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STOCHASTIC INTERSECTION NUMBER AND HOMOLOGICAL BEHAVIORS OF DIFFUSION PROCESSES ON RIEMANNIAN MANIFOLDS

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1. Introduction. Let M be a d-dimensional $(d\geqq 2)$ connected orientable Riemannian manifold and g be its Riemannian metric. Let Δ be the Laplace-Beltrami operator on M and b be a C^∞ vector field on M. Set $L=\Delta/2+b$. Let $X=(X_t, P_x, x\in M)$ be the minimal diffusion process corresponding to L. The main purpose of the present paper is to investigate some homological behaviors of the path of X. Our first objective is to define the intersection number of the path of X and chains and to study its asymptotic behaviors. The usual intersection number of two cycles is defined as the product of their homology classes. But we need in our study the intersection number for chains and its analytical expressions. Theory of harmonic integrals gives such an expression. In passing from smooth cases to stochastic cases, we use the so-called Fisk-Stratonovich integral, which enables us to write down the formulas formally in the same way as in ordinary calculus. Based on the analytic expression of the intersection numbers expressed by the integral of double 1-form with singularity, we shall define the stochastic intersection number with the aid of the integral of 1-form along the path ([8]). By virtue of the approximation theorem for stochastic integrals ([8]), it turns out that stochastic intersection number enjoys analogous properties to those for ordinary intersection numbers. In the study of the asymptotic behavior of the intersection numbers of the path and the cycles, the integrals of harmonic 1-forms play an important role, which is due to the two facts that they depend only on the homology class of the path and that they are martingales.

We then consider the following problem related to the asymptotic behaviors of the intersection numbers. Let M be two-dimensional, compact and let κ be its genus $(1\leqq\kappa<\infty)$. Let X be a Brownian motion on M. Our problem is this. In what manner does the path of X wind holes asymptotically? We formulate this as follows. Let $(A_i, A_{\kappa+i})$, $i=1,\cdots,\kappa$, be a canonical homology basis. For any $x,y\in M$, we choose a smooth curve $\phi_{x,y}$ such that $\phi_{x,y}(0)=x$, $\phi_{x,y}(1)=y$ and $\phi_{x,y}(0,1)$ does not meet any $A_i (i=1,\cdots,2\kappa)$. Set $C=\{\phi_{x,y}; x,y\in M\}$. Consider the homological position of the curve $X[0,t]$, that is, using $\phi_{X(t),X(0)}$, we

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define the cycle $\bar{X}[0, t]=X[0, t]+\psi_{X(t),X(0)}$, where $\psi_{X(t),X(0)}(s)=\phi_{X(t),X(0)}(s/t)$, $0\leqq s\leqq t$ and write

$$\bar{X}[0, t] = \sum_{i=1}^{2\kappa} x_i(t)A_i\,,$$

where the equality means that the both sides are homologous. We call $(x_1(t), \cdots, x_{2\kappa}(t))$ the homological position of $X[0,t]$. The problem is then to investigate the asymptotic behavior of $(x_1(t), \cdots, x_{2\kappa}(t))$. Arnold-Avez [2] formulated an analogous problem for ergodic classical dynamical systems in a similar manner. They studied the behaviors of each $x_i(t)$ and showed the existence of the limit $\lim_{t\to\infty} x_i(t)/t=\mu_i$. They call $\sum_{i=1}^{2\kappa} \mu_i A_i$ the mean homological position of $X[0,t]$.

In our stochastic case, we study the asymptotic not only of each process $x_i(t)$ but also of the N-tuple process $(x_{i_1}(t), \cdots, x_{i_N}(t))$ (where $1\leqq N\leqq 2\kappa$, $1\leqq i_1< \cdots<i_N\leqq 2\kappa$). We say that X winds $(A_{i_1}, \cdots, A_{i_N})$ homologically if $\lim_{t\to\infty}\sum_{\lambda=1}^{N} x_{i_\lambda}(t)^2 =\infty$, a.s. Then our main result can be stated as follows:

X winds $(A_{i_1}, \cdots, A_{i_N})$ homologically if and only if $N\geqq 3$.

McKean [14] considered this problem for Brownian motion on $M=\boldsymbol{R}^2-\{0,1\}$ (genus$=2$). He treated it by considering the covering Brownian motion on the homology surface of M, and reduced it to the problem of recurrence of the covering motion. Our method is similar to McKean's as will now be explained. The main term of $x_i(t)$ is the integral $\int_{X[0,t]}\alpha^{(i)}$ of the harmonic 1-form $\alpha^{(i)}$ (which corresponds to A_i) along the path of X, $i=1,2,\cdots,2\kappa$. By virtue of this, it is sufficient to study $(\int_{X[0,t]}\alpha^{(i_1)}, \cdots, \int_{X[0,t]}\alpha^{(i_N)})$. By using the correspondence defined in §6 (with probability one) between $(\int_{X[0,t]}\alpha^{(i_1)}, \cdots, \int_{X[0,t]}\alpha^{(i_N)})$ and the position $\tilde{X}_I(t)$ of the covering motion $\tilde{X}_I$ on the Abelian covering surface $\tilde{M}(I)$ of M defined in section 6 (where $I=(i_1, \cdots, i_N)$), our problem reduces to the question of recurrence of $\tilde{X}_I$ which in turn leads to us to the type problem for $\tilde{M}(I)$. In the case of Brownian motion on a simply connected Riemann surface, the equivalence of the recurrence of Brownian motion and the type problem was already noted by Kakutani [11]. Since the type of the Abelian covering surface of a compact Riemann surface was already decided by A. Mori [16], we obtain our result.

The organization of this paper is as follows. In section 2, we collect the facts about the integral of 1-form along the path of diffusion. In section 3, we shall define the intersection number of the path of diffusion and a $(d-1)$-chain and study its properties. In section 4, we investigate the asymptotic behaviors of each $x_i(t)$ for the case M is compact. In section 5, we collect several known facts about covering surfaces and covering motions on them. This is a preparatory section to the next one. In section 6, we shall study the asymptotic behavior of $(x_{i_1}(t), \cdots, x_{i_N}(t))$ for Brownian motion on a two-dimensional, compact Riemannian manifold. In section 7, we shall consider the winding problem for a Brownian motion with a drift on Euclidean 2-space.

In a discussion of our results with Professor S.R.S. Varadhan, he gave a purely probabilistic substitute for the proof of Theorem 6.1. His proof not only decided the recurrence of covering motion, but also is interesting itself. We present the proof at the end of section 6.

The summary of sections 3, 4 and 7 was announced in the previous note [13].

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2. Lifted diffusion and a formula for the integral of 1-form along the path. In this section, for later use, we summarize several facts about the integral of 1-form along the path of a diffusion on a Riemannian manifold. Let M be a d-dimensional ($d \geqq 2$) connected orientable Riemannian manifold with the Riemannian metric g. Let Δ be the Laplace-Beltrami operator. Set $L = \Delta/2 + b$, where b is a C^∞ vector field on M. Let $X = (X_t, P_x, x \in M)$ be the minimal diffusion corresponding to L. Denote by ζ the life time of X. We always assume the following condition:

$$\text{(A)} \quad P_x(\zeta = \infty) = 1, \quad \text{for any } x \in M.$$

Let $O(M)$ be the bundle of orthonormal frames on M and $\pi: O(M) \to M$ be the natural projection. We write $r = (x, e)$, where $x = \pi(r)$ and e is an orthonormal frame at x. Let $L(M)$ be the bundle of linear frames. Let $(x^1, \cdots, x^d)$ be a local coordinate system on U. Then any frame $e = (e_1, \cdots, e_d)$ at x can be written as

$$e_i = \sum_{j=1}^{d} e_i^j \frac{\partial}{\partial x^j}, \quad i = 1, \cdots, d.$$

It is easy to see that $(x^1, \cdots, x^d, (e_i^j), 1 \leqq i, j \leqq d)$ is a local coordinate system of $L(M)$. Note that if $(x, e) \in O(M)$, then (e_j^i) satisfies the following relation

$$\sum_{p,q=.}^{d} g_{pq} e_i^p e_j^q = \delta_{ij}, \; i, j = 1, \cdots, d.$$

The diffusion process X corresponding to L can be constructed as follows. Let Γ be an affine connection on M compatible with the Riemannian metric g

corresponding to b in the sense of Ikeda-Watanabe [9. Chap. 5, §4]. Let $B_t=(B_t^1,\cdots,B_t^d)$ be a Brownian motion on $\boldsymbol{R}^d$. Consider the following stochastic differential equation.

$$\text{(2.1)}\qquad \begin{cases} dX_t^i = \sum_{j=1}^d e_j^i(t)\circ dB_t^j\,, \\ de_j^i(t) = -\sum_{p,q=1}^d \Gamma_{pq}^i(X_t)e_j^p(t)\circ dX_t^q\,, \\ r(0)=r, \quad r\in O(M)\,, \end{cases}$$

where Γ_{pq}^i is the coefficients of Γ and $r_t=(X_t^i,e_j^i(t))$ (cf. Ikeda-Watanabe [9]). By solving this, we obtain a diffusion process $r=(r_t,P_r,r\in O(M))$, where P_r is the probability law of the diffusion process r. The probability law of the process $X_t=\pi(r_t)$ depends only on $x=\pi(r)$, which we call P_x. The diffusion corresponding to L is given by $X=(X_t,P_x,x\in M)$.

For the diffusion X, we can define the integral of 1-forms along the path as in [8]. We sketch here the definition. First we consider a simple case. Let U and V be coordinate neighborhoods with $U\subset V$. Let α be a smooth 1-form on M such that $\operatorname{supp}(\alpha)\subset U$. Define the sequences of stopping times $\{\sigma_n\}$, $\{\tau_n\}$ as follows.

$$\sigma=\sigma_0=\inf\{t;\,X_t\notin V\},\ \ \tau=\tau_0=\inf\{t;\,X_t\in U\}^{2)}\,,$$
$$\sigma_n=\tau_n+\sigma\circ\theta_{\tau_n},\ \ \tau_n=\sigma_n+\tau\circ\theta_{\sigma_{n-1}},\ \ n=1,2,\cdots\,,$$

where θ is the shift operator of the process X. We define the integral of α along the path as follows.

$$\int_{X[0,t]}\alpha=\sum_{n=0}^{\infty}\int_{\sigma_n\wedge t}^{\tau_n\wedge t}\sum_{i=1}^d \alpha_i(X_s)\circ dX_s^i\,.$$

For the general case, we define $\int_{X[0,t]}\alpha$ using a partition of unity. We shall omit the details, see [8].

Next we state a formula for the integral of 1-form along the path. Let α be any smooth 1-form on M. Then we have the following formula

$$\text{(2.2)}\qquad \int_{X[0,t]}\alpha=\int_0^t\sum_{i=1}^d \bar{\alpha}_i(r_s)dB_s^i+\int_0^t\Big(\alpha(b)-\frac{1}{2}\,\delta\alpha\Big)(X_s)ds\,,$$

where $(\bar{\alpha}_1(r),\cdots,\bar{\alpha}_d(r))$ is the scalarization of α and δ is the adjoint operator of d (see [8],[9]).

3. Intersection number of the path of a diffusion and a $(d-1)$-chain. In this section, we shall define the intersection number of the path of diffusion and a $(d-1)$-chain by using the integral of 1-form along the path of

2) We set $\inf\phi=\infty$.

diffusion. We retain the assumptions and notations in section 2.

