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LÉVY'S FUNCTIONAL ANALYSIS IN TERMS OF AN INFINITE DIMENSIONAL BROWNIAN MOTION I

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0. Introduction.

In the "Troisième Partie: La notion de moyenne dans l'espace fonctionnel et l'équation de Laplace généralisée" of his book "Problèmes concrets d'analyse fonctionnelle", Paul Lévy has extensively developed a potential theory on an infinite dimensional space. About this work he says in his autobiography (Lévy [2], p. 63): "Aussi la troisième partie de mon lirve est-elle une esquisse à grands traits.".

The purpose of this paper is to give a rigorous formulation of some aspects of "une esquisse à grands traits". That is, we shall construct an infinite dimensional space $\boldsymbol{E}$ and an infinite dimensional Brownian motion $\boldsymbol{B}$ with state space $\boldsymbol{E}$. Then we shall describe some results of his potential theory in terms of the infinite dimensional Brownian motion $\boldsymbol{B}$.

We shall now explain the noticeably different features of Lévy's theory from the ones of recent works (e.g., Daletskii [3], Gross [8], Hida [11]). In his potential theory, such objects as his infinite dimensional Laplacian and harmonic functions are regarded as limits of sequences of the corresponding ones in the finite dimensional Euclidean spaces R^N, as $N\to\infty$. He calls this finite dimensional construction method "la méthode du passage du fini à l'infini". In the potential theory on the space R^N, the volume element, surface elements, the Laplacian Δ_N, Poisson kernels on the balls and so forth are determined in terms of the Riemannian metric $ds_N^2=dx_1^2+\cdots+dx_N^2$. As is seen in Green-Stokes' formula, these objects are compatible with each other. Hence the mutual compatibility could be inherited through his finite dimensional constiuction method by the various objects in Lévy's potential theory. In particular, the coordinates in his infinite dimensional space could be regarded as equally weighted. On the other hand, in Lévy's theory harmonic functions can be discontinuous (see Lévy [1], pp. 305-306). Hence the Laplacian as a differential operator would be rather inconvenient in treating such pathological phenomena.

Now we shall reformulate Lévy's potential theory according to his finite dimensional construction method. Motivated by Gâteaux and Lévy's derivation (see Lévy[1], p. 230) of a mean value formula (the Gâteaux formula) of functionals on balls in a real Hilbert space $L^2([0,1])$, (cf. Hida and Nomoto [10]), we shall choose the following space $\boldsymbol{E}$ of sequences (see §§ 1.1):

$$E = \{x = (x_1, \cdots, x_N, \cdots) \in R^\infty;\ \sup_N \frac{1}{N} \sum_{n=1}^{N} x_n^2 < \infty\}$$

and introduce the following sequence of semi-norms $\{||\cdot||_N;\ 1 \leqslant N \leqslant \infty\}$ to the space $\boldsymbol{E}$:

$$||x||_N = \left(\frac{1}{N} \sum_{n=1}^{N} x_n^2\right)^{1/2} \qquad \text{and} \qquad ||x||_\infty = \overline{\lim_{N\uparrow\infty}}\, ||x||_N$$

for an element $x=(x_1, \cdots, x_N, \cdots) \in E$. We shall *a priori* introduce an infinite dimensional Brownian motion $\boldsymbol{B}=(\Omega, B(t), P^x)$ as follows (see §§ 1.2):

$$B(t, \omega) = (b_1(t, \omega), \cdots, b_n(t, \omega), \cdots),$$

where $\{b_n(t, \omega);\ n \geqslant 1\}$ are mutually independent 1-dimensional Brownian motions. Then under the identification of Lévy's infinite dimensional Laplacian and our infinite dimensional Laplacian Δ_∞ which is defined as twice the infinitesimal generator of the process $\boldsymbol{B}$, we shall describe such pathological phenomena in his potential theory through the behaviour of the sample paths on the space $\boldsymbol{E}$.

From our standpoint, even the above-mentioned pathological phenomena in Lévy's potential theory must be described by a limiting procedure of the finite dimensional potential theory. For example, as is emphasized by P. Lévy (Lévy [1], p. 305) and seen in Section 2 of the present paper, the Laplacian Δ_∞ often behaves like a first order differential operator. But solutions of the Dirichlet problems on a unit ball $D_\infty = \{x \in E;\ ||x||_\infty < 1\}$ with tame boundary function satisfying some integrability conditions can be reconstructed by means of sequences of the corresponding solutions of the Dirichlet problems on the finite dimensional balls with the corresponding boundary functions (see Hasegawa [9], II, Th. 4.1).

Next we shall show our main results.

In Section 1, we shall construct the process $\boldsymbol{B}=(\Omega, B(t), P^x)$ with state space $\boldsymbol{E}$ and show the equality:

$$||B(t, \omega) - B(s, \omega)||_\infty^2 = |t-s| \qquad \text{for all} \quad t, s \geqslant 0,\ \omega \in \Omega .$$

Consequently $\boldsymbol{B}$ is a diffusion process on the space $\boldsymbol{E}$.

In Section 2, we shall define a process $\xi(t, \omega)$ on a sphere $S_\infty = \{x \in \boldsymbol{E};\ ||x||_\infty = 1\}$ by the following formula

$$\xi(t, \omega) = e^{-t/2} B(e^t - 1, \omega), \qquad (t \geqslant 0)$$

for elements $\omega\in\Omega$ such that $B(0, \omega)\in S_\infty$, and it will be called the spherical Brownian motion on the sphere S_∞. Then the process $\boldsymbol{B}$ can be factored as the skew product of its radial part $r(t, \omega)=||B(t, \omega)||_\infty$ and the independent spherical Brownian motion $\xi(t, \omega)$ run with a random clock τ_t:

$$B(t, \omega) = r(t)\cdot\xi(\tau_t)\,.$$

This skew product formula could be identified with Lévy's formula (Lévy [1], p. 305, (5)) for a unit sphere. We notice that the spherical Brownian motion is an infinite dimensional Ornstein-Uhlenbeck process with the Gaussian white noise μ as its unique invariant probability measure and is homogeneous under a transformation group G (see Def. 8).

In Section 3, we shall reformulate the Dirichlet problems discussed in Lévy [1] in terms of our Brownian motion $\boldsymbol{B}$. We shall give an interpretation (see Th. 3.1) to a mean value formula (Lévy [1], p. 316, (24)) in the case of a unit ball $D_\infty=\{x\in\boldsymbol{E}; ||x||_\infty<1\}$. Then we shall discuss the properties of the family $\mathcal{P}=\{\mu_x; x\in D_\infty\}$ of harmonic measures. The Dirichlet problems with discontinuous boundary functions of certain types can be formulated as stochastic Dirichlet problems. This is worth noticing, for discontinuous harmonic functions play essential roles in describing the peculiarities of Lévy's theory.

Now, we shall mention other works, in particular, Daletskii [3], Gross [8] and Hida [11].

Hida's ideal measure around the origin of his space X might be thought of as a system of probability measures $\{\mu_r; r>0\}$, where $\mu_r(A)=P^0(B(r^2, \omega)\in A)$ for $A\subset S_\infty(r)=\{x\in\boldsymbol{E}; ||x||_\infty=r\}$. He defines his Laplacian operator on the space X which seems to agree with our infinite dimensional Laplacian. In particular, our Proposition 1.8 corresponds with Proposition 2 in Hida [11] under the identification $r(x)=||x||_\infty$. Moreover his definition of harmonic functions on the space X is similar to ours.

The framework of this paper is adequate to describe phenomena indicated in Lévy's book [1] and considered to be "paradoxical" in Daletskii's paper [3]. For example, in our framework each component of our Brownian motion could be thought to be equally weighted.

Let $\{\xi_n(u); n\geqslant 1\}$ be a *C.O.N.S.* in the real Hilbert space L^2 ([0, 1]). Then a theorem of Ito and Nisio [13] can be applied to show that the random Fourier series

$$B(t, s, \omega) = \sum_{n=1}^{\infty} b_n(t, \omega)\int_0^s \xi_n(u)du$$

converges uniformly in $(t, s)\in[0, T]\times[0, 1]$ for any fixed positive number T *a.s.* (P^0). Thus we obtain a Brownian motion in the sense of Gross [8].

Concluding the introduction, we shall prove in a forthcoming paper that

Dirichlet solutions on the unit ball D_∞ with tame boundary functions can be reconstructed from Dirichlet solutions on finite dimensional balls, which reflects Lévy's "la méthode du passage du fini à l'infini" (see Hasegawa [9], II, Th. 4.1).

1. Infinite dimensional space $\boldsymbol{E}$ and Infinite dimensional Brownian motion $\boldsymbol{B}$

1.1. Infinite dimensional space $\boldsymbol{E}$.

In this subsection we shall construct an infinite dimensional sequence space $\boldsymbol{E}$ on which we shall reformulate Lévy's infinite dimensional potential theory according to the one on the finite dimensional Euclidean spaces R^N.

