreducible components but one by replacing X by a smallar affine open set with an A^1-fibration over Y similar to that for X. Hence we may assume that $\rho^*(P)$ is an irreducible multiple fiber, i.e., $\rho^*(P) = nC$ with $C \cong A_k^1$ and $n \geqq 2$.

Write $Y := \mathrm{Spec}(k[t])$, and assume that the point P is defined by $t=0$. Let $Z := \mathrm{Spec}(k[\tau]) \to Y$ be the morphism defined by $t = \tau^n$, which is a finite covering ramifying totally over P. Let W be the normalization of $X \times_Y Z$. Then W is a nonsingular affine surface, and the canonical surjective morphism

σ: $W\to Z$ is an A^1-fibration over Z. This can be seen as follows. Let x be a point of X lying over the point P, and find a system of local coordinates (ξ, η) around x such that the curve C is defined by $\xi=0$. Then we have a relation $\xi^n=$at, where a is a unit in $O_{x,X}$. Then ξ/τ is regular at every point $\tilde{x}$ of W lying over x. Analytically, W around $\tilde{x}$ is defined as a hypersurface $(\xi/\tau)^n=a$ in the $(\xi/\tau, \tau, \eta)$-space. By the Jacobian criterion of smoothness, W is nonsingular at every point $\tilde{x}$ lying over x. It is easy to see that W is nonsingular at every point of W lying over $X-\rho^{-1}(P)$. Hence W is nonsingular. By construction, general fibers of σ are isomorphic to A_k^1. Let $\tilde{P}$ be the point of Z lying over P. Every fiber of σ except the fiber $\sigma^*\tilde{P}$ is irreducible and reduced, while $\sigma^*\tilde{P}$ is reduced and reducible with n irreducible components. Let W' be an affine open set of W obtained by deleting all components of $\sigma^*\tilde{P}$ except one. Then $\sigma':=\sigma|_{W'}$: $W'\to Z$ is an A^1-bundle over $Z\cong A_k^1$, whence W' is isomorphic to A_k^2 (cf. Kambayashi-Miyanishi [5]). Let f be the composite of the natural morphisms

$$f\colon A_k^2\cong W' \hookrightarrow W\to X\times_Y Z\to X\,.$$

Since f is apparently a dominant morphism, A is contained in a polynomial ring as a k-subalgebra. Q.E.D.

3.4. **Corollary.** *Let X be a nonsingular affine surface which satisfies the condition in Theorem* 3.3. *Then the torsion part $Pic(X)_{\text{tor}}$ of the Picard group of X is a cyclic group.*

Proof. Let ρ: $X\to Y$ be the A^1-fibration as in Theorem 3.3. Let ρ^*P_i $(0\leqq i\leqq m)$ exhaust all singular fibers of ρ; if there exists a singular fiber of the second kind, we let ρ^*P_0 denote it. Write $\rho^*P_i=\sum_{1\leqq k\leqq s_i} n_{ij}C_{ij}$, where $C_{ij}\cong A_k^1$ and $n_{ij}>0$. Then, since $Y\cong A_k^1$, the Picard group $\text{Pic}(X)$ of X is an abelian group with the following generators and relations:

$$\{\xi_{ij}\,|\,0\leqq i\leqq m,\ 1\leqq j\leqq s_i\}\quad\text{and}\quad \sum_{1\leqq j\leqq s_i} n_{ij}\xi_{ij}=0\qquad\text{for } 0\leqq i\leqq m\,.$$

It is then clear that $\text{Pic}(X)\cong\prod_{i=0}^{m} G_i$, where G_i is an abelian group with generators and relations given as above with i fixed and with $1\leqq j\leqq s_i$. Since $(n_{i1},\cdots,n_{is_i})=1$ for $i\geqq 1$ by assumption, we have $G_i\cong \boldsymbol{Z}^{\oplus(s_i-1)}$. Let $\mu=(n_{01},\cdots,n_{0s_0})$. Then $G_0\cong \boldsymbol{Z}/\mu\boldsymbol{Z}\oplus\boldsymbol{Z}^{\oplus(s_0-1)}$. Hence we have $\text{Pic}(X)_{\text{tor}}\cong\boldsymbol{Z}/\mu\boldsymbol{Z}$. Q.E.D.