First let us recall the expression of the usual intersection number $I^*(c,c')$ of a 1-form c and a $(d-1)$-chain c'. We refer to de Rham [4] for the details. Let $\bar{\mathscr{D}}$ be the space of square integrable 1-currents. Let $\bar{\mathscr{D}}_1$, $\bar{\mathscr{D}}_2$ and $\mathscr{D}_3$ be the subspaces of $\bar{\mathscr{D}}$ consisting of currents which are homologous to zero, cohomologous to zero and harmonic, respectively. Then $\bar{\mathscr{D}}=\bar{\mathscr{D}}_1+\bar{\mathscr{D}}_2+\mathscr{D}_3$. Let H_1, H_2 and H_3 be the orthogonal projections on $\bar{\mathscr{D}}_1$, $\bar{\mathscr{D}}_2$ and $\mathscr{D}_3$, respectively. Denote by $\tilde{\mathscr{D}}$ the space of currents which is continuous in mean at infinity. We extend the operator H_i on $\tilde{\mathscr{D}}$ as follows. For $T\in\tilde{\mathscr{D}}$ and $\phi\in\bar{\mathscr{D}}\cap C^\infty$, we set

$$(H_iT, \phi) = (T, H_i\phi), \ i = 1, 2, 3 .$$

The current $H_iT(i=1,2,3)$ is again continuous in mean at infinity. It is known that every current $T\in\tilde{\mathscr{D}}$ is decomposed as follows.

$$T = H_1T+H_2T+H_3T . \tag{3.1}$$

Let $h_1(x,y)$ be the kernel of H_1 (see [4. p. 170]). We denote by $e(x,y)$ the adjoint form of $h_1(x,y)$:

$$e(x, y) = *_y h_1(x, y) , \tag{3.2}$$

where $*_y$ is the adjoint operator with respect to y.

REMARK 3.1. Since $h_1(x,y)$ is C^∞ except on the diagonal set, $e(x,y)$ is also C^∞ off the diagonal set.

Now let c and c' be a 1-chain and a $(d-1)$-chain respectively such that

$$(c\cap\partial c')\cup(\partial c\cap c') = \phi . \tag{3.3}$$

Then the intersection number $I^*(c,c')$ of a 1-chain c and a $(d-1)$-chain c' can be written of the form

$$I^*(c, c') = \int_{y\in c'}\int_{x\in c} e(x, y) - \int_{x\in c}\int_{y\in c'} e(x, y) . \tag{3.4}$$

Although $e(x,y)$ has singularity on the diagonal set, the above integral is well-defined. Indeed, it is known that $\int_{x\in c} e(x,y)$ is a C^∞ $(d-1)$-form in y for $y\notin\partial c$. So the iterated integral $\int_{y\in c'}\int_{x\in c} e(x,y)$ has the meaning if $c'\cap\partial c=\phi$. Similarly, the second term of the right hand side of (3.4) is also well-defined if $\partial c'\cap c=\phi$ (see de Rham [4. §33]).

To define the intersection number $I(X[0,t],c)$ of the path of diffusion and a $(d-1)$-chain c, we need a more explicit formula for $I^*(c,c')$. For this, we need the following two lemmas. Let Δ be the Hodge-Kodaira's Laplacian acting on 1-forms.

Lemma 3.1 (de Rham [4. §30]). *Given a sufficiently small domain U, there exists a double current $\gamma(x,y)$ on $U\times U$ such that*
(i) *for any differential 1-form β, the equation*

$$\Delta\alpha=\beta \qquad on\ U$$

has a solution α given by

$$\alpha=\int_U \gamma(x,y)\wedge *\beta(y),\ \ x\in U.$$

(ii) *$\gamma(x,y)$ has the following properties.*
(a) *If $x\neq y$, then $\Delta_x\gamma(x,y)=0$,*[3]
(b) *$\gamma(x,y)$ is C^∞ off the diagonal set of $U\times U$.*

We call $\gamma(x,y)$ an elementary solution.

Following lemma shows that $h_1(x,y)$ can be written in terms of γ.

Lemma 3.2. *Let U be a domain determined by Lemma 3.1. Let $\sigma(x,y)$ be a C^∞ function on $U\times U$ with* (i) *supp(σ) is contained in a neighborhood W of the diagonal set of $U\times U$,* (ii) $0\leqq\sigma\leqq 1$, (iii) *$\sigma(x,y)=1$ on a neighborhood $W'(\subset W)$ of the diagonal set and* (iv) *$\sigma(x,y)=\sigma(y,x)$. We set $\gamma'=\sigma\gamma$. Then there exists C^∞ double 1-form $\psi(x,y)$ such that for $x,y\in U$ $(x\neq y)$,*

$$e(x,y)=d_x\delta_x *_y\gamma'(x,y)+*_y\psi(x,y)\,. \tag{3.5}$$

Proof. (see de Rham [4]) Let Γ_1 the operator having the kernel $\gamma'(x,y)$. Then we have $\Delta\Gamma_1=I-\Gamma_2$, where Γ_2 is a smoothing operator. Comparing the d-closed part, we have $d_x\delta_x\gamma'(x,y)=h_1(x,y)-\psi(x,y)$, where ψ is the d-closed part of γ_2. Since ψ is C^∞, the proof is completed.

REMARK 3.2. Since ψ is smooth, we have

$$I^*(c,c')=\int_{y\in c'}\int_{x\in c} d_x\delta_x *_y\gamma'(x,y)-\int_{x\in c}\int_{y\in c'} d_x\delta_x *_y\gamma'(x,y)\,, \tag{3.6}$$

for $c,c'\subset U$.

We need one more lemma.

Lemma 3.3. *Let c be a finite C^∞ $(d-1)$-chain. For any $t>0$, set*

$$\Omega(c,t)=\{\omega\in\Omega;\ \{(X(0)\cup X(t))\cap c\}\cup\{X[0,t]\cap\partial c\}=\phi\}\,.$$

Then we have

$$P_x(\Omega(c,t))=1, \qquad for\ any\ x\notin c\,. \tag{3.7}$$

Proof. Let $\{U_j\}$, $\{V_j\}$ be finite open coverings of c such that (i) (V_j,ϕ_j)

3) In general, for an operator A, $A_x f(x,y)$ means that A operates on x-variable.

is a coordinate neighborhood (ii) $\bar{U}_j \subset V_j$. Define the sequences of stopping times

$$\sigma_j = \inf\{t; X_t \notin V_j\}, \ \tau_j = \inf\{t; X_t \in U_j\},$$
$$\tau_{j,k} = \sigma_{j,k-1} + \tau_j \circ \theta_{\sigma_{j,k-1}}, \ \sigma_{j,k} = \tau_{j,k-1} + \sigma_j \circ \theta_{\tau_{j,k-1}}, \ k = 1\ 2, \cdots .$$

By using local coordinate it is sufficient to consider on $\boldsymbol{R}^d$. We can show that, the diffusion X does not hit $\partial(V_j \cap c)$. Therefore X does not hit ∂c. On the other hand, the minimal fundamental solution $p(t,x,y)$ for $\partial/\partial t - L$ (cf. [14]) is continuous in y. So we have $P_x(X(t) \cap c = \phi) = 1$. Combining these two facts with $x \notin c$, we have (3.7).

Now we proceed to define the intersection number $I(X[0,t]), c)$ of the path of diffusion X and a $(d-1)$-chain c. By Lemma 3.1, we have for any $y \in c$, there exist domains U_y and V_y such that (i) $y \in U_y$, $U_y \subset V_y$ (ii) V_y is an ε-neighborhood of U_y and (iii) an elementary solution $\gamma(x,y)$ exists on V_y. Since c is compact, we can choose a finite number of points $y_1, \cdots, y_m \in c$ such that $c \subset \bigcup_{i=1}^{m} U_i$ (we set $U_i = U_{y_i}$). Choose ρ_j on $V_j \times V_j$ $(V_j = V_{y_j})$, $(j=1,\cdots,m)$ as in Lemma 3.2 and set $\gamma'_j(x,y) = \rho_j(x,y)\gamma_j(x,y)$, $j=1,\cdots,m$. Let ϕ_j, $j=1,\cdots,m$, be a C^∞ function of x with (i) supp $(\phi_j) \subset V_j$, (ii) $\phi_j = 1$ on U_j and (iii) $0 \leqq \phi_j \leqq 1$ on M. Note that if we set

$$F_j(x,y) = \phi_j(x) *_y \psi(x,y) + (1-\phi_j(x)) e(x,y) + (\phi_j(x)-1) d_x \delta_x *_y \gamma'(x,y),$$

then $F_j(x,y)$ is a smooth 1-form of x for $y \in c \cap U_j$.

Therefore the integral $\int_{x \in X[0,t\wedge\sigma_N(c)]} F_j(x,y)$[4] is well-defined as the integral of 1-form along the path ([8]), where $\sigma_N(c) = \inf\{s \geqq 0; \operatorname{dist}(X_s, \partial c) \leqq 1/N\}$, $N = 1,2,\cdots$. Now we define for $y \in c \cap U_j$

$$\int_{x \in X[0,t\wedge\sigma_N(c)]} e(x,y) = \delta_x *_y \gamma'(X_{t\wedge\sigma_N(c)}, y) - \delta_x *_y \gamma'(X_0, y) + \int_{x \in X[0,t\wedge\sigma_N(c)]} F_j(x,y), \ P_x\text{-a.s.} \tag{3.8}$$

It is easy to see that this integral does not depend on the choice of $\{\phi_j\}$. The integral (3.8) is smooth in $y \in c \cap U_j$ for P_x-a.s.ω (cf. de Rham [4. §33. p. 171]). So the integral $\int_{y \in c \cap U_j} \int_{x \in X[0,t\wedge\sigma_N(c)]} e(x,y)$ is well-defined. Define $I_N(X[0,t],c)$ by

$$I_N(X[0,t],c) = \sum_{j=1}^{m} \Big\{ \int_{y \in c \cap U_j} \int_{x \in X[0,t\wedge\sigma_N(c)]} e(x,y) - \int_{x \in X[0,t\wedge\sigma_N(c)]} \int_{y \in c \cap U_j} e(x,y) \Big\}, \ P_x\text{-a.s.} \tag{3.9}$$

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4) For a double 1-form $\alpha(x,y)$, $\int_{x \in X[0,t]} \alpha(x,y)$ means the integral of 1-form $\alpha(\cdot,y)$ in x along the path of X.

The second term of (3.9) is also well-defined as the integral of 1-form along the path, since $\int_{y\in c\cap U_j} e(x,y)$ is a C^∞ 1-form in x for $x\notin\partial(c\cap U_j)$ (cf. de Rham [4. §33. p. 171]).

REMARK 3.3. As in Remark 3.2, $I_N(X[0,t],c)$ can be written of the form

$$(3.10)\qquad \begin{aligned} I_N(X[0,t],c) = \sum_{j=1}^{m}\{\int_{y\in c\cap U_j}(\delta_x *_y \gamma'(X_{t\wedge\sigma_N(c)},y)-\delta_x *_y \gamma'(X_0,y)) \\ -\int_{x\in X[0,t\wedge\sigma_N(c)]}\int_{y\in c\cap U_j} d_x\delta_x *_y \gamma'(x,y)\}\,. \end{aligned}$$

We have the following

Lemma 3.4. *If $x\notin c$, then there exists a limit $I(X[0,t],c)$:*

$$I(X[0,t],c)=\lim_{N\to\infty} I_N(X[0,t],c),\ \ P_x\text{-a.s.}$$

Proof. Define the subset $\Omega_{c\ t,N}$ of by $\Omega_{c,t,N}=\{\omega\in\Omega;\ \sigma_N(\omega)>t\}$. Then by Lemma 3.3, we have $P_x(\Omega(c,t)\cap\bigcup_{N=1}^{\infty}\Omega_{c,t,N})=1$. For $\omega\in\Omega(c,t)\cap\Omega_{c,t,N}$, we have $t\wedge\sigma_{N'}(c)(\omega)=t$ for $N'\geqq N$. Therefore $I_{N'}(X[0,t],c)$ is independent of $N'\geqq N$. Thus there exists a limit $I(X[0,t],c)=\lim_{N\to\infty} I_N(X[0,t],c)$, P_x-a.s.

DEFINITION. We call the above limit $I(X[0,t],c)$ the intersection number of the path of diffusion and a $(d-1)$-chain c.