R. Gâteaux and P. Lévy have established a mean value formula (the Gâteaux formula, see Lévy [1], p. 230) of the real Hilbert space $L^2([0,1])$ by the finite dimensional construction method. In their derivations of the Gâteaux formula, the following projections $\tilde{p}_N$ play essential roles:

$$(\tilde{p}_N x)(t) = \sum_{n=1}^{N} x_n^N \cdot \chi_{I_{N,n}}(t) \qquad \text{for} \quad x(\cdot) \in L^2([0,1]),$$

where

$$x_n^N = N \int_{I_{N,n}} x(s)ds, \quad I_{N,n} = [(n-1)/N, n/N), \qquad (n = 1, \cdots, N).^{1)}$$

Then they have identified the function $\sum_{n=1}^{N} x_n^N \cdot \chi_{I_{N,n}}(t)$ with a point $x^N = (x_1^N, \cdots, x_N^N)$ of the N-dimensional Euclidean space R^N. Since

$$(1.1) \qquad ||\tilde{p}_N x - x|| \to 0 \quad \text{as} \quad N \to \infty \quad \text{for} \quad x(\cdot) \in L^2([0,1]),$$

the quantity

$$(1.2) \qquad ||\tilde{p}_N x|| = \left(\frac{1}{N} \sum_{n=1}^{N} (x_n^N)^2\right)^{1/2}$$

can be thought of the N-*th* approximate semi-norm of the L^2-norm $||\cdot||$.

Consequently, motivated by the formula (1.2) we introduce to the space R^N the following norm r_N:

$$(1.3) \qquad r_N(x) = \left(\frac{1}{N} \sum_{n=1}^{N} x_n^2\right)^{1/2} \qquad \text{for} \quad x = (x_1, \cdots, x_N) \in R^N.$$

By lifting it to the space $\boldsymbol{E}$ we have the following semi-norm $||\cdot||_N$ on the space $\boldsymbol{E}$:

$$(1.4) \qquad ||x||_N = \left(\frac{1}{N} \sum_{n=1}^{N} x_n^2\right)^{1/2} \qquad \text{for} \quad x = (x_1, \cdots, x_N, \cdots) \in \boldsymbol{E},$$

1) The notation χ_A denotes the indicator function of a set A.

and also the following semi-norm $||\cdot||_\infty$ according to the formula (1.1):

$$(1.5) \qquad ||x||_\infty = \overline{\lim_{N\uparrow\infty}} ||x||_N \qquad \text{for} \quad x \in \boldsymbol{E}.$$

We are now ready to obtain the following

DEFINITION 1. We denote by $\boldsymbol{E}$ the space of sequences:

$$(1.6) \qquad \boldsymbol{E} = \{x = (x_1, \cdots, x_N, \cdots) \in R^\infty ; \sup_N \frac{1}{N} \sum_{n=1}^N x_n^2 < \infty\}$$

with the semi-norms $\{||\cdot||_\infty; 1 \leqslant N \leqslant \infty\}$.

Thus we have constructed the space $\boldsymbol{E}$ and the semi-norms $\{||\cdot||_N; 1 \leqslant N \leqslant \infty\}$ according to "la méthode du passage du fini à l'infini". In particular the semi-norm $||\cdot||_\infty$ would play the corresponding role in our potential theory to the one of L^2-norm $||\cdot||$ in Lévy's theory.

DEFINITION 2. O_1 and O_2 denote the topologies induced from the semi-norms $\{||\cdot||_N; 1 \leqslant N \leqslant \infty\}$ and the semi-norms $\{||\cdot||_N; 1 \leqslant N < \infty\}$, respectively.

Then we have the following

Proposition 1.1. 1) *The topological space* $(\boldsymbol{E}, O_1)$ *is non-separable and non-complete, and the space* $(\boldsymbol{E}, O_2)$ *is separable metrizable.*

2) *For any positive number* α, *it holds that*

$$(1.7) \qquad \sum_{n=1}^\infty \frac{x_n^2}{n(\log n)^{1+\alpha}} < +\infty \qquad \textit{for} \quad x = (x_1, \cdots, x_n, \cdots) \in \boldsymbol{E}.$$

Proof. Denoting by $\eta(t)$, $(0 \leqslant t \leqslant 1)$ the following sequences: $\eta(t) = (1, \sqrt{2} \cos(2\pi t), \sqrt{2} \sin(2\pi t), \cdots, \sqrt{2} \cos(2n\pi t), \sqrt{2} \sin(2n\pi t), \cdots)$. Then it is immediate that $||\eta(t)||_\infty = 1$ for all $t \in [0, 1]$, and $||\eta(t) - \eta(s)||_\infty = \sqrt{2}$ for distinct numbers s and t, which shows the non-separability of the space (E, O_1). The other part of this proposition is now clear. (Q.E.D.)

Next we shall introduce to the space $\boldsymbol{E}$ the σ-algebra $\mathcal{E}$ generated by cylinder sets of the space $\boldsymbol{E}$:

$$(1.8) \qquad \mathcal{E} = \sigma(p_n; n = 1, 2, 3, \cdots),$$

where $\{p_n\}$ are projections of the space $\boldsymbol{E}$ such that

$$(1.9) \qquad p_n x = x_n \qquad \text{for} \quad x = (x_1, \cdots, x_n, \cdots) \in \boldsymbol{E}.$$

Since the topological space $(\boldsymbol{E}, O_2)$ is separable and metrizable, the σ-algebra $\mathcal{E}$ agrees with the σ-algebra generated by open subsets in the O_2-topology.

In the sequel we shall use the O_1-topology only without special mentions.

1.2. Infinite dimensional Brownian motion $\boldsymbol{B}$.

In this subsection, we shall introduce an infinite dimensional Brownian motion $\boldsymbol{B}$ on the space $\boldsymbol{E}$ in terms of which we shall reformulate the results of Lévy's infinite dimensional potential theory. He has constructed his theory on the real Hilbert space $(l^2, ||\cdot||)$ according to the one on the N-dimensional Euclidean spaces R^N with respect to the Laplacians Δ_N, (cf. Lévy [1], p. 248, (3)):

$$\Delta_N = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_N^2 . \tag{1.10}$$

Now we shall construct our infinite dimensional Laplacian Δ_∞ on $\boldsymbol{E}$. From our derivation of the semi-norms $\{||\cdot||_N; 1 \leqslant N \leqslant \infty\}$, it is reasonable to regard the sphere $S_{N-1} = \{x \in R^N; r_N(x) = 1\}$ in the space R^N as the corresponding one of the sphere $S_\infty = \{x \in \boldsymbol{E}; ||x||_\infty = 1\}$ in the space $\boldsymbol{E}$. Moreover for a real smooth function $f_N(x)$ on R^N it holds that

$$\Delta_N f_N(x) = \lim_{\rho \downarrow 0} \frac{2}{\rho^2} \left(\int f_N(x+\rho\xi) \mu_{N-1}(d\xi) - f_N(x) \right), \qquad x \in R^N , \tag{1.11}$$

where μ_{N-1} denotes the uniform probability measure on S_{N-1}. On the other hand, the standard Gaussian white noise μ on the sphere S_∞ (see Def. 6) can be regarded as the projective limit of the system $\{(S_{N-1}, \mu_{N-1})\}$ (cf. Hida and Nomoto [10] and Hasegawa [9], I, §1):

$$(S_\infty, \mu) = \varprojlim (S_{N-1}, \mu_{N-1}) . \tag{1.12}$$

Therefore we define our LaplacianΔ_∞ so as to hold

$$\Delta_\infty f(x) = \lim_{\rho \downarrow 0} \frac{2}{\rho^2} \left(\int f(x+\rho\xi) \mu(d\xi) - f(x) \right), \qquad x \in \boldsymbol{E} , \tag{1.13}$$

for a tame function $f(x) = \tilde{f}(x_1, \cdots, x_p)$, $x = (x_1, \cdots, x_p, \cdots) \in \boldsymbol{E}$, where $\tilde{f}(x_1, \cdots, x_p)$ is a real smooth function on R^p, $(p \geqslant 1)$. Then we have the following

$$\Delta_\infty f(x) = \sum_{n=1}^{\infty} \frac{\partial^2}{\partial x_n^2} \tilde{f}(x_1, \cdots, x_p) , \quad x = (x_1, \cdots, x_p, \cdots) \in \boldsymbol{E} . \tag{1.14}$$

Therefore we shall define our Brownian motion $\boldsymbol{B} = (\Omega, B(t), P^x)$ on the space $\boldsymbol{E}$ with the infinitesimal generator $(1/2)\Delta_\infty$ as follows:

$$B(t) = (b_1(t), \cdots, b_N(t), \cdots) , \tag{1.15}$$

where $\{b_n(t); n \geqslant 1\}$ are mutually independent 1-dimensional Brownian motions. We have to show that the process $\boldsymbol{B}$ can be actually constructed on the space $\boldsymbol{E}$.

Now, we introduce a family $\{w_n(t, \hat{\omega}); n \geqslant 1\}$ of mutually independent 1-dimensional standard Wiener processes on a complete probability space $(\hat{\Omega}, \hat{\mathcal{B}}, \hat{P})$. Then we have the following

Proposition 1.2. *For a given element $x = (x_1, \cdots, x_n, \cdots) \in \boldsymbol{E}$, the sample* *paths of the processes*

$$\left\{\frac{1}{N}\sum_{n=1}^{N} w_n(t, \hat{\omega})^2;\ N\geqslant 1\right\} \quad \textit{and} \quad \left\{\frac{1}{N}\sum_{n=1}^{N} x_n w_n(t, \hat{\omega});\ N\geqslant 1\right\}$$

converge to the function t and to the function zero, respectively, uniformly on compact subsets of the interval $[0, \infty)$ *almost surely.*

Proof. Since the processes $\frac{1}{N}\sum_{n=1}^{N} w_n(t, \hat{\omega})^2-t$ and $\frac{1}{N}\sum_{n=1}^{N} x_n w_n(t, \hat{\omega})$ are $\left\{\bigvee_{n=1}^{N}\sigma(w_n(s);\ 0\leqslant s\leqslant t),\ \hat{P}\right\}$-martingales, the submartingale inequality shows the following ones:

$$\hat{P}\left(\sup_{0\leqslant t\leqslant T}\left|\frac{1}{N}\sum_{n=1}^{N} w_n(t, \hat{\omega})^2-t\right|\geqslant \varepsilon\right)\leqslant \frac{T^4}{\varepsilon^4}\cdot\left(\frac{12}{N^2}+\frac{48}{N^3}\right)$$

$$\hat{P}\left(\sup_{0\leqslant t\leqslant T}\left|\frac{1}{N}\sum_{n=1}^{N} x_n w_n(t, \hat{\omega})\right|\geqslant \varepsilon\right)\leqslant \frac{T^2}{\varepsilon^4 N^2}\cdot\left(\frac{x_1^2+\cdots+x_N^2}{N}\right)^2$$

for any positive numbers ε, T. Then the Borel-Cantelli's lemma completes the proof.