3.5. We shall prove:

Theorem. *Let $X:=Spec(A)$ be a nonsingular affine surface with an A^1-fibration ρ: $X\to Y$, where $Y\cong P_k^1$. Assume that A is contained in a polynomial ring as a k-subalgebra. Then the fibration ρ has at most three multiple fibers. If*

ρ has three multiple fibers, their multiplicities $\{\mu_1, \mu_2, \mu_3\}$ are given, up to permutation, by one of the following triplets: $\{2, 2, n\}$ $(n \geqq 2)$, $\{2, 3, 3\}$, $\{2, 3, 4\}$ *and* $\{2, 3, 5\}$.

Proof. Suppose that ρ has three or more multiple fibers. Let $\rho^*P_i := \mu_i F_i$ $(1 \leqq i \leqq 3)$ be a multiple fiber of multiplicity $\mu_i > 1$. Let $f: \boldsymbol{A}_k^2 := \mathrm{Spec}(k[u_1, u_2]) \to X$ be a dominant morphism as in 3.1. Then $\rho \cdot f(\boldsymbol{A}_k^2) \cong \boldsymbol{A}_k^1$ or $\rho \cdot f(\boldsymbol{A}_k^2) = Y$. If $\rho \cdot f(\boldsymbol{A}_k^2) \cong \boldsymbol{A}_k^1$, we may assume that $P_1, P_2 \in \rho \cdot f(\boldsymbol{A}_k^2)$. However, this assumption leads to a contradiction by the argument in the step (I) of the proof of Theorem 3.3. Hence $\rho \cdot f(\boldsymbol{A}_k^2) = Y$. Then f^*F_i $(1 \leqq i \leqq 3)$ is defined by an equation $f_i = 0$, where f_i is a non-constant polynomial in $k[u_1, u_2]$. Since $\mu_1 f^*F_1 \sim \mu_2 f^*F_2 \sim \mu_3 f^*F_3$ (linear equivalence), we have

$$\frac{f_3^{\mu_3}}{f_1^{\mu_1}} = a \frac{f_2^{\mu_2}}{f_1^{\mu_1}} + b\,,$$

where $a, b \in k^*$. Without loss of generality, we may assume that $a = b = -1$. Namely, we have a relation

$$f_1^{\mu_1} + f_2^{\mu_2} + f_3^{\mu_3} = 0\,.$$

Note that $f^*(F_i) \cap f^*(F_j) = \phi$ whenever $i \neq j$. The assignment $x_i \mapsto f_i$ defines a non-constant morphism

$$\psi: \boldsymbol{A}_k^2 \to S^*_{\mu_1, \mu_2, \mu_3} \subset \mathrm{Spec}(k[x_1, x_2, x_3]/(x_1^{\mu_1} + x_2^{\mu_2} + x_3^{\mu_3}))\,.$$

Hence $\{\mu_1, \mu_2, \mu_3\}$ is, up to permutation, one of the following triplets: $\{2, 2, n\}$ $(n \geqq 2)$, $\{2, 3, 3\}$, $\{2, 3, 4\}$ and $\{2, 3, 5\}$ (cf. 2.14).