To clarify the relation between the above intersection number and the usual one, we shall show the following approximation theorem. Choose locally finite open coverings $\{W_n\}_{n\in N}$, $\{U_n\}_{n\in N}$ and $\{V_n\}_{n\in N}$ of M which satisfy the following conditions:

(i) For any $n\in N$, W_n is a coordinate neighborhood.
(ii) For any $n\in N$, $\bar{U}_n\subset V_n\subset\bar{V}_n\subset W_n$.
(iii) For any $n\in N$ and $x,y\in W_n$, there exists a unique minimal geodesic $\gamma_{x,y}$ such that $\gamma_{x,y}(0)=x$, $\gamma_{x,y}(1)=y$ and $\{\gamma_{x,y}(s);\ 0\leqq s\leqq 1\}\subset W_n$. Let $\sigma_{n,k}$and $\tau_{n,k}$ be the stopping times defined by

$$\sigma_n=\inf\{t;\ X_t\notin V_n\},\ \ \tau_n=\inf\{t;\ X_t\in U_N\},\ \ \sigma_{n,-1}=0\,,$$
$$\sigma_{n,k}=\tau_{n,k}+\sigma_n\circ\theta_{\tau_{n,k}},\ \ \tau_{n,k}=\sigma_{n,k-1}+\tau_n\circ\theta_{\sigma_{n,k-1}}\,,$$

for $n=1,2,\cdots$, and $k=0,1,\cdots$.
Let $\pi(m)=\{0=t_0^{(m)}<t_1^{(m)}<\cdots\}$ be a subdivision of $[0,\infty)$ and $\tilde{\pi}(m)=\{0=s_0^{(m)}<s_1^{(m)}<\cdots\}$ be the refinement of $\pi(m)$ obtained by adding $\{\sigma_{n,k}\}_{n,k}$ and $\{\tau_{n,k}\}_{n,k}$. Let X_m be the polygonal geodesic whose restriction on $[s_k^{(m)},s_{k+1}^{(m)}]$ is the minimal geodesic joining $X_{s_k^{(m)}}$ and $X_{s_{k+1}^{(m)}}$.

Lemma 3.5. *$X_m[0,t]$ can be regarded as a C^∞ singular 1-chain C_m;*

$$\begin{aligned} C_m(s) &= \sum_{k=1}^{\infty} C_{m,k}(s), \quad 0 \leqq s \leqq t\,, \\ C_{m,k}(s) &= X_m((s_{k+1}^{(m)} - s_k^{(m)})s/t + s_k^{(m)}) \end{aligned} \tag{3.11}$$

and the following equality holds.

$$\int_{X_m[0,t]} \alpha = \int_{C_m[0,t]} \alpha, \quad \text{for any 1-form } \alpha\,. \tag{3.12}$$

Proof. This lemma follows from the following fact: Let f_0, f_1, f_2: $[0,t] \to M$ be singular 1-chains such that

$$f_0(s) = \begin{cases} f_1(s/t_0), & \text{for } 0 \leqq s \leqq t_0\,, \\ f_2((s-t_0)/(t-t_0)), & \text{for } t_0 \leqq s \leqq t, \ (0 < t_0 < t)\,, \end{cases}$$

then f_0 is homologous to $f_1 + f_2$ (cf. Spanier [19]).

By this lemma, the usual intersection number $I^*(X_m[0, t\wedge\sigma_N(c)], c) = I^*(C_m[0, t\wedge\sigma_N(c)], c)$ is already defined. Now we have the following approximation theorem.

Theorem 3.1. *Let c be a finite C^∞ $(d-1)$-chain and $x \notin c$. Then there exists a subsequence $m_1 < m_2 < \cdots \uparrow \infty$ such that*

$$\lim_{j\to\infty} I^*(X_{m_j}[0, t\wedge\sigma_N(c)], c) = I_N(X[0,t], c), \ P_x\text{-a.s.} \tag{3.13}$$

Proof. Since $\int_{y\in c\cap U_j} d_x\delta_x *_y \gamma'(x,y)$ is a smooth 1-form (in x), by the approximation theorem (see [8]), there exists a subsequence $\{m_k\}$ such that

$$\begin{aligned} &\int_{x\in X_{m_k}[0,t\wedge\sigma_N(c)]} \int_{y\in c\cap U_j} d_x\delta_x *_y \gamma'(x,y) \\ &\to \int_{x\in X[0,t\wedge\sigma_N(c)]} \int_{y\in c\cap U_j} d_x\delta_x *_y \gamma'(x,y), \ P_x\text{-a.s.}, \end{aligned} \tag{3.14}$$

$j=1,\cdots,l$. On the other hand, $\delta_x *_y \gamma'(X_{m_k}(t\wedge\sigma_N(c)),y) - \delta_x *_y \gamma'(X_{m_k}(0),y)$ converges to $\delta_x *_y \gamma'(X(t\wedge\sigma_N(c)),y) - \delta_x *_y \gamma'(X(0),y)$ uniformly in y $(\in c\cap U_j)$. Therefore we have

$$\begin{aligned} &\int_{y\in c\cap U_j} \{\delta_x *_y \gamma'(X_{m_k}(t\wedge\sigma_N(c)),y) - \delta_x *_y \gamma'(X_{m_k}(0),y)\} \\ &\to \int_{y\in c\cap U_j} \{\delta_x *_y \gamma'(X(t\wedge\sigma_N(c)),y) - \delta_x *_y \gamma'(X(0),y)\}\,. \end{aligned}$$

By Remark 3.2 and Remark 3.3, we have (3.13).

Using this theorem, we can show that $I(X[0,t],c)$ has similar properties as the usual intersection number.

Proposition 3.1. *$I(X[0,t],c)$ has the following properties for almost all ω (P_x) $(x\notin c)$.*
(i) *If $x\notin c_1\cup c_2$, then*

$$I(X[0,t],\lambda_1c_1+\lambda_2c_2)=\lambda_1I(X[0,t],c_1)+\lambda_2I(X[0,t],c_2),\ \lambda_1,\lambda_2\in\boldsymbol{R}\,.$$

(ii) *If c is a cycle, then $I(X[0,t],c)$ depends only on the homology class of $X[0,t]$.*
(iii) *If $X[0,t]\cap c=\phi$, then $I(X[0,t],c)=0$.*
(iv) *If c is a $(d-1)$-chain with $\boldsymbol{Z}$-coefficients, then $I(X[0,t],c)$* is an integer.

Proof. In the proof, we write $\{m\}$ for $\{m_j\}$ in Theorem 3.1 and omit the term almost surely (P_x). To prove (i), let $\sigma_N=\sigma_N(c_1)\wedge\sigma_N(c_2)$. Then we have

$$I^*(X_m[0,t\wedge\sigma_N],\ \lambda_1c_1+\lambda_2c_2)=\lambda_1I^*(X_m[0,t\wedge\sigma_N],c_1)+\lambda_2I^*(X_m[0,t\wedge\sigma_N],c_2)\,.$$

By Theorem 3.1 we obtain

$$I_N(X[0,t],\ \lambda_1c_1+\lambda_2c_2)=\lambda_1I_N(X[0,t\wedge\sigma_N],c_1)+\lambda_2I_N(X[0,t\wedge\sigma_N],c_2)\,.$$

Letting $N\to\infty$, in view of Lemma 3.4, we have

$$I(X[0,t],\ \lambda_1c_1+\lambda_2c_2)=\lambda_1I(X[0,t],c_1)+\lambda_2I(X[0,t],c_2)\,.$$

For (ii), we consider ω which belongs to $\Omega(c,t)\cap\Omega_{c,t,N}$. It is known that $I^*(X_m[0,t],c)$ depends only on the homology class of $X_m[0,t]$ (see de Rham [4]). Let $X[0,t]$ be homologous to a C^∞ singular 1-chain ψ. Since $X_m[0,t]$ is homotopic to $X[0,t]$ for sufficiently large m, $X_m[0,t]$ is homologous to $X[0,t]$. Therefore $X_m[0,t]$ is homologous to ψ for sufficiently large m. So there exists a C^∞ singular 2-chain ψ_1 such that $X_m[0,t]=\psi+\partial\psi_1$. This implies that $I^*(X_m[0,t],c)=I^*(\psi,c)$ for sufficiently large m. By letting $m\to\infty$, in view of Theorem 3.1, we have $I(X[0,t],c)=I^*(\psi,c)$, which proves (ii). Next we show (iii). If $X[0,t]\cap c=\phi$, then for sufficiently large m, it holds that $X_m[0,t]\cap c=\phi$. This leads to $I^*(X_m[0,t],c)=0$. Using Theorem 3.1, we have $I(X[0,t],c)=0$. Finally let c be a $(d-1)$-chain with $\boldsymbol{Z}$-coefficients. Then we have $I^*(X_m[0,t\wedge\sigma_N(c)],c)\in\boldsymbol{Z}$. By Theorem 3.1, we have $I_N(X[0,t],c)\in\boldsymbol{Z}$. Letting $N\to\infty$, we have $I(X[0,t],c)\in\boldsymbol{Z}$, which proves (iv).

4. Asymptotic behavior of the homological position (I). Throughout this section, we assume that (M,g) is a compact connected orientable Riemannian manifold. In this section, we shall study the asymptotic behavior of the each of the homological position of the path. First we give the definition of the homological position of $X[0,t]$. Let $c_1,\cdots,c_k$ be a basis of the $(d-1)$-dimensional homology group $H_{d-1}(M)$ of M, where $k=\dim H_{d-1}(M)$. Let $c_1',\cdots,c_k'$ be a basis of 1-dimensional homology group $H_1(M)$ of M. For any point $x,y\in M$, we choose a smooth curve $\phi_{x,y}$ such that $\phi_{x,y}(0)=x$, $\phi_{x,y}(1)=y$ and $\phi_{x,y}(0,1)\cap\bigcup_{i=1}^{k}$

$c_i=\phi$. Set $C=\{\phi_{x,y};\, x,y\in M\}$. We make a cycle $\bar{X}[0,t]$ as a sum of chains:

$$\bar{X}[0,t]=X[0,t]+\psi_{X(t),X(0)},$$

where $\psi_{X(t),X(0)}(s)=\phi_{X(t),X(0)}(s/t)$, $0\leqq s\leqq t$. Then the cycle $\bar{X}[0,t]$ can be written of the form

$$\bar{X}[0,t]=\sum_{i=1}^{k}x_i(t)c'_i,$$

where the equality means the both sides are homologous.

DEFINITION. We call $(x_1(t),\cdots,x_k(t))$ the homological position of $X[0,t]$ with respect to $(\{c_i\},\{c'_i\},C)$ (or $(\{c'_i\},C)$ for short).

Before proceeding to study the asymptotic behavior of the homological position, we discuss the relation between the stochastic intersection number and the homological position of the path. To do this, we rewrite the stochastic intersection number. Let Δ be the Hodge-Kodaira's Laplacian acting on 1-forms. There exists a unique operator G (called Green operator) such that $G\beta$ is the solution of

$$\begin{aligned}\Delta\alpha&=\beta-H_3\beta,\\ H_3\alpha&=0,\end{aligned}$$

where H_3 was the operator defined in section 2. This leads to the orthogonal decomposition

$$\beta=d\delta G\beta+\delta dG\beta+H_3\beta.$$

Since G has a kernel $g(x,y)$, we have (with the same notation in section 2) $h_1(x,y)=d_x\delta_x g(x,y)$. Using

$$d_x\delta_x g(x,y)=-\delta_y d_y g(x,y)-h_3(x,y)\quad(x\neq y),$$

we can show that the formula (3.6) is written as

$$\begin{aligned}I^*(c,c')=&\int_{x\in c}\int_{y\in\partial c'}\delta_y *_y g(x,y)+\int_{y\in c'}\int_{x\in\partial c}\delta_x *_y g(x,y)\\ &+\int_{x\in c}\int_{y\in c'} *_y h_3(x,y),\end{aligned}\tag{4.1}$$

(cf. de Rham [4. §33. p. 174]).

Now the kernel h_3 of H_3 can be written as follows. Let $h_1,\cdots,h_k$ be an orthonormal basis of the one-dimensional cohomology group $H^1(M)$ $(k=\dim H^1(M))$. Then $h_3(x,y)$ can be written as $h_3(x,y)=\sum_{j=1}^{k}h_j(x)h_j(y)$. Let $c_1,\cdots,c_k$ be a basis of the $(d-1)$-dimensional homology group $H_{d-1}(M)$. We consider the intersection number $I(X[0,t],c_i)$ of $X[0,t]$ and c_i. Let Q be a countable dense subset of $(0,\infty)$ and $\Omega_0=\bigcap_{i=1}^{k}\bigcap_{t\in Q}\Omega(c_i,t)$. We note that by Lemma 3.1, if $x\notin\bigcup_{i=1}^{k}c_i$, then $P_x(\Omega_0)=1$. Using (4.1), we can write $I(X[0,t],c_i)$ more convenient form.