Proposition 1.3. 1) *For a given element* $x=(x_1, \cdots, x_n, \cdots)\in \boldsymbol{E}$, *we have* $\overline{\lim_{N\uparrow\infty}}\frac{1}{N}\sum_{n=1}^{N}(x_n+w_n(t, \hat{\omega}))^2=t+\|x\|_\infty^2$ *for all* $t\geqslant 0$, *a.s.* $(\hat{P})$.

2) *For a non-negative number s, the sample paths of the processes* $\left\{\frac{1}{N}\sum_{n=1}^{N}(w_n(t, \hat{\omega})-w_n(s, \hat{\omega}))^2;\ N\geqslant 1\right\}$ *converge to the function* $|t-s|$ *uniformly on compact subsets of the interval* $[0, \infty)$ *almost surely as* $N\to\infty$.

Proposition 1.3 comes easily from Proposition 1.2. Hence we have the following $\boldsymbol{E}$-valued process

$$W(t, \hat{\omega}) = (w_1(t, \hat{\omega}), \cdots, w_n(t, \hat{\omega}), \cdots) \tag{1.16}$$

on the probability space $(\hat{\Omega}, \hat{\mathscr{B}}, \hat{P})$, which is called the infinite dimensional Wiener process or simply the Wiener process.

Then the following theorem shows the Hölder continuity of the sample paths of the Wiener process.

Theorem 1.4. *Let* $W(t, \hat{\omega})$ *be the infinite dimensional Wiener process. Then it holds that*:

$$\lim_{N\uparrow\infty}\frac{1}{N}\sum_{n=1}^{N}(w_n(t, \hat{\omega})-w_n(s, \hat{\omega}))^2 = |t-s| \tag{1.17}$$

for any pair of non-negative numbers s, t almost surely.

Proof. We denote by Λ the totality of the elements $\hat{\omega}\in\hat{\Omega}$ such that for every non-negative rational number r, the processes $\left\{\frac{1}{N}\sum_{n=1}^{N}(w_n(t, \hat{\omega})-w_n(r, \hat{\omega}))^2;\ N\geqslant 1\right\}$ converge to the function $|t-r|$ uniformly on compact subsets of the half line $[0, \infty)$ as $N\to\infty$. Then Proposition 1.3 shows $\hat{P}(\Lambda)=1$. Now for any element $\hat{\omega}\in\Lambda$, any pair (t, s), $(t, s\geqslant 0)$ and any rational number r, $(0\leqslant r\leqslant t\wedge s)$, it holds that

$$\begin{aligned}
&\overline{\lim_{N\uparrow\infty}}\left(\frac{1}{N}\sum_{n=1}^{N}(w_n(t, \hat{\omega})-w_n(s, \hat{\omega}))^2\right)^{1/2}\\
&\leqslant \overline{\lim_{N\uparrow\infty}}\left(\left|\frac{1}{N}\sum_{n=1}^{N}(w_n(t\vee s, \hat{\omega})-w_n(r, \hat{\omega}))^2\right|^{1/2}\right.\\
&\qquad\left.+\left|\frac{1}{N}\sum_{n=1}^{N}(w_n(t\wedge s, \hat{\omega})-w_n(r, \hat{\omega}))^2\right|^{1/2}\right)\\
&=\sqrt{t\vee s-r}+\sqrt{t\wedge s-r}\,,
\end{aligned}$$

where $t\vee s$ and $t\wedge s$ denote the maximum and the minimum of the pair of the numbers t and s, respectively. Hence, by increasing the rational number r to the number $t\wedge s$, we have the following inequality:

$$\overline{\lim_{N\uparrow\infty}}\left(\frac{1}{N}\sum_{n=1}^{N}(w_n(t, \hat{\omega})-w_n(s, \hat{\omega}))^2\right)^{1/2}\leqslant\sqrt{|t-s|}\,. \tag{1.18}$$

For the above pair (t, s) and an element $\hat{\omega}\in\Lambda$, we have also the following inequality:

$$\underline{\lim_{N\uparrow\infty}}\left(\frac{1}{N}\sum_{n=1}^{N}(w_n(t, \hat{\omega})-w_n(s, \hat{\omega}))^2\right)^{1/2}\geqslant\sqrt{|t-s|}\,,$$

which completes the proof together with (1.18).

Now we shall construct the infinite dimensional Brownian motion $\boldsymbol{B}$ by means of the infinite dimensional Wiener process. We denote by $\tilde{\Omega}$ the space of O_1-continuous maps ω; $[0, \infty)\to\boldsymbol{E}$ and by B_t, $t\in[0, \infty)$ the coordinate maps $B_t(\omega)=\omega(t)$, respectively, and introduce to the space $\tilde{\Omega}$ a σ-algebra $\mathcal{F}^0=\sigma(B_t; t\in[0, \infty))$. Now we define maps Φ_x; $\hat{\Omega}\to\tilde{\Omega}$, $(x\in\boldsymbol{E})$ as follows:

$$(\Phi_x(\hat{\omega}))(t)=x+\hat{\omega}(t) \qquad \text{for}\quad t\in[0, \infty)\,,\quad \hat{\omega}\in\hat{\Omega}\,. \tag{1.19}$$

Then for each point $x\in\boldsymbol{E}$, the map Φ_x; $(\hat{\Omega}, \hat{\mathcal{B}})\to(\tilde{\Omega}, \mathcal{F}^0)$ is measurable. Hence we have probability measures $\{P^x; x\in\boldsymbol{E}\}$ on the measurable space $(\tilde{\Omega}, \mathcal{F}^0)$:

$$P^x(\Lambda)=\hat{P}(\Phi_x^{-1}(\Lambda)) \qquad \text{for elements}\quad \Lambda\in\mathcal{F}^0\,. \tag{1.20}$$

The following subset Ω belongs to $\mathcal{F}^0$ by the O_1-continuity of sample paths and satisfies $P^x(\Omega)=1$ for all $x\in\boldsymbol{E}$ by Proposition 1.3 and Theorem 1.4:

$$\Omega = \left\{\omega \in \tilde{\Omega};\ \begin{array}{ll} \|B(t, \omega)-B(s, \omega)\|_\infty^2 = |t-s| & \text{for all} \quad t, s \in [0, \infty) \\ \|B(t, \omega)\|_\infty^2 = t+\|B(0, \omega)\|_\infty^2 & \text{for all} \quad t \in [0, \infty) \end{array}\right\}.$$

Therefore, we can restrict the process B_t and the σ-algebra $\mathcal{F}^0$ to the space Ω, and the restricted process and the restricted σ-algebra will be also denoted by B_t and $\mathcal{F}^0$, respectively. Next we shall introduce the following two σ-algebras to the space Ω:

$$\mathcal{F}_t^0 = \sigma(B(s); s \leqslant t) \quad \text{and} \quad \mathcal{F}_{t+}^0 = \bigcap_{s>t} \mathcal{F}_s^0 \qquad \text{for all} \quad t \geqslant 0 .$$

Then we have the following

Proposition 1.5. *The collection $\boldsymbol{B}=(\Omega, \mathcal{F}^0, \mathcal{F}_{t+}^0, P^x, B_t)$ is a Markov process with state space $(E, \mathcal{E})$. Moreover it is of Feller property in the following sense: For a bounded O_2-continuous function f on the space $\boldsymbol{E}$ the function $E^x[f(B_t)]$, $(0 \leqslant t < \infty)$ is O_2-continuous on the space $\boldsymbol{E}$.*

Proof. Noticing the O_2-continuity of sample paths $\omega \in \Omega$, we can see the σ-algebras $\mathcal{F}_{s+}^0$ and $\sigma(B_t - B_s; s \leqslant t < \infty)$ are mutually P^x-independent, $(x \in \boldsymbol{E})$. The Feller property is evident because of the spatial homogeneity of the process $\boldsymbol{B}$. (Q.E.D.)

As usual we denote by $\mathcal{F}$ and $\mathcal{F}_t$, $(t \geqslant 0)$ the completion of the σ-algebra $\mathcal{F}^0$ and the one of $\mathcal{F}_t^0$, respectively. We are now in position to state the strong Markov property of the process $\boldsymbol{B}$.

Theorem 1.6. *The collection $\boldsymbol{B}=(\Omega, \mathcal{F}, \mathcal{F}_t, P^x, B_t)$ is a strong Markov process with state space $(\boldsymbol{E}, \mathcal{E})$ and the σ-algebras $\{\mathcal{F}_t; 0 \leqslant t < \infty\}$ is right continuous. Moreover it holds that*

$$\|B(t, \omega)-B(s, \omega)\|_\infty^2 = |t-s| \qquad \textit{for all} \quad t, s \geqslant 0, \quad \omega \in \Omega, \tag{1.21}$$

$$\|B(t, \omega)\|_\infty^2 = t+\|B(0, \omega)\|_\infty^2 \qquad \textit{for all} \quad t \geqslant 0, \quad \omega \in \Omega. \tag{1.22}$$

Proof. The proof of this theorem is immediate because of the Feller property of the process $\boldsymbol{B}$. (Q.E.D.)