Suppose that ρ has four multiple fibers $\rho^*P_i = \mu_i F_i$ with multiplicity μ_i $(1 \leqq i \leqq 4)$. Let f^*F_i be defined by $f_i = 0$, where f_i is a non-constant polynomial in $k[u_1, u_2]$. Then we obtain relations of the following form:

$$f_1^{\mu_1} + f_2^{\mu_2} + f_3^{\mu_3} = 0$$
$$a f_1^{\mu_1} + f_2^{\mu_2} + f_4^{\mu_4} = 0\,,$$

where $a \in k - \{0, 1\}$. In view of the above observations on possible multiplicities of three multiple fibers of ρ, we know that $\{\mu_1, \mu_2, \mu_3, \mu_4\}$ is, up to permutation, one of the following quadruplets: $\{2, 2, 2, n\}$ $(n \geqq 2)$, $\{2, 2, 3, 3\}$, $\{2, 2, 3, 4\}$ and $\{2, 2, 3, 5\}$. The induced relations provide a non-constant morphism

$$\psi: \boldsymbol{A}_k^2 \to \Sigma^*_{\mu_1, \mu_2, \mu_3, \mu_4}\,.$$

This is impossible by 2.5 and 2.15. Q.E.D.

3.6. **Corollary.** *Let X be the same surface as in* 3.5. *Then $Pic(X)_{\mathrm{tor}}$ has at most two cyclic components. If $Pic(X)_{\mathrm{tor}}$ has two cyclic components, it is of the form*:

$$\mathrm{Pic}(X)_{\mathrm{tor}} \cong \boldsymbol{Z}/2\boldsymbol{Z} \oplus \boldsymbol{Z}/2s\boldsymbol{Z} \qquad (s \geqq 1).$$

Proof. An argument similar to that in Corollary 3.4.

3.7. We shall prove:

Theorem. *Let $X := Spec(A)$ be a nonsingular affine surface with an $\boldsymbol{A}^1$-fibration $\rho: X \to Y$, where $Y \cong \boldsymbol{P}^1_k$. Assume that ρ satisfies the following conditions:*

(1) *ρ has no singular fibers of the second kind but at most three multiple fibers with a single irreducible component;*

(2) *if ρ has three multiple fibers, the set of multiplicities $\{\mu_1, \mu_2, \mu_3\}$ is one of the following triplets:* $\{2, 2, n\}$ $(n \geqq 2)$, $\{2, 3, 3\}$, $\{2, 3, 4\}$ *and* $\{2, 3, 5\}$.

Then A is contained in a polynomial ring as a k-subalgebra.

Proof. (I) By performing the same operation as we did in the second step of the proof of Theorem 3.3, we may assume that ρ has no singular fibers of the first kind. Suppose that ρ has at most two multiple fibers. Let P be a point of Y such that ρ^*P is a multiple fiber (if such a fiber exists at all), and let $X' := X - \rho^{-1}(P)$. Then the nonsingular affine surface X' with an $\boldsymbol{A}^1$-fibration $\rho' := \rho|_{X'}$ over $Y' := Y - \{P\}$ has at most one singular fiber of the second kind. By Theorem 3.3, there exist a dominant morphism $\boldsymbol{A}^2_k \to X'$, and hence a dominant morphism $\boldsymbol{A}^2_k \to X$. Therefore A is contained in a polynomial ring as a k-subalgebra.

(II) Suppose that ρ has three multiple fibers $\rho^*P_i = \mu_i F_i$ $(1 \leqq i \leqq 3)$ with multiplicity μ_i. We consider first the case where $\{\mu_1, \mu_2, \mu_3\} = \{2, 2, n\}$ $(n \geqq 2)$. Let $Y' \to Y$ be a double covering of Y which ramifies over the points P_1 and P_2; then $Y' \cong \boldsymbol{P}^1_k$. Let X' be the normalization of $X \times_Y Y'$ and let $\rho': X' \to Y'$ be the natural projection. Then X' is a nonsingular affine surface and ρ' is an $\boldsymbol{A}^1$-fibration over Y' (cf. the proof of Theorem 3.3). Moreover, $\rho'^*P'_i$ $(i=1, 2)$ is a reduced singular fiber with two irreducible components, P'_i being the unique point of Y' lying over P_i, and ρ'^*Q_i $(i=1, 2)$ is a multiple fiber of multiplicity n with single irreducible component, Q_1 and Q_2 being two points of Y' lying over P_3. Replacing X' by an affine open set, we may assume that ρ' has no singular fibers of the first kind. Let $Y'' \to Y'$ be an n-ple covering which ramifies totally over Q_1 and Q_2, let X'' be the normalization of $X' \times_{Y'} Y''$, and let $\rho'': X'' \to Y''$ be the natural projection. Then X'' is a nonsingular affine surface and ρ'' is an $\boldsymbol{A}^1$-fibration over $Y'' \cong \boldsymbol{P}^1_k$. The fibration ρ'' has two reduced singular fibers $\rho''^*Q'_i$ $(i=1, 2)$ with n irreducible components, where Q'_i $(i=1, 2)$ is the unique point of Y'' lying over Q_i. Then, by virtue of the step (I), there exist a dominant morphism $\boldsymbol{A}^2_k \to X''$, and hence a dominant morphism $\boldsymbol{A}^2_k \to X$. Therefore, A is contained in a polynomial ring as a k-subalgebra.