Lemma 4.1. *For any $\omega\in\Omega_0$ and for each $i=1,\cdots,k$, we have*

$$I(X[0,t],c_i)=\int_{y\in c_i}*_y\delta_x g(x,y)\Big|_{x=X(0)}^{X(t)}+\int_{x\in X[0,t]}\alpha^{(i)}\,, \tag{4.2}$$

[5]

where $\alpha^{(i)}=\int_{y\in c_i}*_y h_3(x,y)$.

Proof. Noting $\partial c_i=0$, we have by (4.1)

$$I_N(X[0,t],c_i)=\int_{y\in c_i}*_y\delta_x g(x,y)\Big|_{x=X(0)}^{X(t\wedge\sigma_N(c_i))}+\int_{x\in X[0,t\wedge\sigma_N(c_i)]}\alpha^{(i)}\,.$$

Letting $N\to\infty$, we obtain (4.2).

Now we can state the relation between the stochastic intersection number and the homological position of the path.

Proposition 4.1. *Let $c_1',\cdots,c_k'$ be a basis of $H_1(M)$ corresponding to $\alpha^{(1)},\cdots,\alpha^{(k)}$, that is, $\int_{c_i'}\alpha^{(j)}=\delta_{ij}$, $i,j=1,\cdots,k$. Let $(x_1(t),\cdots,x_k(t))$ be the homological position of $X[0,t]$ with respect to $(\{c_i\},\{c_i'\},C)$. Then we have for $x\notin\bigcap_{i=1}^{k}c_i$*

$$P_x(x_i(t)=I(X[0,t],c_i),\ i=1,\cdots,k,\ t\in Q)=1\,. \tag{4.3}$$

Proof. Let $\bar{X}[0,t]=X[0,t]+\psi_{X(t),X(0)}$. Since $I(\bar{X}[0,t],c_i)=I(X[0,t],c_i)$, using (4.2), we have

$$\int_{y\in c_i}*_y\delta_x g(x,y)\Big|_{x=X(0)}^{X(t)}=\int_{\psi_{X(t),X(0)}}\alpha^{(i)}\,. \tag{4.4}$$

On the other hand, since

$$x_i(t)=\int_{\bar{X}[0,t]}\alpha^{(i)}=\int_{X[0,t]}\alpha^{(i)}+\int_{\psi_{X(t),X(0)}}\alpha^{(i)}\,,$$

comparing this with (4.2), by (4.4) we have (4.3).

Now we proceed to study the limit behavior of $x_i(t)$. Set $f_i(x)=\alpha^{(i)}(b)(x)$, $i=1,\cdots,k$. We note that $\alpha^{(i)}$ is a harmonic 1-form and f_i is a C^∞ function on $M(i=1,\cdots,k)$. Let μ be the invariant measure of X with $\mu(M)=1$. Let R be the potential operator:

$$Rf(x)=\int_0^\infty(T_tf(x)-\bar{f})dt\,,$$

---

5) $f(x)\Big|_{x=a}^{b}=f(b)-f(a)$.

where $T_t f(x)=E_x[f(X_t)]$, $f\in C^\infty(M)$ and $\bar{f}=\int_M f(x)\mu(dx)$ (cf. [22]).

Theorem 4.1. (1) $\overline{\lim_{t\to\infty}}\frac{x_i(t)}{t}=\bar{f}_i,\ i=1,\cdots,k.$

(2) *In case* $\bar{f}_i=0$, *we have*

$$\overline{\lim_{t\to\infty}}\frac{x_i(t)}{\sqrt{2\,t\log\log t}}=-\underline{\lim}_{t\to\infty}\frac{x_i(t)}{\sqrt{2\,t\log\log t}}=\bar{a}_i\,,$$

where $\bar{a}_i$ *is constant defined by*

$$\bar{a}_i=\left(\int_M\langle\alpha^{(i)}+dRf_i,\ \alpha^{(i)}+dRf_i\rangle(x)\mu(dx)\right)^{1/2} \tag{4.5}$$

and $\langle\ ,\ \rangle(x)$ *denotes the inner product of* $T_x^*(M)$.

Before proceeding to the proof, we state remarks.

REMARK 4.1. Let M be a compact Riemann surface with genus $\kappa(\geqq 1)$. Let $(A_i,A_{\kappa+i})_{i=1,\cdots,\kappa}$ be a canonical homology basis and $(\alpha^{(i)},\alpha^{(\kappa+i)})_{i=1,\cdots,\kappa}$ be the corresponding 1-forms:

$$\int_{A_j}\alpha^{(i)}=\delta_{ij},\ i,j=1,\cdots,2\kappa\,.$$

We take $\{c_i\}$ and $\{c_i'\}$ as follows: $c_i=A_{\kappa+i}$, $1\leqq i\leqq\kappa$, $=A_{i-\kappa}$, $\kappa+1\leqq i\leqq 2\kappa$ and $c_i'=A_i$, $1\leqq i\leqq 2\kappa$. Intuitively, Theorem 4.1 implies the following. Denote by C_i the hole corresponding to $(A_i,A_{\kappa+i})$, $i=1,\cdots,\kappa$. For simplicity, we set $C_{\kappa+i}=C_i$, $i=1,\cdots,\kappa$.

If $\int_M\alpha^{(i)}(b)(x)\mu(dx)>0$ (or <0), then for almost all $\omega(P_x)$, the path $X[0,t]$ winds C_i infinitely often in the positive (or negative) direction along A_i. If $\int_M\alpha^{(i)}(b)(x)\mu(dx)=0$, then for almost all $\omega(P_x)$, the path $X[0,t]$ winds C_i infinitely often in both directions along A_i.

REMARK 4.2. (i) In the above theorem, we can give examples for which each of two cases really occurs.

(ii) If X is symmetric with respect to some measure ν, then only the case (2) of Theorem 4.1 can occur.

For the proof, we note that has a density $q(x)$ with respect to the Riemannian volume m and q satisfies the following equation:

$\int_M Lu(x)q(x)m(dx)=0$, for any $u\in C^\infty(M)$. It follows that $\delta(q\beta-\frac{1}{2}dq)=0$, where β is the 1-form such that β_x is the dual element of b_x for any $x\in M$. Therefore $q\beta$ s can be written

(4.7) $q\beta=\frac{1}{2}dq+\delta\beta_1+\beta_2$, where β_1 is a 2-form and β_2 is a harmonic 1-form[6]. Then we have $f_i=(\alpha^{(i)}, \beta_2)$, where $(\alpha,\beta)=\int_M \langle\alpha,\beta\rangle(x)m(dx)$. Since $\alpha^{(1)},\cdots,\alpha^{(k)}$ is a basis of $H^1(M)$, for prescrived real numbers $a_1,\cdots,a_k$, we can choose a harmonic 1-form β_2 so that $(\alpha^{(i)},\beta)=a_i$, which proves (i). To prove (ii), we note that the symmetricity of X implies $\delta\beta_1=\beta_2=0$ in (4.7). Thus we have $\bar{f}_i=0$, $i=1,\cdots,k$.

Proof of Theorem 4.1. Since $x_i(t)$ is of the form

$$x_i(t)=\int_{X[0,t]}\alpha^{(i)}+\int_{\psi_{X(t),X(0)}}\alpha^{(i)},$$

it is sufficient to consider $\int_{X[0,t]}\alpha^{(i)}$. Moreover since $\alpha^{(i)}$ is harmonic, it holds $\delta\alpha^{(i)}=0$ and by virtue of (2,2), we have

$$\int_{X[0,t]}\alpha^{(i)}=\sum_{j=1}^{d}\int_0^t\bar{\alpha}_j^{(i)}(r_s)dB_s^j+\int_0^t f_i(X_s)ds. \tag{4.8}$$

By the ergodic theorem, we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f_i(X(s))ds=\bar{f}_i. \tag{4.9}$$

Set $M(t)=\sum_{j=1}^{d}\int_0^t\bar{\alpha}_j^{(i)}(r_s)dB_s^j$. Then M is a continuous martingale and its quadratic variational process $\langle M\rangle(t)$ is

$$\langle M\rangle(t)=\int_0^t\langle\alpha^{(i)},\alpha^{(i)}\rangle(X_s)ds.$$

So by the ergodic theorem

$$\lim_{t\to\infty}\frac{\langle M\rangle(t)}{t}=\int_M\langle\alpha^{(i)},\alpha^{(i)}\rangle(x)\mu(dx)\equiv\rho_i. \tag{4.10}$$

Since $(h_1,\cdots,h_k)$ is an orthonormal basis of $H^1(M)$, we get $\rho_i>0$. By the representation theorem for continuous martingales, there exists a one-dimensional Brownian motion $\tilde{B}(t)$ such that

$$M(t)=\tilde{B}(\langle M\rangle(t)). \tag{4.11}$$

Using the law of the iterated logarithm for $\tilde{B}$, (4.10) and (4.11), we have

$$\overline{\lim_{t\to\infty}}\frac{M(t)}{\sqrt{2t\log\log t}}=-\underline{\lim_{t\to\infty}}\frac{M(t)}{\sqrt{2t\log\log t}}=\sqrt{\rho_i}. \tag{4.12}$$

Combining (4.8), (4.9) and (4.12), we have

6) The use of the Hodge-Kodaira's decomposition for $q\beta$ is due to S. Watanabe.

$$\lim_{t\to\infty}\frac{1}{t}\int_{X[0,t]}\alpha^{(i)} = \bar{f}_i ,$$

which proves (1). Next we show (2). If $\bar{f}_i=0$, then Rf_i is a smooth bounded function and satisfy the equation $LRf_i(x)=-f_i(x)$. Therefore by using Itô's formula, we have

$$\text{(4.13)} \qquad \int_{X[0,t]}\alpha^{(i)} = -(Rf_i(X_t)-Rf_i(X_0)) + \sum_{j,l=1}^{d}\int_0^t(\alpha_j^{(i)}(X_s)+\frac{\partial(Rf_i)}{\partial x^j})e_l^j dB_s^l ,$$

where e_l^j is a coordinate of orthonormal frames (see §2). The quadratic variational process of the second term of the right hand side of (4.13) is

$$a_i(t) = \int_0^t\langle\alpha^{(i)}+dRf_i,\ \alpha^{(i)}+dRf_i\rangle(X_s)ds$$

(cf. Ikeda-Manabe [8]). By the ergodic theorem, we have

$$\text{(4.14)} \qquad \lim_{t\to\infty}\frac{a_i(t)}{t} = \int_M\langle\alpha^{(i)}+dRf_i,\ \alpha^{(i)}+dRf_i\rangle(x)\mu(dx) = \bar{a}_i^2 .$$

Here $\bar{a}_i\neq 0$, in fact if $\bar{a}_i=0$, then $\langle\alpha^{(i)}+dRf_i, \alpha^{(i)}+dRf_i\rangle(x)\equiv 0$. Thus $\alpha^{(i)}+dRf_i\equiv 0$. From this, we conclude that $\alpha^{(i)}\equiv 0$. In fact, if $f_i\equiv 0$, then $\alpha^{(i)}\equiv 0$. In case $f_i\not\equiv 0$, the above equality implies $\alpha^{(i)}$ is exact. Since $\alpha^{(i)}$ is harmonic, it follows that $\alpha^{(i)}\equiv 0$. This is a contradiction. So we have $\bar{a}_i\neq 0$. Taking into account that $Rf_i(X_t)-Rf_i(X_0)$ is bounded, we have

$$\text{(4.15)} \qquad \overline{\lim_{t\to\infty}}\,(2\,t\log\log t)^{-1/2}\int_{X[0,t]}\alpha^{(i)} = -\underline{\lim_{t\to\infty}}\,(2t\log\log t)^{-1/2}\int_{X[0,t]}\alpha^{(i)} = \bar{a}_i ,$$

which proves the assertion (2).