DEFINITION 3. We call the process $\boldsymbol{B}$ the infinite dimensional Brownian motion or simply the Brownian motion.

Now we proceed to the measurability of the hitting times and the entrance times of subsets of the space $\boldsymbol{E}$.
For a subset A of the space $\boldsymbol{E}$, we define three functions

$$D_A(\omega) = \inf \{t \geqslant 0; B(t, \omega) \in A\}, \tag{1.23}$$

$$T_A(\omega) = \inf \{t > 0; B(t, \omega) \in A\}, \tag{1.24}$$

$$\tau_A(\omega) = \inf \{t \geqslant 0; B(t, \omega) \notin A\}. \tag{1.25}$$ [1]

1) The infimum of the empty set is understood to be the infinity.

We call D_A the first entrance time of A, T_A the first hitting time of A and τ_A the first exit time from A, respectively.

Proposition 1.7. 1) *If a subset $A\subset E$ is O_2-open or O_2-closed, the first entrance time D_A of A and the first hitting time T_A of A are $\{\mathcal{F}^0_{t+}\}$ stopping times.*

2) *If a subset $G\subset E$ is O_1-open and O_2-measurable, the first entrance time D_G of G and the first hitting time T_G of G are $\{\mathcal{F}^0_{t+}\}$ stopping times and agree with each other.*

3) *Suppose that for an O_1-closed subset $A\subset \boldsymbol{E}$, there exists a sequence $\{G_n\}$ of O_1-open and O_2-measurable subsets $G_n\subset \boldsymbol{E}$ which satisfy the following conditions:*

$$G_1\supset \bar{G}_2\supset\cdots\supset G_n\supset \bar{G}_{n+1}\supset G_{n+1}\supset\cdots\supset A\,,\quad \text{and}\quad \bigcap_{n=1}^{\infty} G_n = A\,, \tag{1.26}$$

where each $\bar{G}_n$ denotes the closure of the set G_n in the O_1-topology. Then the first entrance time D_A of A and the first hitting time T_A of A are $\{\mathcal{F}^0_{t+}\}$ stopping times.

Proof. Notice that the σ-algebra $\mathcal{E}$ of the space $\boldsymbol{E}$ is generated by O_2-open sets, not by O_1-open sets.

1.3. Infinitesimal generator of the infinite dimensional Brownian motion $\boldsymbol{B}$.

First of all, we denote by $B(\boldsymbol{E}, \mathcal{E})$ the space of bounded O_2-measurable functions on the space $\boldsymbol{E}$ and we introduce the uniform convergence topology to the space $B(\boldsymbol{E}, \mathcal{E})$. Second we denote by $\{T_t;\, t\geqslant 0\}$ the semi-group induced from the Brownian motion $\boldsymbol{B}$ on the Banach space $B(\boldsymbol{E}, \mathcal{E})$ as follows:

$$T_t f(x) = E^x[f(B_t)] \qquad \text{for elements } f\in B(\boldsymbol{E}, \mathcal{E})\,. \tag{1.27}$$

In the first step of investigations of the infinitesimal generator of the semi-group $\{T_t;\, t\geqslant 0\}$, the function space under our considerations will be restricted to the one of rather special type. Some of the phenomena occuring in a less restrictive function space are too pathological to treat analytically. We denote by $\mathcal{D}$ the space of functions $\phi(r, x)$, $(r\in[0, \infty)$, $x=(x_1, \cdots, x_m)\in R^m)$ which satisfy the following conditions:

1) Each function $\phi(r, x)$ is bounded on the space $[0, \infty)\times R^m$ and derivatives $(\partial^2\phi/\partial x_i\partial x_j)(r, x)$, $(i, j=1, \cdots, m)$ are bounded continuous in $(r, x)\in[0, \infty)\times R^m$.

2) For a given positive number ε, there exists such a positive number δ that inequalities

$$\left|\frac{\partial^2\phi}{\partial x_i\partial x_j}(r, x)-\frac{\partial^2\phi}{\partial x_i\partial x_j}(r, y)\right|<\varepsilon\,,\quad (i, j = 1, \cdots, m) \tag{1.28}$$

hold for any $r\in[0, \infty)$, $x, y\in R^m$ with $|x-y|<\delta$.

3) If we set

$$(1.29)\qquad h(r, x)=\begin{cases}\dfrac{1}{2r}\cdot\dfrac{\partial\phi}{\partial r}(r, x) & \text{for } r>0,\\ \lim\limits_{r\downarrow 0}\dfrac{f(r, x)-f(0, x)}{r^2} & \text{for } r=0,\end{cases}$$

the function $h(\sqrt{r}, x)$ is bounded and uniformly continuous in $(r, x)\in[0, \infty)\times R^m$.

Then from the formula (1.22) we have the following

Proposition 1.8. *If a function $f(x)$ on the space $\boldsymbol{E}$ is of the form*

$$(1.30)\qquad f(x)=\phi(||x||_\infty, x_1, \cdots, x_m), \quad (x=(x_1, \cdots, x_m, \cdots)\in\boldsymbol{E})$$

with the aid of a function ϕ of the space $\mathcal{D}$, it holds that

$$(1.31)\quad \lim_{t\downarrow 0}\frac{T_t f(x)-f(x)}{t}=\frac{1}{2}\sum_{n=1}^{m}\frac{\partial^2\phi}{\partial x_n^2}(||x||_\infty, x_1, \cdots, x_m)+h(||x||_\infty, x_1, \cdots, x_m)$$

on the space $\boldsymbol{E}$, where the convergence is understood to be the uniform convergence on the space $\boldsymbol{E}$.

2. Infinite dimensional spherical Brownian motion Ξ

2.1. Infinite dimensional spherical Brownian motion Ξ.

As is well known, we have the following polar expression of the N-dimensional Laplacian Δ_N on the space R^N:

$$(2.1)\qquad \Delta_N=(1/N)\cdot\partial^2/\partial r_N^2+((N-1)/(Nr_N))\cdot\partial/\partial r_N+(1/r_N^2)\cdot\bar{\Delta}_{N-1},$$

where $\bar{\Delta}_{N-1}$ denotes the spherical Laplacian on the $(N-1)$-dimensional sphere $S_{N-1}=\{x\in R^N; r_N(x)=1\}$. Denoting by the same notation $\bar{\Delta}_{N-1}$ the lift-up of the spherical Laplacian $\bar{\Delta}_{N-1}$ to the space $\boldsymbol{E}$, we shall formally define an infinite dimensional spherical Laplacian $\bar{\Delta}_\infty$ on the sphere $S_\infty=\{x\in\boldsymbol{E}; ||x||_\infty=1\}$ as follows (cf. Umemura [17]):

$$(2.2)\qquad \bar{\Delta}_\infty=\lim_{N\uparrow\infty}\bar{\Delta}_N.$$

On the other hand, from the derivation of the infinite dimensional Laplacian Δ_∞ on $\boldsymbol{E}$ we have the following:

$$(2.3)\qquad \Delta_\infty=\lim_{N\uparrow\infty}\Delta_N.$$

Observing the polar expression (2.1), we have therefore the following formal polar expression of the Laplacian Δ_∞:

$$\Delta_\infty = (1/r)\partial/\partial r + (1/r^2)\bar{\Delta}_\infty \,, \tag{2.4}$$

where $r(x) = ||x||_\infty$ for $x \in \boldsymbol{E}$. This polar expression of the Laplacian Δ_∞ could be identified with Lévy's formula (Lévy [1], p. 305, (5)). On the other hand, the polar expression (2.1) of the Laplacian Δ_N can be expressed by the factorization of the finite dimensional Brownian motion as the skew product of its radial part and an independent spherical Brownian motion run with the random clock (cf. Ito and McKean [12], p. 270).

In this subsection, in order to interpret Lévy's formula we shall introduce an infinite dimensional spherical Brownian motion Ξ on the sphere S_∞ with the infinitesimal generator (1/2) $\bar{\Delta}_\infty$ and factor the infinite dimensional Brownian motion $\boldsymbol{B}$ as the skew product of its radial part and the infinite dimensional spherical Brownian motion Ξ.

Now we set $\hat{\boldsymbol{E}} = \{x \in \boldsymbol{E};\ ||x||_\infty > 0\}$ and associate with it the restriction $\hat{\mathcal{E}}$ of the σ-algebra $\mathcal{E}$ to the O_1-open set $\hat{\boldsymbol{E}}$. Moreover, we introduce a strong Markov process $\hat{\boldsymbol{B}} = (\Omega_0, \hat{B}_t, \hat{P}^x)$ with state space $(\hat{\boldsymbol{E}}, \hat{\mathcal{E}})$ which is equivalent to the part of the process $\boldsymbol{B}$ on the set $\hat{\boldsymbol{E}}$, (see Fukushima [7], pp. 149–152), and the process $\hat{\boldsymbol{B}}$ is called the Brownian motion on the space $\hat{\boldsymbol{E}}$. Here we set

$$\Omega_0 = \{\omega \in \Omega;\ \omega(0) \in \hat{\boldsymbol{E}}\} \,, \quad \hat{B}_t(\omega) = B_t(\omega) \qquad \text{for all } t \geqslant 0 \,, \quad \omega \in \Omega_0 \tag{2.5}$$

and $\hat{P}^x$, $(x \in \hat{\boldsymbol{E}})$ is the restriction of P^x to the space $\Omega_0 (\subset \Omega)$.