(III) The other cases except the last one can be treated in a similar fashion, that is, by choosing suitable multiple coverings $\boldsymbol{P}_k^1 \to \boldsymbol{P}_k^1$ and then taking the normalizations of the fiber products with respect to such multiple coverings. The following diagram will indicate roughly the necessary steps:

$$\{2, 3, 3\} \xrightarrow[\text{covering}]{\text{triple}} \{2, 2, 2\} \to \text{the former case,}$$

$$\{2, 3, 4\} \xrightarrow[\text{covering}]{\text{double}} \{2, 3, 3\} \to \text{the former case.}$$

(IV) In the case where $\{\mu_1, \mu_2, \mu_3\} = \{2, 3, 5\}$, we know by the theory of Kleinian singularities that there exists a ramified covering τ: $Y' \to Y$ of degree 60 with 30 points over P_1 with ramification index 2, 20 points over P_2 with ramification index 3 and 12 points over P_3 with ramification index 5, where $Y' \cong \boldsymbol{P}_k^1$. Let X' be the normalization of $X \times_Y Y'$ and ρ': $X' \to Y'$ be the natural $\boldsymbol{A}^1$-fibration. Then ρ' has no multiple fibers of the second kind. So, we are done. Q.E.D.

References

[1] E. Brieskorn: *Rationale Singularitäten komplexer Flächen,* Invent. Math. **4** (1968), 336–358.
[2] T. Fujita: *On Zariski problem,* Proc. Japan Acad. **55** (1979), 106–110.
[3] W.H. Hayman: Meromorphic functions, Oxford, Clarendon Press, 1964.
[4] S. Iitaka: *Homogeneous Lüroth theorem, and logarithmic classification of algebraic surfaces,* Proc. of Kinosaki Symposium on Algebraic Geometry, 1977 (in Japanese).
[5] T. Kambayashi and M. Miyanishi: *On flat fibrations by the affine line,* Illinois J. Math. **22** (1978), 662–671.
[6] M. Miyanishi: Lectures on curves on rational and unirational surfaces, Tata Institute of Fundamental Research, Vol. 60, Berlin-Heidelberg-New York, Springer Verlag, 1978.
[7] M. Miyanishi: *Regular subrings of a polynomial ring,* Osaka J. Math. **17** (1980), 329–338.
[8] M. Miyanishi and T. Sugie: *Affine surfaces containing cylinderlike open sets,* J. Math. Kyoto Univ. **20** (1980), 11–42.
[9] D. Mumford: Geometric invariant theory, Berlin-Heidelberg-New York, Springer Verlag, 1965.
[10] P. Orlik and P. Wagreich: *Algebraic surfaces with k*-action,* Acta Math. **138** (1977), 43–81.
[11] H. Sumihiro: *Equivariant completion,* J. Math. Kyoto Univ. **14** (1974), 1–28.

Department of Mathematics
Osaka University
Toyonaka, Osaka 560
Japan