5. Covering motions. In this section, we prepare several known facts about covering motions on normal covering surfaces of M, since in the next section, covering motions on Abelian covering surfaces play an important role. We refer to Ahlfors-Sario [1] for covering surfaces of a Riemann surface and to Itô-McKean [10] for covering motions on covering surfaces. Throughout this section, we assume that (M,g) is a two-dimensional compact connected orientable Riemannian manifold with genus $\kappa(\geqq 1)$. Under this assumption, (M,g) can be regarded as a compact Riemann surface with genus κ by taking isothermal coordinates.

Let $\tilde{M}$ be a normal covering surface of M. Let $\tilde{G}=\pi_1(\tilde{M})$ be its fundamental group and $\tilde{\Gamma}$ be its covering transformation group. It is known that $\tilde{\Gamma}\cong\pi_1(M)/\tilde{G}$.

Let $\pi_{\tilde{G}}: \tilde{M} \to M$ be the natural projection. $\pi_{\tilde{G}}$ is a local conformal mapping. As is described in Itô-McKean [10. Chap. 7], we can define a covering Brownian motion $\tilde{X} = (\tilde{X}_t, \tilde{P}_{\tilde{x}}, \tilde{x} \in \tilde{M})$ on $\tilde{M}$ by continuation along X.

By using the uniformization theorem, we can give a concrete construction of covering motions (in case $\kappa \geqq 2$). We begin with the following

Lemma 5.1 (Lévy, see [14]). *Let (M, g) and (M', g') be compact, connected and orientable Riemannian manifolds of dimension two. We regard them Riemann surfaces. Assume that they are conformally equivalent. Let f be a conformal homeomorphism of M onto M'. Let X and X' be Brownian motions on M and M', respectively. Then $f^{-1}(X')$ can be obtained by a time change of X.*

Following lemma is well-known (see Springer [20]).

Lemma 5.2. *Let $\kappa \geqq 2$. There exists a one-to-one correspondence between normal covering surfaces of $\tilde{M}$ and normal subgroups $\tilde{G}$ of $\pi_1(M)$ in such a way that $\tilde{M} \cong \tilde{G} \backslash H$ (conformally equivalent), where H is the upper-half plane and $\pi_1(\tilde{M}) \cong \tilde{G}$.*

Now we proceed to construct a Brownian motion on a normal covering surface $\tilde{M}$ of M (in case $\kappa \geqq 2$). By the above two lemmas, it is sufficient to consider the case $M = \tilde{G} \backslash H$, $G \cong \pi_1(M)$. Let $\hat{g}$ be the Poincaré metric in H. Then the distance $d(\hat{x}, \hat{y})$ of $\hat{x}$ and $\hat{y}$ in H is of the form

$$d(\hat{x}, \hat{y}) = \cosh^{-1}\left[1 + \frac{(\hat{x}^1 - \hat{y}^1)^2 + (\hat{x}^2 - \hat{y}^2)^2}{2\hat{x}^2 \hat{y}^2}\right], \tag{5.1}$$

and satisfies

$$d(\gamma\hat{x}, \gamma\hat{y}) = d(\hat{x}, \hat{y}) \qquad \text{for } \gamma \in G. \tag{5.2}$$

Let $\tilde{G}$ be a normal subgroup of G and $\tilde{M}$ be the corresponding covering surface of M. Let $\rho_{\tilde{G}}: H \to \tilde{M}$ and $\rho: H \to M$ be the natural projections. Since $\rho_{\tilde{G}}$ and ρ are local diffeomorphisms, we can give the Riemannian metrics $g_{\tilde{G}}$ and g on $\tilde{M}$ and M, respectively, by the pull-back of $\hat{g}$. Let Δ_H be the Laplace-Beltrami operator corresponding to $\hat{g}$ and let $\hat{p}(t, \hat{x}, \hat{y})$ be the fundamental solution of the heat equation (see [15):

$$\hat{p}(t, \hat{x}, \hat{y}) = \frac{\sqrt{2}e^{-t/8}}{(2\pi t)^{3/2}} \int_{d(\hat{x}, \hat{y})}^{\infty} \frac{b e^{-b^2/2t}}{\sqrt{\cosh b - \cosh d(\hat{x}, \hat{y})}} \, db. \tag{5.3}$$

Let $X = (\hat{X}_t, \hat{P}_{\hat{x}}, \hat{x} \in H)$ be the diffusion process corresponding to $\Delta_H/2$ i.e. Brownian motion on H. Denote by D and $D(\tilde{G})$ fundamental domains of G and $\tilde{G}$, respectively. Define

$$p_M(t, x, y) = \sum_{\gamma \in G} \hat{p}(t, \hat{x}, \gamma\hat{y}), \tag{5.4}$$

where $\hat{x}, \hat{y} \in D$ and $x = \rho(\hat{x})$, $y = \rho(\hat{y})$. Then $p_M(t, x, y)$ is the minimal fundamental solution of $\partial/\partial t-\frac{1}{2}\Delta_M$. Similarly define

(5.5) $$p_{\tilde{M}}(t,\tilde{x},\tilde{y})=\sum_{\gamma\in\tilde{G}}\hat{p}(t,\hat{x},\gamma\hat{y}),$$

where $\hat{x},\hat{y}\in D(\tilde{G})$ and $\tilde{x}=\rho_{\tilde{G}}(\hat{x})$, $y=\rho_{\tilde{G}}(\hat{y})$. Then $p_{\tilde{M}}(t,\tilde{x},\tilde{y})$ is the minimal fundamental solution of $\partial/\partial t-\frac{1}{2}\Delta_{\tilde{M}}$. Let X and $\tilde{X}$ be the diffusion processes on M and $\tilde{M}$ defined by $X_t=\rho(\hat{X}_t)$ and $\tilde{X}_t=\rho_{\tilde{G}}(\hat{X}_t)$, $t\geqq 0$, respectively. Then the transition probability densities of X and $\tilde{X}$ are p_M and $p_{\tilde{M}}$, respectively. X and $\tilde{X}$ are nothing but a Brownian motion on M and a covering Brownian motion on $\tilde{M}$, respectively.

6. Asymptotic behavior of homological position (II). Throughout this section, $X=(X_t,P_x,x\in M)$ is a Brownian motion on a two-dimensional compact connected Riemannian manifold (M,g) with genus $\kappa(\geqq 1)$. We formulate our problem as follows (cf. Arnold-Avez [2. Appendix 16]). Let $(A_i, A_{\kappa+i})$, $i=1,\cdots,\kappa$, be a canonical homology basis and $\alpha^{(i)},\alpha^{(\kappa+i)}$, $i=1,\cdots,\kappa$, be the harmonic 1-forms corresponding to A_i, $A_{\kappa+i}$, respectively, i.e.

$$\int_{A_j}\alpha^{(i)}=\delta_{ij}\quad(i,j=1,\cdots,2\kappa).$$

We define $\{c_i\}$ and $\{c_i'\}$ as in Remark 4.1: $c_i=A_{\kappa+i}$, $1\leqq i\leqq\kappa$, $=A_{i-\kappa}$, $\kappa+1\leqq i\leqq 2\kappa$ and $c_i'=A_i$, $1\leqq i\leqq 2\kappa$. Consider the homological position $(x_1(t),\cdots,x_{2\kappa}(t))$ of $X[0,t]$ with respect to $(\{c_i\},\{c_i'\},C)$ defined in section 4.

DEFINITION. Let N be any integer with $1\leqq N\leqq 2\kappa$ and let $1\leqq i_1<\cdots<i_N\leqq 2\kappa$. We call the path of X winds $(A_{i_1},\cdots,A_{i_N})$ homologically if

$$\lim_{t\to\infty}\sum_{\lambda=1}^{N}x_{i_\lambda}(t)^2=\infty,\ P_x\text{-a.s.}$$

We shall prove

Theorem 6.1. *X winds $(A_{i_1},\cdots,A_{i_N})$ homologically if and only if $N\geqq 3$. Especially the path of X winds $(A_1,\cdots,A_{2\kappa})$ homologically if and only if $\kappa\geqq 2$.*

Proof. Note that (M,g) can be regarded as a compact Riemann surface with genus κ by taking isothermal coordinates. Denote $I=(i_1,\cdots,i_N)$. Let $\tilde{M}(I)$ be the Abelian covering surface of M on which $\left(\int_{p_0}^{p}\alpha^{(i_1)},\cdots,\int_{p_0}^{p}\alpha^{(i_N)}\right)$ is single-valued (see Tsuji [21. Chap. X. §15]). Let π_I: $\tilde{M}(I)\to M$ be the natural projection. For any curve γ in M starting at p_0, set $\Psi_I(\gamma)=\left(\int_\gamma\alpha^{(i_1)},\cdots,\int_\gamma\alpha^{(i_N)}\right)$. Let $\tilde{p}_0$ be a point of $\tilde{M}(I)$ which lies over p_0, i.e. $\pi_I(\tilde{p}_0)=p_0$. Note that, for the lift $\tilde{\gamma}(I)$ of γ starting at $\hat{p}_0$, it holds that

$$\Psi_I(\gamma) = \left(\int_{\tilde{\gamma}(I)} \pi_I^* \alpha^{(i_1)}, \cdots, \int_{\tilde{\gamma}(I)} \pi_I^* \alpha^{(i_N)}\right).$$

Since this depends only on the end point $\tilde{p}$ of $\tilde{\gamma}(I)$, we can define a mapping $\tilde{\Phi}_I(\tilde{p})$ from $\tilde{M}(I)$ into $\boldsymbol{R}^N$ by $\tilde{\Phi}_I(\tilde{p}) = \Psi_I(\gamma)$. $\tilde{\Phi}_I$ is a continuous mapping from $\tilde{M}(I)$ into $\boldsymbol{R}^N$. Because $\tilde{M}(I)$ is a normal covering surface of M, we can define a covering Brownian motion $\tilde{X}_I$ of X as in section 5. Now let

$$\tilde{Y}_i(t) = \int_{X[0,t]} \alpha^{(i)},\ i = 1, \cdots, 2k \text{ and } \tilde{Y}_I(t) = (\tilde{Y}_{i_1}(t), \cdots, \tilde{Y}_{i_N}(t)).$$

Then we obtain

$$\tilde{Y}_I(t) = \tilde{\Phi}_I(\tilde{X}_I(t)). \tag{6.1}$$

Indeed, let X_n be a polygonal approximation of X and set

$$\tilde{Y}_i^{(n)}(t) = \int_{X_n[0,t]} \alpha^{(i)},\ \tilde{Y}_I^{(n)}(t) = (\tilde{Y}_{i_1}^{(n)}, \cdots, \tilde{Y}_{i_N}^{(n)}(t)).$$

Because X_n is piecewise smooth, it holds that $\tilde{Y}_I^{(n)}(t) = \tilde{\Phi}_I(\tilde{X}_n(t))$. On the other hand, note that there exists a positive integer $n_0(\omega)$ such that $X_n[0,t]$ and $X[0,t]$ are homologous for $n \geqq n_0(\omega)$. Since $\alpha^{(i)}$ is harmonic, $\tilde{Y}_I(t)$ depends only on the homology class of $X[0,t]$ (see [7]), so we have $\tilde{Y}_I(t) = \tilde{Y}_I^{(n)}(t)$ for $n \geqq n_0(\omega)$. Hence (6.1) holds. By the definition of $\tilde{M}(I)$ ([1], Chap. 1]), we see that for any compact set K of $\tilde{\Phi}_I(\tilde{M}(I))$, there exists a compact set $\tilde{K}$ of $\tilde{M}(I)$ such that $\tilde{\Phi}_I(\tilde{M}(I) \backslash \tilde{K}) \subset \boldsymbol{R}^N \backslash K$. By using this fact and (6.1), we can easily see that the continuous stochastic process $\tilde{Y}_I$ tends to infinity or not with probability one i.e., $\|\tilde{Y}_I(t)\|$[7]$\to \infty$ or not a.s. as $t \to \infty$ according as $\tilde{X}_I$ is transient or not. Since each $x_i(t)$ differs from $\tilde{Y}_i(t)$ by only a bounded term ($i = 1, \cdots, 2\kappa$), X winds $(A_{i_1}, \cdots, A_{i_N})$ homologically or not according as $\tilde{Y}_I$ tends to infinity or not with probability one. Hence X winds $(A_{i_1}, \cdots, A_{i_N})$ homologically if and only if $\tilde{X}_I$ is transient. If $\kappa = 1$, then $\tilde{M}(I) = \boldsymbol{R}^2$ for $I = (1,2)$. Therefore in this case, $\tilde{X}_I$ is nothing but a time change of a two-dimensional Brownian motion. So $\tilde{X}_I$ is recurrent, that is X does not wind (A_1, A_2). Now let $\kappa \geqq 2$. In this case, since $\tilde{M}(I)$ is a normal covering surface of M, there exists the transition probability density $p_{\tilde{M}(I)}$ of $\tilde{X}_I$ as is given by (5.5). For simplicity, write p_I for $p_{\tilde{M}(I)}$. Set

$$g_I(\tilde{x}, \tilde{y}) = \int_0^\infty p_I(t, \tilde{x}, \tilde{y}) dt. \tag{6.2}$$