DEFINITION 4. A subset $\{x \in \hat{\boldsymbol{E}};\ ||x||_\infty = 1\}$ of the space $\hat{\boldsymbol{E}}$ is denoted by S_∞ and called the infinite dimensional unit sphere or simply the unit sphere. We associate with it the restriction $\mathcal{S}_\infty$ of the σ-algebra $\hat{\mathcal{E}}$ to the set S_∞.

Now we define the radial process r_t of the process $\hat{\boldsymbol{B}}$ as follows:

$$r_t(\omega) = ||\hat{B}(t, \omega)||_\infty \qquad \text{for all } \ t \geqslant 0 \,, \quad \omega \in \Omega_0 \,. \tag{2.6}$$

Next the radial process r_t is associated with the following continuous additive functional (random clock) τ_t of the process $\hat{\boldsymbol{B}}$:

$$\tau_t(\omega) = \int_0^t \frac{ds}{r(s, \omega)^2} \qquad \text{for all } \ t \geqslant 0 \ \text{ and } \ \omega \in \Omega_0 \,. \tag{2.7}$$

Then, from (1.22), we have the following equality:

$$\tau_t(\omega) = \log (1 + (t/r(0, \omega)^2)) \qquad \text{for all } \ t \geqslant 0 \ \text{ and } \ \omega \in \Omega_0 \,. \tag{2.8}$$

Finally we define a process $\xi_t(\omega)$ on the unit sphere S_∞ so as to hold

$$(2.9)\qquad \hat{B}(t, \omega) = r_t(\omega)\cdot\hat{\xi}(\tau_t(\omega), \omega) \qquad \text{for all} \quad t\geqslant 0 \quad \text{and} \quad \omega\in\Omega_0 .$$

Since the radial process r_t is deterministic, the processes r_t and $\hat{\xi}_t$ are mutually independent with respect to the probability measures $\hat{P}^x$, $(x\in\hat{\boldsymbol{E}})$. It holds also that

$$(2.10)\qquad \hat{\xi}_t(\omega) = \frac{e^{-t/2}}{r(0, \omega)}\cdot\hat{B}((e^t-1)\cdot r(0, \omega)^2, \omega) \qquad \text{for all} \quad t\geqslant 0 , \quad \omega\in\Omega_0 .$$

We assume now that the asked Markov process Ξ on the unit sphere S_∞ is of the form $(\xi_t, \hat{P}^\xi)$, $(\xi\in S_\infty)$. Therefore we define the process ξ_t as follows:

$$(2.11)\qquad \xi_t(\omega) = e^{-t/2}\cdot\hat{B}(e^t-1, \omega) \qquad \text{for all} \quad t\geqslant 0 \quad \text{and} \quad \omega\in\Omega_1 ,$$

where

$$(2.12)\qquad \Omega_1 = \{\omega\in\Omega_0;\ ||\hat{B}(0, \omega)||_\infty = 1\} .$$

Then we are in position to state the following

Proposition 2.1. 1) *The collection* $\Xi=(\Omega_1, \xi_t, \hat{P}^\xi)$ *is a strong Markov process with state space* S_∞ *having the Feller property. It holds that*

$$(2.13)\qquad ||\xi_t(\omega)-\xi_s(\omega)||_\infty^2 = 2(1-e^{-|t-s|/2}) \qquad \textit{for all} \quad t, s\geqslant 0,\ \textit{a.s.}\ (\hat{P}^\xi) , \quad (\xi\in S_\infty) .$$

2) *For a given function* $f(\xi)$ *on* S_∞ *of the form*

$$(2.14)\qquad f(\xi) = \phi(\xi_1, \cdots, \xi_m) \quad \textit{for} \quad \xi = (\xi_1, \cdots, \xi_m, \cdots)\in S_\infty ,$$

where $\phi(\xi_1, \cdots, \xi_m)\in C^3(R^m)$ *of compact support, it holds that in the uniform convergence topology on* S_∞:

$$(2.15)\qquad \lim_{t\downarrow 0}\frac{\hat{E}^\xi[f(\xi_t)]-f(\xi)}{t} = \frac{1}{2}\sum_{n=1}^{m}\left(\frac{\partial^2}{\partial\xi_n^2}-\xi_n\frac{\partial}{\partial\xi_n}\right)\phi(\xi_1, \cdots, \xi_m) .$$

Proof. The Hölder continuity of the sample paths of the process Ξ comes immediately from Proposition 1.3 and Theorem 1.4, and the Markovian properties of the process Ξ follows from the Brownian scaling property (cf. Ito and McKean [12], p. 18). Next for the above -mentioned function $f(\xi)$ on S_∞, it holds that:

$$(2.16)\qquad \hat{E}^\xi[f(\xi_t)] = \int_{R^m}\phi(e^{-t/2}(\xi_1+u_1, \cdots, \xi_m+u_m))\frac{\exp(-u^2/(2T))}{(2\pi T)^{m/2}}du$$

where $T=e^t-1$, $\xi=(\xi_1, \cdots, \xi_m, \cdots)\in S_\infty$, $u^2=u_1^2+\cdots+u_m^2$. From this expression we have the last assertion. (Q.E.D.)

DEFINITION 5. The Markov process Ξ is called the infinite dimensional spherical Brownian motion on the unit sphere S_∞ or simply the spherical Brownian motion.

We have thus factored the Brownian motion to obtain the spherical Brownian motion Ξ. Conversely we shall now construct the process $\hat{\boldsymbol{B}}$ as the skew product of a radial process and the spherical Brownian motion, which are mutually independent.

To begin with, we shall introduce a radial process and a random clock according to the preceding ones. First we introduce a family of probability measures $\{\dot{P}^r; 0<r<\infty\}$ on a space $R_+=(0, \infty)$ such that $\dot{P}^r(A)=\chi_A(r)$ for measurable subsets $A(\subset R_+)$ and also introduce a process $r_t(\dot{\omega})=(\dot{\omega}^2+t)^{1/2}$ for elements $\dot{\omega}\in R_+$. Next we associate with the radial process r_t a random clock $\tau_t(\dot{\omega})$ as follows:

$$\tau_t(\dot{\omega}) = \int_0^t \frac{ds}{r(s, \dot{\omega})^2} \quad \text{for all} \quad t\geqslant 0\,, \quad \dot{\omega}\in R_+\,. \tag{2.17}$$

Then by the Brownian scaling property we have the following

Theorem 2.2. *The skew product process* $\hat{\hat{\boldsymbol{B}}}=(\hat{\hat{\Omega}}, \hat{\hat{B}}_t, \hat{\hat{P}}^x)$:

$$\hat{\hat{B}}(t, (\dot{\omega}, \omega)) = r_t(\dot{\omega})\cdot\xi(\tau_t(\dot{\omega}), \omega) \quad \textit{for all} \quad t\geqslant 0\,, \quad (\dot{\omega}, \omega)\in\hat{\hat{\Omega}}=R_+\times\Omega_1\,, \tag{2.18}$$

$$\hat{\hat{P}}^x = \dot{P}^{\|x\|_\infty}\times\dot{P}^{x/\|x\|_\infty}\,, \quad (x\in\hat{\boldsymbol{E}})\,, \tag{2.19}$$

is a Markov process with state space $(\hat{\boldsymbol{E}}, \hat{\mathcal{E}})$ *and is equivalent to the part of the Brownian motion* $\boldsymbol{B}$ *on the* O_1*-open set* $\hat{\boldsymbol{E}}$.

We notice that the spherical Brownian motion Ξ on the unit sphere S_∞ is an infinite dimensional Ornstein-Uhlenbeck process. Thus Theorem 2.2 gives a new approach to investigation of infinite dimensional Brownian motions and Ornstein-Uhlenbeck processes (cf. Malliavin [15]).

2.2. Properties of the spherical Brownian motion Ξ.

In this subsection, we shall investigate further properties of the spherical Brownian motion Ξ.

DEFINITION 6. We call the following probability measure μ on the unit sphere S_∞ the standard Gaussian white noise or simply the white noise:

$$\mu(A) = P^0(B(1, \omega)\in A) \quad \text{for any element} \quad A\in\mathcal{S}_\infty\,. \tag{2.20}$$

Then we have the following ergodic theorem.

Theorem 2.3.

1) *For a bounded* O_1*-continuous and* O_2*-measurable function* $f(\xi)$ *on the unit*

sphere S_∞, the following equality holds at each point $\xi\in S_\infty$,

$$\lim_{t\uparrow\infty}\hat{E}^{\xi}[f(\xi_t)]=\int f(\zeta)\mu(d\zeta)\,. \tag{2.21}$$

2) *The white noise μ is the unique invariant probability measure of the spherical Brownian motion Ξ.*

Proof. The Brownian scaling property gives us the following equalities for the above-mentioned function $f(\xi)$ at each point $\xi\in S_\infty$:

$$\begin{aligned}\hat{E}^{\xi}[f(\xi_t)]&=E^0[f(e^{-/2}\cdot\xi+e^{-t/2}B(e^t-1,\,\omega))]\\&=E^0[f(e^{-t/2}\cdot\xi+\sqrt{1-e^{-t}}B(1,\,\omega))]\,.\end{aligned}$$

Hence Lebesgue's convergence theorem can be applied to show the following equalities:

$$\lim_{t\uparrow\infty}\hat{E}^{\xi}[f(\xi_t)]=E^0[f(B(1,\,\omega))]=\int f(\zeta)\mu(d\zeta)\,,$$

which prove the first assertion. Moreover for a bounded O_2-measurable function $f(\xi)$ on the unit sphere S_∞, we have the following:

$$\begin{aligned}\int\hat{E}^{\xi}[f(\xi_t)]\mu(d\xi)&=E^0[f(e^{-t/2}B(e^t,\,\omega)]\\&=E^0[f(B(1,\,\omega))]=\int f(\zeta)\mu(d\zeta)\,,\end{aligned}$$

which prove the second assertion in combination with (2.21). (Q.E.D.)