Then it holds that (i) $g_I(\tilde{x}, \tilde{y}) \equiv \infty$, or (ii) $g_I(\tilde{x}, \tilde{y}) < \infty$, for $\tilde{x} \neq \tilde{y}$. Moreover, in this case, g_I is the Green function of $\tilde{M}(I)$ (for the definition of the Green function, see [21. p. 18]). By virtue of the result of [11] and Theorem 6C of [1], we see that the case (i) is equivalent to the recurrence of $\tilde{X}_I$ and the case (ii) is

7) $\|y\| = \left(\sum_{i=1}^N (y^i)^2\right)^{1/2}$ for $y = (y^1, \cdots, y^N) \in \boldsymbol{R}^N$.

equivalent to the transience of $\tilde{X}_I$. Let us recall that the type of a Riemann surface is hyperbolic (resp. parabolic) if there exists (resp. does not exist) a Green function. In view of this, it remains to determine the type of $\tilde{M}(I)$. Applying the following theorem due to Mori to our situation, that is, $F=\tilde{M}(I)$ and $r=N$, the proof of Theorem 6.1 is completed.

Mori's theorem ([16]). *Let F be an Abelian covering surface of M whose covering transformation group $\Gamma(F)$ has rank r. Then F is hyperbolic if and only if $r\geqq 3$.*

In case $\kappa\geqq 2$, using a concrete form of the transition probability density, we have the following result.

REMARK 6.1. Let $\kappa\geqq 2$. Then by Lemma 5.2, $M\cong G\backslash H(G\cong\pi_1(M))$. G is a Fuchsian group whose fundamental domain D is compact. It holds that $\tilde{M}(I)$ is parabolic according as $\sum_{\gamma\in\tilde{G}}||\gamma||^{-2}=\infty$, where $||\gamma||^2=a^2+b^2+c^2+d^2$, $\gamma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ and $\tilde{G}$ is the normal subgroup of G corresponding to $\tilde{M}(I)$.

Proof. Let $g_I(\tilde{x},\tilde{y})$ be the function given by (6.2). In view of (5.5), we must check the convergence of the series $\sum_{\gamma\in\tilde{G}}\int_0^\infty \hat{p}(t,\hat{x},\gamma\hat{y})dt$. Fix a positive number a_0. Since $\#\{\gamma\in\tilde{G};\ d(\hat{x},\gamma\hat{y})<a_0\}<\infty$ (see Lehner [12. p. 117]), we have only to examine the series $\sum_{\gamma\in\tilde{G}_1}\int_0^\infty \hat{p}(t,\hat{x},\gamma\hat{y})dt$, where $\tilde{G}_1=\{\gamma\in\tilde{G};\ d(\hat{x},\gamma\hat{y})\geqq a_0\}$. By a formula of Laplace transform (see [5]), we have

$$\int_0^\infty \hat{p}(t,\hat{x},\gamma\hat{y})dt=\frac{1}{\pi\sqrt{2}}\int_{d(\hat{x},\gamma\hat{y})}^\infty \frac{e^{-b/2}db}{\sqrt{\cosh b-\cosh d(\hat{x},\gamma\hat{y})}}\,. \tag{6.3}$$

In view of the following inequalities

$$\frac{1}{\sqrt{\cosh b-\cosh a}}\geqq e^{-b/2},\quad \text{for } b>a\,,$$

and

$$\frac{1}{\sqrt{\cosh b-\cosh a}}\leqq\sqrt{\frac{2}{1-e^{-2a_0}}}\,e^{-a/2}\frac{1}{\sqrt{b-a}}\,,\quad \text{for } b>a\geqq a_0\,,$$

we have

$$\exp\{-d(\hat{x},\gamma\hat{y})\}\leqq\int_0^\infty \hat{p}(t,\hat{x},\gamma\hat{y})dt\leqq K\exp\{-d(\hat{x},\gamma\hat{y})\}\,, \tag{6.4}$$

where K is a positive constant depends only on a_0. Using the inequality

$$||\gamma||^{-2}\leqq\exp\{-d(\sqrt{-1},\gamma\sqrt{-1})\}\leqq 2||\gamma||^{-2}\,,$$

(5.2) and the triangular inequality, we have

$$C^{-1}||\gamma||^{-2} \leqq \exp\{-d(\hat{x}, \gamma\hat{y})\} \leqq 2C||\gamma||^{-2}, \tag{6.5}$$

where $C=\exp\{d(\hat{x}, \sqrt{-1})+d(\sqrt{-1}, \hat{y})\}$. By these two inequalities (6.4) and (6.5), it follows that

$$C^{-1}\sum_{\gamma\in\tilde{G}_1}||\gamma||^{-2} \leqq \sum_{\gamma\in\tilde{G}_1}\int_0^\infty p(t, \hat{x}, \gamma\hat{y})dt \leqq 2CK\sum_{\gamma\in\tilde{G}_1}||\gamma||^{-2},$$

which proves our assertion.

REMARK 6.2. We can consider the homotopic behavior. Let $\tilde{X}[0,t]$ as before. Let $\sigma_1, \cdots, \sigma_{2\kappa}$ be a generator of $\pi_1(M)$. We write $\tilde{X}[0,t]=\sigma_{j_1}^{\varepsilon_1}\cdots\sigma_{j_l}^{\varepsilon_l}(\varepsilon_1, \cdots, \varepsilon_l=\pm 1)$, where the equality means that $\tilde{X}[0,t]$ and $\sigma_{j_1}^{\varepsilon_1}\cdots\sigma_{j_l}^{\varepsilon_l}$ are homotopic. Set

$$l(t) = \min\{l;\ \tilde{X}[0,t] = \sigma_{j_1}^{\varepsilon_1}\cdots\sigma_{j_l}^{\varepsilon_l},\ \varepsilon_1, \cdots, \varepsilon_l = \pm 1\}.$$

We call $l(t)$ the length of $\tilde{X}[0,t]$. We say that the path of X winds homotopic if the length of $\tilde{X}[0,t]$ tends to infinity as $t\to\infty$.

(i) If $\kappa=1$, then $\pi_1(M)$ is isomorphic to $H_1(M)$, the problem is the same as the homological behavior.

(ii) In case $\kappa>1$, then the path of X winds homotopic. In fact, since the Brownian motion $\hat{X}$ on H approaches to the boundary of H, the length of $\tilde{X}[0,t]$ tends to infinity as $t\to\infty$.

Before closing this section, we present a probabilistic proof of Theorem 6.1, which is due to Professor S.R.S. Varadhan.[8)]

In the sequel, we write $\tilde{Y}(t)$ for $\tilde{Y}_I(t)$ and $\alpha^{(j)}$ for $\alpha^{(i_j)}$ $(j=1, \cdots, N)$ for simplicity. Let $Y(t;y)=y+\tilde{Y}(t)$, $Z_t=(X_t, Y_t)$ and $P_{x,y}(Z.\in A)=P_x((X., y+Y.)\in A)$. Then $Z=(Z_t, P_{x,y})$ is a diffusion process on $M\times \boldsymbol{R}^N$. We give a canonical coordinate $(y^1, \cdots, y^N)$ in $\boldsymbol{R}^N$. If $((x^1, x^2), U)$ be any local coordinate system on M, then the generator of Z on $C^2(U\times\boldsymbol{R}^N)$ is given by

$$L = \frac{1}{2}\Delta + \frac{1}{2}\sum_{j=1}^{2}\sum_{i=1}^{N} b^{ij}(x)\frac{\partial^2}{\partial x^j\partial y^i} + \frac{1}{2}\sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2}{\partial y^i\partial y^j},$$

where $a^{ij}(x)=\langle\alpha^{(i)}, \alpha^{(j)}\rangle(x)$, $b^{ij}(x)=\sum_{l=1}^{2}\alpha_l^{(i)}(x)g^{jl}(x)$. We set $a(x)=(a^{ij}(x))$. We note that $a^{ij}(x)$ is invariant under the changes of local coordinate (x^1, x^2). Without loss of generality, we can assume

$$\int_M \langle\alpha^{(i)}, \alpha^{(j)}\rangle(x)m(dx) = \delta_{ij},\quad i,j = 1, \cdots, N. \tag{6.6}$$

Indeed, for given $A_1, \cdots, A_N$, it is sufficient to take a suitable 1-chains $B_1, \cdots, B_N$ such that corresponding 1-forms $\beta^{(i)}$, $1\leqq i\leqq N$, satisfies (6.6) by Schmidt orthogonalization.

8) Private communication.

First we show the following lemma.

Lemma 6.1. *Let $\lambda>0$ and $Y_\lambda(t;\sqrt{\lambda}y)=Y(\lambda t;\sqrt{\lambda}y)/\sqrt{\lambda}$. Denote by W the space of continuous functions from $[0,\infty)$ into $\boldsymbol{R}^N$. Let $P_{x,y}^\lambda$ be the probability measure on W induced by Y_λ. Then $P_{x,y}^\lambda$ converges weakly to the probability measure of N-dimensional Brownian motion $\bar{P}_y$. This convergence is uniform in $x\in M$ and $y\in K$ for any compact set K in $\boldsymbol{R}^N$.*

Proof. Set $\theta=(\theta_1,\cdots,\theta_N)\in\boldsymbol{R}^N$. By Ito's formula

$$\exp\left[i\langle\theta,\frac{Y(\lambda t;\sqrt{\lambda}y)}{\sqrt{\lambda}}\rangle-i\langle\theta,y\rangle+\frac{1}{2\lambda}\sum_{i,j=1}^{N}\int_0^{\lambda t}a^{ij}(X_s)ds\theta_i\theta_j\right]$$

is a $P_{x,y}$-martingale. So we have

$$e^{-i\langle\theta,y\rangle}E_{x,y}[e^{i\langle\theta,\frac{Y(\lambda t)}{\sqrt{\lambda}}\rangle+\frac{1}{2}\langle\theta,\frac{1}{\lambda}\int_0^{\lambda t}a(X_s)ds\,\theta\rangle}]=1\,.$$

By the ergodic theorem, we have

$$\lim_{\lambda\to\infty}\langle\theta,\frac{1}{\lambda}\int_0^{\lambda t}a(X_s)ds\theta\rangle=t\langle\theta,\bar{a}\theta\rangle\,,$$

where $\bar{a}=(\bar{a}^{ij})$ and $\bar{a}^{ij}=\int_M a^{ij}(x)m(dx)=\delta_{ij}$. Thus we obtain

$$\lim_{\lambda\to\infty}E_{x,y}[e^{i\left\langle\theta,\frac{Y(\lambda t)}{\sqrt{\lambda}}\right\rangle}]=\exp\left[-\frac{t}{2}\langle\theta,\bar{a}\theta\rangle+i\langle\theta,y\rangle\right].\tag{6.7}$$

By martingale inequality, there exists a positive constant C such that

$$\sup_{x\in M,y\in K}E_{x,y}\left[\left\|\frac{Y(\lambda t)}{\sqrt{\lambda}}-\frac{Y(\lambda s)}{\sqrt{\lambda}}\right\|^4\right]\leqq C\,|t-s|^2\,.\tag{6.8}$$

Therefore $P_{x,y}^\lambda$ converges weakly to $\bar{P}_y$. Next we show the uniformity of convergence. We shall show that

$$\lim_{\lambda\to\infty}\sup_{x\in M,y\in K}|E_{x,y}^\lambda[f]-E^{\bar{P}_y}[f]|=0\,,\tag{6.9}$$

for any bounded continuous function f on W. Suppose (6.9) does not hold for some $f\in C(W)$. Then there exist $\varepsilon>0$ and $\lambda_1<\lambda_2<\cdots<\lambda_n\uparrow\infty$ such that

$$\sup_{x\in M,y\in K}|E_{x,y}^{\lambda_n}[f]-E^{\bar{P}_y}[f]|\geqq\varepsilon\,,\qquad n=1,2,\cdots.\tag{6.10}$$

Note that $E_{x,y}^{\lambda_n}[f]-E^{\bar{P}_y}[f]$ is continuous in (x,y). Indeed, it is sufficient to show that $E_{x,y}^{\lambda_n}[f]$ is continuous in (x,y). Let $x_m\to x_0$ and $y_m\to y_0$. Then by virtue of (6.8), there exists a subsequence $(x_{m(j)},y_{m(j)})$ such that $P_{x_{m(j)},y_{m(j)}}^{\lambda_n}$ converges weakly to some Q. But by the strong Feller property of X, we see that the finite dimensional distribution of Q is $P^{\lambda_n}_{x_0,y_0}$. So $Q=P^{\lambda_n}_{x_0,y_0}$. This implies the continuity of $E^{\lambda_n}_{x,y}[f]$. Therefore there exist $x_n\in M$, $y_n\in K$ such that

$$|E^{\lambda_n}_{x_n,y_n}[f]-E^{\bar{P}_{y_n}}[f]| = \sup_{x\in M, y\in K} |E^{\lambda_n}_{x,y}[f]-E^{\bar{P}_y}[f]| .$$

We can choose subsequences $\{x_{n(j)}\}$ and $\{y_{n(j)}\}$ such that they converge to some points x_0 and y_0, respectively. Then we have

$$\lim_{j\to\infty} |E^{\lambda_{n(j)}}_{x_{n(j)},y_{n(j)}}[f]-E^{\bar{P}_{y_0}}[f]| = 0 .$$

This contradicts to (6.10), which proves the lemma.

Now we proceed to prove the theorem. Let τ_n be the stopping time defined by

$$\tau_n = \inf\{t;\ ||Y_t|| = 2^{n-1} \text{ or } 2^{n+1}\} , \qquad n = 1, 2, \cdots,$$

where $||y||=\langle y, y\rangle^{1/2}$ for $y\in \boldsymbol{R}^N$. Define $u_n(x, y)=P_{x,y}(||Y(\tau_n)||=2^{n+1})$. Set $Y_n(t)=\dfrac{Y(2^{2n}t)}{2^n}$ and $\tau_n'=\inf\left\{t;\ ||Y_n(t)||=\dfrac{1}{2} \text{ or } 2\right\}$, then it holds that

$$\tau_n = 2^{2n}\tau_n' , \quad Y_n(\tau_n') = Y(\tau_n)/2^n \tag{6.11}$$

and

$$u_n(x, y) = P_{x,y}(||Y_n(\tau_n')|| = 2) . \tag{6.12}$$

Proof of Theorem. Case of $N\geqq 3$. First we note that each point of the set $A=\{x\in \boldsymbol{R}^N;\ ||x||=2^{-1} \text{ or } ||x||=2\}$ is a regular point (of the Brownian motion) for A. Set $\tau(w)=\inf\{t;\ ||w(\tau)||=2^{-1} \text{ or } 2\}$ for $w\in W$. Since the set $\{w\in W;\ ||w(\tau)||=2^{-1}\}$ is a $\bar{P}_y$-continuity set, using Lemma 6.1, we have

$$\lim_{n\to\infty} \sup_{x\in M, ||y||=1} |P_{x,2^{n+1}y}(||Y_{\tau_{n+1}}|| = 2^n)-\bar{P}_y(||w(\tau)|| = 2^{-1})| = 0 . \tag{6.13}$$

Therefore, since $N\geqq 3$, by using (6.13) and the well-known formula $\bar{P}_y(||w(\tau)||=2^{-1})=(||y||^{2-N}-2^{2-N})/(2^{N-2}-2^{-N+2})$, we conclude that there exist positive constants n_0, α_N and β_N with $4\alpha_N\beta_N<1$ and $2^{-1}<\beta_N$ such that for any $n\geqq n_0$,

$$\sup_{x\in M, ||y||=2^{n+1}} P_{x,y}(||Y(\tau_{n+1})|| = 2^n)\leqq \alpha_N \tag{6.14a}$$

and

$$\sup_{x\in M, ||y||=2^{n+1}} P_{x,y}(||Y(\tau_{n+1})|| = 2^{n+2})\leqq \beta_N . \tag{6.14b}$$

Define $\sigma_n=\inf\{t;\ ||Y(t)||=2^n\}$ and $\eta_n=Y(\tau_n)$, $n=1, 2, \cdots$. Then using the strong Markov property and (6.14), we have for $x\in M$, $||y||=2^{n+1}$,

$$P_{x,y}(\sigma_n<\infty)=P_{x,y}(\eta_{n+1}=2^n)$$
$$+\sum_{k=2}^{\infty}P_{x,y}(\eta_{n+1}>2^n,\ \eta_{n+2}>2^n,\ \cdots,\ \eta_{n+2k-2}>2^n,\ \eta_{n+2k-1}=2^n)$$
$$=P_{x,y}(\eta_{n+1}=2^n)+\sum_{k=2}^{\infty}\sum_{\substack{\varepsilon_2+\cdots+\varepsilon_{2k-2}=-1\\ \varepsilon_2,\cdots,\varepsilon_{2k-2}=\pm 1}}$$
$$P_{x,y}(\eta_{n+1}=2^{n+2},\ \eta_{n+2}=2^{n+2+\varepsilon_2},\ \cdots,\ \eta_{n+2k-2}=2^{n+2+\varepsilon_2+\cdots+\varepsilon_{2k-2}},$$
$$\eta_{n+2k-1}=2^n)$$
$$\leqq\alpha_N+\sum_{k=2}^{\infty}\sum_{\substack{\varepsilon_2+\cdots+\varepsilon_{2k-2}=-1\\ \varepsilon,\cdots,\varepsilon_{2k-2}=\pm 1}}\alpha_N{}^k\beta_N{}^{k-1}.$$

Because $\sum_{\substack{\varepsilon_2+\cdots+\varepsilon_{2k-2}=-1\\ \varepsilon_2,\cdots,\varepsilon_{2k-2}=\pm 1}} 1=k^{-1}\binom{2k-2}{k-1}$, we have

$$P_{x,y}(\sigma_n<\infty)\leqq\sum_{k=1}^{\infty}k^{-1}\binom{2k-2}{k-1}\alpha_N{}^k\beta_N{}^{k-1}.$$

Using $(\alpha_N\beta_N)^k\leqq 2^{-2k}$, $2\beta_N>1$ and the following formula $\sum_{k=1}^{\infty}k^{-1}\binom{2k-2}{k-1}2^{-2k}=2^{-1}$, we conclude $P_{x,y}(\sigma_n<\infty)<(2\beta_N)^{-1}<1$. Thus $\lim_{t\to\infty}\|Y(t)\|=\infty$, a.s.

Next we prove the case $N=2$. We will show that the following estimate (6.15), from which our theorem follows.

$$\sup_{x\in M,\,\|y\|=2^n}|u_n(x,y)-\frac{1}{2}|=O(2^{-n}),\qquad n\to\infty. \tag{6.15}$$

Let $r(y)=\langle y,y\rangle^{1/2}$ and $r_t=r(Y_t)$. Then by Itô's formula, we have $\log r_t-\log r_0-\int_0^t(L\log r)(Z_s)ds$ is a martingale. Therefore if $\|y\|^n=2$, then by a simple calculation we get

$$E_{x,y}[\log r_{\tau_n}-\log r_0]=(\log 2)[2u_n(x,y)-1].$$

So we have

$$\sup_{x\in M,\,\|y\|=2^n}\left|u_n(x,y)-\frac{1}{2}\right| \tag{6.16}$$
$$\leqq\frac{1}{2\log 2}\sup_{x\in M,\,\|y\|=2^n}|E_{x,y}(L\log r)(Z_{\tau_n})|.$$

Note that

$$(L\log r)(x,y)=\frac{Tr[a(x)-I]}{2r(y)^2}-\frac{\langle(a(x)-I)y,y\rangle}{r(y)^4}. \tag{6.17}$$

Since each term of (6.17) is of the form $V(x)\phi(y)$ with

$$\int_M V(x)m(dx)=0, \tag{6.18}$$

(6.19) $\phi(y)$ is a homogeneous function of degree -2,
it is sufficient to consider

$$\sup_{x\in M,\|y\|=2^n}\left|E_{x,y}\left[\int_0^{\tau_n}[V(X_s)\phi_s(Y)ds\right]\right|$$

in order to estimate the right hand side of (6.16). Let $A(t)=\int_0^t V(X_s)ds$. Then we have

$$E_{x,y}\left[\int_0^{\tau_n}V(X_s)\phi(Y_s)ds\right]=E_{x,y}[A(\tau_n)\phi(Y_{\tau_n})]-E_{x,y}\left[\int_0^{\tau_n}A(s)\psi(X_s,\ Y_s)ds\right],$$

where $\psi(x, y)=L\phi$. By (6.18), there exists a function $v(x)$ such that $Lv(x)=V(x)$. Therefore by Itô's formula,

$$\begin{aligned}A(t)&=\int_0^t V(X_s)ds=\int_0^t Lv(X_s)ds\\&=v(X_t)-v(X_0)+\text{a martingale}.\end{aligned}$$

Set $A^*(t)=\sup_{0\leqq s\leqq t}|A(s)|$. By virtue of martingale inequality, we have

$$E_{x,y}[A^*(\tau_n)^p]\leqq K+K_p(E_{x,y}[\tau_n])^{p/2},\qquad p=1,2. \tag{6.20}$$

Using Lemma 6.1 and (6.11), we obtain

$$\sup_{x\in M,\|y\|=2^n}E_{x,y}[\tau_n{}^p]\leqq C_p2^{2np},\qquad p=1,2. \tag{6.21}$$

Using (6.19), (6.20) and (6.21), we get

$$\begin{aligned}|E_{x,y}[A(\tau_n)\phi(Y_{\tau_n})]|&\leqq\sup_{\|y\|=1}|\phi(y)|E_{x,y}[A^*(\tau_n)]2^{-2(n-1)}\\&\leqq\sup_{\|y\|=1}|\phi(y)|[K_0+K_1\sqrt{C_1}2^n]2^{-2(n-1)}\leqq K_32^{-n}.\end{aligned} \tag{6.22}$$

Since ψ is a homogeneous function in y of degree -4, by using (6.20) and (6.21), it holds

$$\begin{aligned}&E_{x,y}\left[\int_0^{\tau_n}A(s)\psi(X_s,\ Y_s)ds\right]\\&\leqq\sup_{x\in M,\|y\|=1}|\psi(x,y)|E_{x,y}[\tau_nA^*(\tau_n)]2^{-4(n-1)}\\&\leqq\sup_{x\in M,\|y\|=1}|\psi(x,y)|(E_{x,y}[\tau_n{}^2])^{1/2}(E_{x,y}[A^*(\tau_n)^2])^{1/2}2^{-4(n-1)}\\&\leqq K_42^{-n}.\end{aligned} \tag{6.23}$$

Combining (6.22) and (6.23), we conclude that

$$\sup_{x\in M,\|y\|=2^n}\left|E_{x,y}\left[\int_0^{\tau_n}V(X_s)\phi(Y_s)ds\right]\right|\leqq K_52^{-n},$$

which implies (6.15). Now we proceed to the proof. We use the same notation as in the proof of the case $N\geqq 3$. By (6.15), we have

(6.24) $$\inf_{x\in M,\,||y||=2^n} P_{x,y}(||Y(\tau_n)||=2^{n+1})\geqq 2^{-1}-K2^{-n} \qquad \text{for } n\geqq n_0 .$$