Consequently we have a contraction semi-group $\{\hat{T}_t;\,t\geqslant0\}$, (Yosida [18], Chap. IX) on the complex Hilbert space $L^2(S_\infty,\,\mu)$ as follows:

$$\hat{T}_tf(\xi)=\hat{E}^{\xi}[f(\xi_t)]\qquad\text{for elements}\quad f\in L^2(S_\infty,\,\mu)\,. \tag{2.22}$$

Here we shall show that the infinitesimal generator of the semi-group is actually a self-adjoint operator.

DEFINITION 7. For a sequence $K=(k_1,\,\cdots,\,k_p)$ of non-negative integers we define a Fourier-Hermite polynomial H_K of degree $|K|=k_1+\cdots+k_p$ as follows:

$$H_K(x_1,\,\cdots,\,x_p)=\prod_{n=1}^{p}H_{k_n}(x_n)/\sqrt{k_n!} \tag{2.23}$$

where H_n denote the Hermite polynomials with the generating function

$$\sum_{n=1}^{\infty}H_n(x)\cdot t^n/n!=\exp\,(t\cdot x-(x^2/2))\,. \tag{2.24}$$

Proposition 2.4. *The infinitesimal generator A of the strongly continuous semi-group $\{\hat{T}_t;\,t\geqslant0\}$ is a self-adjoint operator with pure-point spectrum*

$\{-n/2; n \geqslant 0\}$ *and the eigenspace of the eigenvalue* $-n/2$ *is spanned by the functions* $\{H_K; |K|=n\}$:

$$H_K(\xi) = H_K(\xi_1, \cdots, \xi_p) \quad \text{for} \quad \xi = (\xi_1, \cdots, \xi_p, \cdots) \in S_\infty . \tag{2.25}$$

Proof. By applying the formula (2.16) to the generating function (2.24) of the Hermite polynomials, we can easily see

$$\hat{T}_t H_K = \exp(-|K|t/2) \cdot H_K \quad \text{for any} \quad K = (k_1, \cdots, k_p) .$$

Since the family $\{H_K; K\}$ constitutes a *C.O.N.S.* of the Hilbert space $L^2(S_\infty, \mu)$, we have the assertion. (Q.E.D.)

REMARK. 1) The second assertion of Proposition 2.1 holds also in the Hilbert space $L^2(S_\infty, \mu)$.

2) The infinitesimal generator A has been called the infinite dimensional Laplacian operator and been studied by Y. Umemura [17] and others.

3) Set $V = \{\xi \in S_\infty; ||\xi - a||_\infty < \delta\}$, $(\delta < \sqrt{2}, a \in S_\infty)$. Then V is an O_1-open and O_2-measurable subset of the unit sphere S_∞ and it holds that $\mu(V)=0$.

Next we show homogeneity of the spherical Brownian motion Ξ under a transformation group G.

DEFINITION 8. We denote by G a group of linear O_1-homeomorphisms g of the space $\boldsymbol{E}$ onto itself such that

1) maps g and g^{-1} are O_2-measurable,
2) maps g preserve the probability measure μ:

$$\mu(gA) = \mu(A) \quad \text{for elements} \quad A \in \mathcal{S}_\infty , \tag{2.26}$$

3) maps g preserve the semi-norm $||\cdot||_\infty$:

$$||gx||_\infty = ||x||_\infty \quad \text{for elements} \quad x \in \boldsymbol{B} . \tag{2.27}$$

Then by the Brownian scaling property we have the following

Proposition 2.5. *The spherical Brownian motion* Ξ *is homogeneous under the group* G*: for any set* $A \in \mathcal{S}_\infty$ *and element* $g \in G$*, it holds that*

$$\hat{P}^{g\xi}(\xi_t \in gA) = \hat{P}^{\xi}(\xi_t \in A) \quad \text{on the unit sphere} \quad S_\infty , \quad (t \geqslant 0) . \tag{2.28}$$

We shall also consider the following transformation group described in the book (Lévy [1], p. 259). For a given sequence of integers $(p_1, \cdots, p_n, \cdots)$ such that

$$2 < p_1 < p_2 < \cdots < p_n < \cdots , \quad \text{and} \quad \lim_{n \uparrow \infty} p_{n+1}/p_n = 1 ,$$

we are given a bijection σ of the set $\{1, 2, 3, \cdots\}$ onto itself which is also a permutation of the set $\{p_n+1, p_n+2, \cdots, p_{n+1}\}$ for each $n=1, 2, 3, \cdots$. We then define a map g_σ of the space $\boldsymbol{E}$ onto itself as follows:

$$(g_\sigma x)_n = x_{\sigma(n)} \quad \text{for} \quad x = (x_1, \cdots, x_n, \cdots) \in \boldsymbol{E}. \tag{2.29}$$

Then we have the following

Proposition 2.6. *The maps g_σ of the above-mentioned type form a subgroup G_0 of the group G.*

3. The Dirichlet problems on the space $\boldsymbol{E}$

3.1. The Dirichlet problems on the unit ball.

In this subsection, we shall reformulate the results of Paul Lévy [1] on the Dirichlet problems on his infinite dimensional space in terms of our infinite dimensional Brownian motion $\boldsymbol{B}$.

DEFINITION 9. A subset A of the space $\boldsymbol{E}$ is said to be semi-bounded, if the set $\{||x||_p; x \in A\}$ is bounded for some semi-norm $||\cdot||_p$, $(1 \leqslant p \leqslant \infty)$.

DEFINITION 10. For a given domain $G(\subset \boldsymbol{E})$, we denote by $\mathcal{U}(G)$ a family of semi-bounded domains U such that the closures $\bar{U}$ of U in the O_1-topology are included in G and that the first exit times τ_U from U are $\{\mathcal{F}_t\}$ stopping times. Then a real-valued O_2-measurable function $f(x)$ on G is said to be harmonic on the domain G, if for any subdomain $U \in \mathcal{U}(G)$ the following equality holds on G:

$$E^x[f(B(\tau_U))] = f(x). \tag{3.1}$$

DEFINITION 11. An O_1-open subset $\{x \in \boldsymbol{E}; ||x||_\infty < 1\}$ of the space $\boldsymbol{E}$ is is denoted by D_∞ and called the infinite dimensional unit ball or simply the unit ball. We denote by $S_\infty(a, r)$ a sphere $\{x \in \boldsymbol{E}; ||x-a||_\infty = r\}$, $(a \in D_\infty, 0 < 2r < 1 - ||a||_\infty)$, and by $T_{a,r}\xi$ the point of $S_\infty(a, r)$ on the segment between two points a and ξ, $(\xi \in S_\infty)$:

$$T_{a,r}\xi = (1-(r/||\xi - a||_\infty)) \cdot a + r \cdot \xi / ||\xi - a||_\infty \in S_\infty(a, r). \tag{3.2}$$

Clearly the unit sphere S_∞ is the boundary of the unit ball D_∞ in the O_1-topology and the first exit time τ from the unit ball D_∞ it an $\{\mathcal{F}^0_{t+}\}$ stopping time. The map $T_{a,r}$, $(a \in D_\infty, 0 < 2r < 1 - ||a||_\infty)$ is an O_2-bimeasurable O_1-homeomorphism from the unit sphere S_∞ onto the another sphere $S_\infty(a, r)$.

DEFINITION 12. We define a harmonic measure μ_x relative to the unit ball D_∞ and a point $x \in D_\infty$ by putting

$$\mu_x(A) = P^x(B_\tau \in A) \quad \text{for} \quad A \in \mathcal{S}_\infty. \tag{3.3}$$

The Poisson integral formulas for the finite dimensional balls are of fundamental importance in the potential-theoretic sense and also in the group representation-theoretic sense (cf. Kashiwara et al., [14]). Accordingly we shall pay special attention to the Dirichlet problems on the unit ball D_∞. First of all, we shall give an interpretation to Lévy's mean value formula (Lévy [1], p. 316, (24)) in the case of the unit ball D_∞.

Theorem 3.1. *Let $\phi(\xi)$ be a real bounded O_1-continuous and O_2-measurable function on the unit sphere S_∞.*

Then the following function $f(x)$

$$f(x) = E^x[\phi(B_\tau)] \tag{3.4}$$

on the unit ball D_∞ is the unique one satisfying the following conditions:

1) *The function $f(x)$ is bounded, O_1-continuous and harmonic on the unit ball D_∞.*

2) *For any point $\xi \in S_\infty$, the function $f(x)$ converges to $\phi(\xi)$ as $x(\in D_\infty)$ tends to ξ in the O_1-topology.*

3) *We have also the following peculiar equality*

$$\mu_a(\{\xi \in S_\infty;\ T_{a,r}\xi \in A\}) = \mu((A-a)/r) \tag{3.5}$$

for any measurable subset A of the sphere $S_\infty(a, r)$, $(a \in D_\infty,\ 0 < 2r < 1 - ||a||_\infty)$.