Set $S^{(n)}=\inf_{x\in M,\,||y||=2^{n+1}} P_{x,y}(\sigma_n<\infty)$. Using the strong Markov property and (6.24), we obtain

$$\begin{aligned}
&P_{x,y}(\sigma_n<\infty)\\
&=P_{x,y}(\eta_{n+1}=2^n)+\sum_{k=2}^{\infty}P_{x,y}(\eta_{n+1}>2^n,\ \eta_{n+2}>2^n,\ \cdots,\ \eta_{2k-2}>2^n,\ \eta_{n+2k-1}=2^n)\\
&=P_{x,y}(\eta_{n+1}=2^n)+\sum_{k=2}^{\infty}\sum_{\substack{\varepsilon_2+\cdots+\varepsilon_{2k-2}=-1\\ \varepsilon_2,\cdots,\varepsilon_{2k-2}=\pm 1}}P_{x,y}(\eta_{n+1}=2^{n+2},\ \eta_{n+2}=2^{n+2+\varepsilon_2},\\
&\cdots,\ \eta_{n+2k-2}=2^{n+2+\varepsilon_2+\cdots+\varepsilon_{2k-2}},\ \eta_{n+2k-1}=2^n)\\
&\geqq(1-K2^{1-n})/2+\sum_{k=2}^{\infty}\sum_{\substack{\varepsilon_2+\cdots+\varepsilon_{2k-2}=-1\\ \varepsilon_2,\cdots,\varepsilon_{2k-2}=\pm 1}}2^{1-2k}(1-K2^{1-n})^{2k-1}\\
&=(1-K2^{1-n})/2+\sum_{k=2}^{\infty}k^{-1}\binom{2k-2}{k-1}2^{1-2k}(1-K2^{1-n})^{2k-1}.
\end{aligned}$$

Thus we have

$$S^{(n)}\geqq\sum_{k=1}^{\infty}k^{-1}\binom{2k-2}{k-1}2^{1-2k}(1-K2^{1-n})^{2k-1}.$$

Applying the formula $(1-x)^{1/2}=1-\sum_{k=1}^{\infty}k^{-1}\binom{2k-2}{k-1}2^{1-2k}x^k$ for $x=(1-K2^{1-n})^2$, we can estimate the right hand side of the above inequality: $S^{(n)}\geqq 1-K_6 2^{-n/2}$, $n\geqq n_0$. By the strong Markov property, we have

$$\inf_{x\in M,\,||y||=2^n}P_{x,y}(\sigma_{n_0}<\infty)\geqq\prod_{j\geqq n_0+1}^{n}S^{(j)}, \qquad n>n_0 .$$

Since the series $\sum_{j=n_0+1}^{\infty}(1-S^{(j)})$ converges, the infinite product $\prod_{j=n_0+1}^{\infty}S^{(j)}$ is convergent. Thus $\prod_{j=n_0+1}^{\infty}S^{(j)}\equiv\delta>0$. So we obtain

$$\inf_{x\in M,\,||y||=2^n}P_{x,y}(\sigma_{n_0}<\infty)\geqq\delta>0 \qquad \text{for } n>n_0 .$$

From this inequality, it follows that

(6.25) $$\inf_{x\in M,\,||y||\leqq 2^{n_0}}P_{x,y}(\sigma_{n_0}<\infty)\geqq\delta_1>0 .$$

Now we can complete the proof as follows. Let $||y||>2^{n_0}$. Then using the strong Markov property and (6.25), we have

$$P_{x,y}(t<\sigma_{n_0}<\infty) = E_{x,y}[P_{Z_t}(\sigma_{n_0}<\infty);\ \sigma_{n_0}>t]$$
$$\geqq \delta_1 P_{x,y}(\sigma_{n_0}>t)\,.$$

By letting $t\to\infty$, we have $\delta_1 P_{x,y}(\sigma_{n_0}=\infty)\leqq 0$. Since $\delta_1>0$, we have $P_{x,y}(\sigma_{n_0}=\infty)=0$. Q.E.D.

7. Winding problem for R^2. In this section, we discuss the winding of a two-dimensional Brownian motion with a drift around the origin. Let X be the minimal diffusion corresponding to L which is written in polar coordinate as

$$L=\frac{1}{2}\Big(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\Big)+b(r)\frac{\partial}{\partial\theta}\,. \tag{7.1}$$

Let $c=[0,\infty)$. We define the intersection number $I(X[0,t],c)$ by

$$I(X[0,t],c)=\lim_{n\to\infty} I(X[0,t],c_n)\,,\qquad \text{where } c_n=\{(x,0);\ 0\leqq x<n\}\,.$$

It is easy to see that $I(X[0,t],c)$ can be written of the form

$$I(X[0,t],c)=-\frac{1}{2\pi}\int_{X[0,t]} d\theta+I_1\,,$$

where I_1 is bounded. We note that $\arg X(t)=\int_{X[0,t]} d\theta$, (see [7]). Therefore $I(X[0,t],c)$ represents the winding number of $X[0,t]$ around the origin. Set $\phi(t)=\int_0^t r_s^{-2}ds$, where r_s is the radial part of X_s. Let $\gamma=\log r$ and $\xi(t)=\int_{X[0,\phi^{-1}(t)]} d\gamma$. Then the process $\xi=\{\xi(t),t\geqq 0\}$ is a one-dimensional Brownian motion starting from 0.

We have the following

Theorem 7.1. *Let $x\neq 0$. (i) If $b(r)\in L^1([0,\infty), rdr)$, then $X(t)$ winds infinitely many times both clockwise and counterclockwise around the origin, P_x-a.s.* (ii) *If $b(r)=r^{-\beta}$, $0\leqq\beta\leqq 2$, then $X(t)$ winds infinitely many times counterclockwise only around the origin, P_x*-a.s.

This theorem follows from the following

Lemma 7.1. *Let $x\neq 0$.* (i) *If $b(r)\in L^1([0,\infty),rdr)$, then*

$$\varlimsup_{t\to\infty}\frac{\arg X(t)}{L(\phi(t))}=-\varliminf_{t\to\infty}\frac{\arg X(t)}{L(\phi(t))}=\infty,\qquad P_x\text{-}a.s\,. \tag{7.2}$$

(ii) *If $b(r)=r^{-\beta}$, $0\leqq\beta\leqq 2$, then for any $0<\delta<1$,*

$$\lim_{t\to\infty}\frac{\arg X(t)}{L(\phi(t))^2\{\log L(\phi(t))\}^{-\delta}}=\infty,\qquad P_x\text{-}a.s\,. \tag{7.3}$$

In the above, $L(t)$ *is the local time at* 0 *of the process* ξ.

Proof. Set

$$\begin{cases} B^{(1)}(t) = r(t) - r_0 - \int_0^t \frac{ds}{2r_s}, \\ B(t) = \int_{X[0,\phi^{-1}(t)]} d\theta - \int_0^{\phi^{-1}(t)} b(r_s)ds, \end{cases} \tag{7.4}$$

where $r_0=r(0)$. Then $\{(B^{(1)}(t), B(t)), t\geqq 0\}$ is a two-dimensional Brownian motion. Set $\Theta(t)=\arg X(t)$. Then it holds that

$$\Theta(t) = B\left(\int_0^t r_s^{-2}ds\right) + \int_0^t b(r_s)ds \tag{7.5}$$

and

$$\Theta(\phi^{-1}(t)) = B(t) + \int_0^t b(r_0 e^{\xi(s)}) r_0^2 e^{2\xi(s)} ds\,. \tag{7.6}$$

Now let $b\in L^1([0, \infty), rdr)$. By (7.6), we have

$$\frac{\Theta(\phi^{-1}(t))}{L(t)} = \frac{B(t)}{L(t)} + \frac{1}{L(t)} \int_0^t b(r_0 e^{\xi(s)}) r_0^2 e^{2\xi(s)}\, ds\,. \tag{7.7}$$

Since the process ξ is independent of the Brownian motion B, the process $C(t)=B(L^{-1}(t))$ is a symmetric Cauchy process. Therefore since $\overline{\lim}_{t\to\infty} \frac{C(L(t))}{L(t)} = -\underline{\lim}_{t\to\infty} \frac{C(L(t))}{L(t)} = \infty$ (see [6]), we obtain $\overline{\lim}_{t\to\infty} \frac{B(t)}{L(t)} = -\underline{\lim}_{t\to\infty} \frac{B(t)}{L(t)} = \infty$. For the second term of (7.7), we note that $b(r_0e^x)e^{2x} \in L^1((-\infty, \infty), dr)$, since $b(r)\in L^1([0, \infty), rdr)$. Therefore we have

$$\lim_{t\to\infty} \frac{1}{L(t)} \int_0^t b(r_0 e^{\xi(s)}) r_0^2 e^{2\xi(s)} ds = \int_{-\infty}^{\infty} b(r_0 e^x) r_0^2 e^{2x}\, dx$$

(see [10. Chap. 6]). It follows from these two facts that

$$\overline{\lim_{t\to\infty}} \frac{\Theta(\phi^{-1}(t))}{L(t)} = -\underline{\lim_{t\to\infty}} \frac{\Theta(\phi^{-1}(t))}{L(t)} = \infty\,.$$

This implies (7.2)

Next let $b(r)=r^{-\beta}$, $0\leqq\beta\leqq 2$. We write $\int_0^{\phi^{-1}(t)} r_s^{-\beta} ds$ as follows: $\int_0^{\phi^{-1}(t)} b(r_s)ds = \int_{-\infty}^{\infty} r_0^{2-\beta} e^{(2-\beta)x} L(t, x)dx$, where $L(t,x)$ is the local time of the process ξ at x. Define $J_1(t)$ and $J_2(t)$ by

$$J_1(t)=\int_{-\infty}^{\infty}\chi_{(-\infty,0]}(x)r_0^{2-\beta}e^{(2-\beta)x}L(t,x)dx$$

and

$$J_2(t)=\int_{-\infty}^{\infty}\chi_{(0,\infty)}(x)r_0^{2-\beta}e^{(2-\beta)x}L(t,x)dx\,,$$

where $\chi_A(x)=1$, for $x\in A$, $=0$, for $x\notin A$. For J_1, since $\chi_{(-\infty,0]}(x)r_0^{2-\beta}e^{(2-\beta)x}\in L^1$, we have

$$\lim_{t\to\infty}\frac{J_1(t)}{L(t)}=\int_{-\infty}^{\infty}\chi_{(-\infty,0]}(x)r_0^{2-\beta}e^{(2-\beta)x}dx \tag{7.8}$$

(see [10, Chap. 6]).

To estimate J_2, set $\psi(t)=r_0^{2-\beta}\int_{-\infty}^{\infty}\chi_{(0,\infty)}(x)L(t,x)dx$, then $J_2(t)\geqq\psi(t)=r_0^{2-\beta}\int_0^t\chi_{(0,\infty)}(\xi(s))ds$. If we write $L_+^{-1}(t)=\int_0^{L^{-1}(t)}\chi_{(0,\infty)}(\xi(s))ds$, then $\{L_+^{-1}(t),\ t\geqq 0\}$ is a one-sided stable process with index 1/2 (see [10. Chap. 6]). Since it holds (see [3])

$$\varlimsup_{t\to\infty}\frac{L_+^{-1}(t)}{t^2(\log t)^{-\delta}}=\infty,\qquad \text{for any } 0<\delta<1\,,$$

we see that

$$\varlimsup_{t\to\infty}\frac{J_2(t)}{L(t)^2\{\log L(t)\}^{-\delta}}=\infty\,. \tag{7.9}$$

On the other hand, since $\lim_{t\to\infty}\frac{|C(t)|}{t(\log t)^{1+\varepsilon}}=0$, for any ε $(0<\varepsilon\leqq 1)$(see [6]), we obtain

$$\lim_{t\to\infty}\frac{B(t)}{L(t)^2\{\log L(t)\}^{-\delta}}=\lim_{t\to\infty}\frac{C(L(t))}{L(t)^2\{\log L(t)\}^{-\delta}}=0\,. \tag{7.10}$$

Combining (7.8), (7.9) and (7.10), we conclude that

$$\varlimsup_{t\to\infty}\frac{\Theta(\phi^{-1}(t))}{L(t)^2\{\log L(t)\}^{-\delta}}=\infty\,,$$

which proves (7.3).

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