Proof. Because of the equality

$$\tau(\omega) = \tau_{D_\infty}(\omega) = 1 - ||x||_\infty^2 \qquad a.s.\ P^x, \quad (x \in D_\infty) \tag{3.6}$$

and the Brownian scaling property, we have the following equalities on the unit ball D_∞:

$$f(x) = E^x[\phi(B_\tau)] = \int \phi(x + \sqrt{1 - ||x||_\infty^2} \cdot \xi)\mu(d\xi),$$

where μ denotes the white noise on the unit sphere S_∞. Hence we have the assertions 1,2 of the theorem. By the Brownian scaling property we have the following equalities for a measurable subset A of the sphere $S_\infty(a, r)$:

$$\mu_a(T_{a,r}\xi \in A) = P^a\left(\frac{B_\tau - a}{\sqrt{\tau}} \in \frac{A-a}{r}\right) = \mu((A-a)/r),$$

which prove the assertion 3 of the theorem. (Q.E.D.)

Here we pause to show the homogeneity of the family $\mathcal{P} = \{\mu_x;\ x \in D_\infty\}$ of harmonic measures under the group G (see Def. 8) and the mutual singularity of $\mathcal{P}$.

Proposition 3.2. *For any $g \in G$, $x \in D_\infty$ and $A \in \mathcal{S}_\infty$, we have*

$$\mu_{gx}(gA) = \mu_x(A). \tag{3.7}$$

Proof. The proof is immediate by the Brownian scaling property.

REMARK. The family $\mathcal{P}$ seems to be an intertwining operator between two representations of the group G on the unit sphere S_∞ and the unit ball D_∞ *i.e.,* for the above boundary function ϕ on S_∞ and the associated harmonic function f on D_∞, we have the following equality on D_∞ for any element $g \in G$

$$f(g^{-1}x) = \int \phi(g^{-1}\xi)\mu_x(d\xi). \tag{3.8}$$

Proposition 3.3. *Harmonic measures μ_x and $\mu_y(\in \mathcal{P})$, $(x, y \in D_\infty)$ are mutually equivalent [resp., mutually singular], if the sequence $x-y=(x_1-y_1, \cdots, x_n-y_n, \cdots)$ is square summable [resp., not square summable]. In particular, for an element $x=(x_1, \cdots, x_n, \cdots) \in l^2 \subset D_\infty$, we have the following density formula:*

$$\mu_x(d\xi) = \exp(x \cdot \xi - (x^2/2))\mu(d\xi). \tag{3.9}$$

Here $x^2 = \sum_{n=1}^{\infty} x_n^2$ and $x \cdot \xi$ denotes the following μ-almost surely convergent random series:

$$x \cdot \xi = \sum_{n=1}^{\infty} x_n \xi_n \quad \text{for} \quad \xi = (\xi_1, \cdots, \xi_n, \cdots) \in S_\infty. \tag{3.10}$$

Proof. The proof is easy (cf. Rozanov [16], pp. 38, 66).

REMARK. For an element $x \in l^2 \subset E$ and an element $g \in G$ it holds that $gx \in l^2$ and $x^2 = (gx)^2$, for we have $gx \in l^2$ by Propositions 3.2, 3.3 and next the asked equality by using the density formula (3.9).

Next we proceed to the stochastic Dirichlet problems on the unit ball D_∞ with discontinuous boundary functions.

Proposition 3.4. *For given elements e_j, $(\|e_j\|_\infty > 0, j=1, \cdots, p)$ and a given real bounded measurable function $h(u_1, \cdots, u_p, x_1, \cdots, x_m)$ on the space $[0, \infty)^p \times R^m$, we define a function $\phi(\xi)$ on the unit sphere S_∞ as follows:*

$$\phi(\xi) = h(\|\xi + e_1\|_\infty, \cdots, \|\xi + e_p\|_\infty, \xi_1, \cdots, \xi_m) \tag{3.11}$$

for elements $\xi = (\xi_1, \cdots, \xi_m, \cdots) \in S_\infty$. Then the function $f(x)$

$$f(x) = E^x[\phi(B_\tau)] \tag{3.12}$$

on D_∞ is the unique one satisfying the following conditions:

1) *The function $f(x)$ is bounded harmonic on D_∞.*

2) *For any point $x \in D_\infty$, the function $f(B_t(\omega))$ is continuous in t in the interval $[0, \tau(\omega))$ a.s. P^x and*

(3.13) $$\lim_{t\uparrow\tau(\omega)} f(B_t(\omega)) = \phi(B_\tau(\omega)) \qquad a.s. \quad P^x .$$

Proof. By Theorem 1.6 we have the following equalities:

(3.14) $$f(x) = E^x[\phi(B_\tau)]$$
$$= \int_{R^m} h(X_1(x), \cdots, X_p(x), u_1, \cdots, u_m) \frac{\exp[-(u-\hat{x})^2/(2(1-||x||_\infty^2))]}{(2\pi(1-||x||_\infty^2))^{m/2}} du$$

on the unit ball D_∞, where $X_k(x)=\sqrt{||x+e_k||_\infty^2+1-||x||_\infty^2}$, $(1\leqslant k\leqslant p)$, $(u-\hat{x})^2=(u_1-x_1)^2+\cdots+(u_m-x_m)^2$ for points $x=(x_1, \cdots, x_m, \cdots)\in D_\infty$. Also by Theorem 1.6 it holds that

$$X_k(B_t(\omega)) = X_k(x) \qquad \text{for all} \quad t\geqslant 0\,, \quad a.s.P^x\,, \quad (x\in D_\infty)\,, \quad (k=1, \cdots, p)\,.$$

Hence the function $f(B_t(\omega))$ in t is continuous in the interval $[0, \tau(\omega))$ $a.s.P^x$, $(x\in D_\infty)$.

Now we introduce a sequence $\{\tau_n; n\geqslant 2\}$ of the first exit times from subdomains $U_n=\{x\in D_\infty; ||x||_\infty<1-(1/n)\}$. Then by the convergence theorem of martingales we have the following equality:

$$\lim_{n\uparrow\infty} f(B(\tau_n, \omega)) = E^x[\phi(B_\tau)| \bigvee_{n=2}^{\infty} \mathcal{F}_{\tau_n}] \qquad \text{a.s.} \quad P^x\,, \quad (x\in D_\infty)\,.$$

Hence the sample continuity of the process $\boldsymbol{B}$ shows the following:

$$\lim_{t\uparrow\tau(\omega)} f(B_t(\omega)) = \phi(B_\tau) \qquad \text{a.s.} \quad P^x, \quad (x\in D_\infty)\,,$$

which completes the proof.

Bearing in mind the pathological phenomena of "harmonic" functions which are described in Lévy [1] (pp. 305–306) and will be shown in Proposition 3.7 partly, it is worth noticing that the stochastic Dirichlet problems can be formulated also for discontinuous boundary functions of the above-mentioned type.

Proposition 3.5. *For a function* $\tilde{\phi}(x_1, \cdots, x_m)$

(3.15) $$\tilde{\phi}\in L^2(R^m, \exp(-(x_1^2+\cdots+x_m^2)/2)dx_1\cdots dx_m)\,,$$

define a real function $\phi(\xi)$ *on the unit sphere* S_∞ *by*

(3.16) $$\phi(\xi) = \tilde{\phi}(\xi_1, \cdots, \xi_m) \qquad \textit{for} \quad \xi = (\xi_1, \cdots, \xi_m, \cdots)\in S_\infty\,.$$

Then the function $f(x)=E^x[\phi(B_\tau)]$ *on the unit ball* D_∞ *is the unique one satisfying the following conditions:*

1) *The function* $f(x)$ *is* O_1*-continuous and harmonic on the unit ball* D_∞.

2) *We are given a sequence* $\{U_n\}$ *of subdomains of* D_∞ *such that* $\bigcup_{n=1}^{\infty} U_n=D_\infty$,

$\bar{U}_n \subset U_{n+1}$ and the first exit times τ_n from U_n are $\{\mathcal{F}_t\}$ stopping times. Then the family $\{f(B(\tau_n)); n \geqslant 1\}$ is uniformly P^x-integrable, $(x \in D_\infty)$.

Proof. By Hölder's inequality and Jensen's one we have $(n \geqslant 1)$

$$E^x[|f(B(\tau_n))|^{3/2}] \leqslant E^x[|\phi(B(\tau))|^{3/2}] < \infty\ , \quad (x \in D_\infty)\ ,$$

which implies the uniqueness of the function $f(x)$. (Q.E.D.)

In case of $\tilde{\phi}(x_1, \cdots, x_m) = H_K(x_1, \cdots, x_m)$, $(K = (k_1, \cdots, k_m))$, the associated Dirichlet solution $f(x)$ is the restriction to D_∞ of the following harmonic function $h(x)$ on $\boldsymbol{E}$:

$$(3.17) \qquad h(x) = \begin{cases} ||x||_\infty^{|K|} H_K((x_1, \cdots, x_m)/||x||_\infty) & \text{on} \quad \{x \in \boldsymbol{E};\ ||x||_\infty > 0\} \\ x_1^{k_1} \cdots x_m^{k_m} / \sqrt{k_1!, \cdots, k_m!} & \text{on} \quad \{x \in \boldsymbol{E};\ ||x||_\infty = 0\}\ . \end{cases}$$

This is of its own interst (cf. Dobrushin and Minlos [4]) and the restriction of the function $h(x)$ to the domain $\{x \in \boldsymbol{E};\ ||x||_\infty > 0\}$ is just identical with Hida's Fourier-Hermite polynomial, (cf. Hida [11]).

3.2. The Dirichlet problems on other domains.

In this subsection we shall consider the Dirichlet problems on some other domains than the unit ball D_∞.

First we shall consider the following Dirichlet problem on a domain the first exit time from which is not deterministic.

Proposition 3.6. *Let D be a convex domain:*

$$(3.18) \qquad D = \{x_1 = (x_1, \ldots, x_n, \cdots) \in \boldsymbol{E};\ ||x||_\infty < 1,\ 0 < x_1 < 1\}\ ,$$

and $\tilde{\phi}$ be a real bounded continuous function on a subset

$$(3.19) \quad (\{x=0\} \times \{0 \leqslant r \leqslant 1\}) \cup (\{0 \leqslant x \leqslant 1\} \times \{r=1\}) \cup (\{x=1\} \times \{0 \leqslant r \leqslant 1\})$$

of the space R^2. Then, for the function $\phi(\xi)$

$$(3.20) \qquad \phi(\xi) = \phi(\xi_1, ||\xi||_\infty) \qquad \text{for} \quad \xi = (\xi_1, \cdots, \xi_n, \cdots) \in \partial D$$

on the boundary ∂D of D, the function $f(x) = E^x[\phi(B_\tau)]$, $(\tau = \tau_D)$ on D is the unique one satisfying the following conditions:

1) *The function $f(x)$ is bounded O_1-continuous and harmonic on the domain D.*

2) *For any point $\xi \in \partial D$, the function $f(x)$ converges to $\phi(\xi)$ as $x (\in D)$ tends to ξ in the O_1-topology.*

Proof. By Poisson's summation formula and calculations of passage times of 1-dimensional Brownian motion, we have (cf. Ito and McKean [12], pp. 29, 30):

$$f(x) = \Phi_1(x_1, r) + \Phi_2(x_1, r) + \Phi_3(x_1, r)\ , \quad r = ||x||_\infty$$

for $x = (x_1, \cdots, x_n, \cdots) \in D$, where

$$\Phi_1(x_1, r) = 2\sum_{n=1}^{\infty}\int_0^1 \tilde{\phi}(u, 1)\sin(n\pi u)du \sin(n\pi x_1)\exp(-n^2\pi^2(1-r^2)/2),$$

$$\Phi_2(x_1, r) = \sum_{n=1}^{\infty}\int_0^{1-r^2} \tilde{\phi}(1, \sqrt{r^2+t})e^{-n^2\pi^2 t/2}dt\, n\pi(-1)^{n-1}\sin(n\pi x_1),$$

$$\Phi_3(x_1, r) = \sum_{n=1}^{\infty}\int_0^{1-r^2} \tilde{\phi}(0, \sqrt{r^2+t})e^{-n^2\pi^2 t/2}dt\, n\pi \sin(n\pi x_1).$$

Hence we have the first assertion and also the second assertion by the usual way (cf. Dynkin [5], II, p. 33). (Q.E.D.)

Now the unicity principle does not hold, peculiar to our infinite dimensional potential theory. In fact, by Theorem 1.6 we have the following

Proposition 3.7. *For a real bounded continuous function ϕ on the real line R and an element $\xi \in \boldsymbol{E}$, ($\|\xi\|_\infty > 0$), we define a function $f(x)$ on the space $\boldsymbol{E}$ by*

(3.21) $$f(x) = \phi(\langle x, \xi\rangle_\infty),$$

where

(3.22) $$4\langle x, \xi\rangle_\infty = \|x+\xi\|_\infty^2 - \|x-\xi\|_\infty^2.$$

Then the function $f(x)$ is harmonic on the space $\boldsymbol{E}$.

REMARK. We have another dissimilarity with finite dimensional cases, i.e., the regularity of a point of a boundary in the probabilistic sense does not imply the one in the classical potential-theoretic sense in our case. In fact we have the following example. Let D be a semi-ball: $D = \{x \in \boldsymbol{E};\ \|x\|_\infty < 1, \langle x, \xi\rangle_\infty > 0\}$, ($\xi \in \boldsymbol{E}$, $\|\xi\|_\infty = 1$). Then the first exit time $\tau_D(\omega)$ from the semi-ball D satisfies

$$\tau_D(\omega) = 1 - \|x\|_\infty^2 \qquad \text{a.s. } (P^x), \quad (x \in D).$$

On the other hand, the first hitting time $\sigma(\omega) = T_{D^c}(\omega)$ of D^c satisfies $\sigma(\omega) = 0$ a.s. (P^ζ) for $\zeta \in \partial D$ with $\|\zeta\|_\infty < 1$, $\langle \zeta, \xi\rangle_\infty = 0$.

Finally we have the following

Proposition 3.8. *Let $f(x)$ be a tame function on the space $\boldsymbol{E}$ given by*

(3.23) $$f(x) = \phi(x_1, \cdots, x_n) \quad \textit{for points} \quad x = (x_1, \cdots, x_n, \cdots) \in \boldsymbol{E}.$$

If $\phi(x_1, \cdots, x_n)$ is a harmonic polynomial on the space R^n, the function $f(x)$ is harmonic on the space $\boldsymbol{E}$.

Proof. Clearly we have only to show the equality

(3.24) $$f(x) = E^x[f(B_\tau)]$$

on the space $\boldsymbol{E}$ for the first exit time τ from the domain $\hat{D}$: $\hat{D} = \{x = (x_1, \cdots, x_p,$

$\cdots, x_n, \cdots)\in \boldsymbol{E}; 0<x_j<d,\ j=1, \cdots, p\}$, $(p<n,\ d>0)$. We denote by $(b(t), \tilde{P}^x)$ the n-dimensional Brownian motion and by σ the first exit time from the n-dimensional cylinder $\tilde{D}=D\times R^{n-p}$, $D=\{x=(x_1, \cdots, x_p)\in R^p;\ 0<x_j<d, j=1, \cdots, p\}$.

As will be shown separetely (cf. Étienne et Thunus [6]), Green function $G(a, b; x, y)$, $(x, a\in D; y, b\in R^{n-p})$, and Poisson kernel $P(a, b;\ \xi, \eta)$, $(a\in D;\ \xi\in\partial D;\ b, \eta\in R^{n-p})$ of the cylinder $\tilde{D}$ decay exponentially in y as $|y-b|\to\infty$ and in η as $|\eta-b|\to\infty$, respectively. Then Dynkin's formula can be applied to show the following equality for the harmonic polynomial ϕ for points $c=(a, b)\in\tilde{D}$:

$$\tilde{E}^c[\phi(b_\sigma)] = \phi(c)\,.$$

Hence it holds that on the space E

$$E^x[f(B(\tau))] = f(x)\,. \qquad \text{(Q.E.D.)}$$

From this proposition, it is clear that the totality of the continuous harmonic functions on the space $\boldsymbol{E}$ separates each point of the space $\boldsymbol{E}$.

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References

[1] Paul Lévy: Problèmes concrets d'analyse fonctionnelle, Gauthier-Villars, Paris, 1951.

[2] ————: Quelques aspects de la pensée d'un mathématicien, Albert Blanchard, Paris, 1970.

[3] Yu. L. Daletskii: *Infinite-dimensional elliptic operators and parabolic equations connected with them,* Russian Math. Surveys **22** (1967), 1–53.

[4] R.L. Dobrushin and R.A. Minlos: *Polynomials in linear random functions,* Russian Math. Surveys **32** (1971), 7–127.

[5] E.B. Dynkin: Markov processes I, II, Springer-Verlag, Berlin, 1965.

[6] J. Étienne et J. Thunus: *Fonctions de Green de l'opérateur métaharmonique pour les problèmes de Dirichlet et Neumann posés dans un intervalle,* Bull. Soc. Roy. Sci. Liège **28** (1959), 198–206.

[7] M. Fukushima: Dirichlet forms and Markov processes, Kinokuniya, Tokyo, 1975, (in Japanese).

[8] L. Gross: *Potential theory on Hilbert space,* J. Funct. Anal. **1** (1967), 123–181.

[9] Y. Hasegawa: *Lévy's functional analysis in terms of an infinite dimensional Brownian motion,* I, II, Proc. Japan Acad. **56** (1980), 109–113; 114–118.

[10] T. Hida and H. Nomoto: *Gaussian measure on the projective limit space of spheres,*

Proc. Japan Acad. **40** (1964), 301–304.
[11] T. Hida: *Harmonic analysis on the space of generalized functions,* Theory Prob Appl. **15** (1970), 119–124.
[12] K. Ito and H.P. McKean Jr.: Diffusion processes and their sample paths, Springer-Verlag, Berlin, 1974.
[13] K. Ito and M. Nishio: *On the convergence of sums of independent Banach space valued random variables,* Osaka J. Math. **5** (1968), 35–48.
[14] M. Kashiwara, A. Kowata. K. Minemura, K. Okamoto, T. Oshima and M. Tanaka: *Eigenfunctions of invariant differential operators on a symmetric space,* Ann. of Math. **107** (1978), 1–39.
[15] P. Malliavin: *Stochastic calculus of variation and hypoelliptic operators,* Proc. of Intern. Symp. SDE,. Kinokuniya, Tokyo, 1976, 195–263.
[16] Ju. Rozanov: Infinite dimensional Gaussian distributions, Amer. Math. Soc., 1971.
[17] Y. Umemura: *On the infinite dimensional Laplacian operator,* J. Math. Kyoto Univ. **4** (1965), 477–492.
[18] K. Yosida: Functional analysis, Springer-Verlag, Berlin, 1971.

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