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# ON THE DEGREES OF IRREDUCIBLE REPRESENTATIONS OF FINITE GROUPS

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## 1. Introduction

Let $G$ be a finite group of order $|G|$ and $F$ be an algebraically closed field of characteristic 0. Let $T$ be an irreducible representation of $G$ over $F$ and $d_T$ be the degree of $T$. As is well know, $d_T$ divides $|G|$. Furthermore there exists a sharper result due to Ito [2], namely, $d_T$ divides the index in $G$ of every abelian normal subgroup of $G$. Let $s_T$ be the order of det $T$, that is, $s_T$ is the smallest natural number such that $|T(x)|^{s_T}=1$ for all $x\in G$, where $|T(x)|$ is the determinant of $T(x)$. In Lemma of [4] we showed the first part of the following

**Theorem 1.** *Let $T$ be an irreducible representation of $G$ over $F$. Then we have*

(i) $d_T s_T \,\big|\, 2|G|$,
(ii) *if $d_T$ or $s_T$ is odd then* $d_T s_T \,\big|\, |G|$.

The second part follows from (i) by considering the 2-part of $d_T s_T$, since both $d_T$ and $s_T$ divide $|G|$.

The purpose of the present paper is to prove the following theorems.

**Theorem 2.** *If $G$ has an irreducible representation $T$ over $F$ with $d_T s_T \nmid |G|$, then the following holds.*

(i) *A 2-Sylow subgroup $P$ of $G$ is cyclic and $P\neq 1$. Hence $G$ has the normal 2-complement $K$.*
(ii) $C_P(K)=1$.
(iii) *$T$ is induced from a representation of $K$.*

The converse of Theorem 2 is also true:

**Theorem 3.** *If $G$ satisfies* (i) *and* (ii) *in Theorem* 2, *then $G$ has an irreducible representation $T$ with $d_T s_T \nmid |G|$.*

We also have the following

**Theorem 4.** *Let $T$ be an irreducible representation of $G$ over $F$. Then we have*

$$d_T s_T \leqslant |G|.$$

*If $d_T s_T = G$, then $G$ is cyclic.*

I express my thanks to Professor H. Nagao for his valuable advice.

## 2. Proofs of the theorems

To prove our theorems we need the following Lemma.

**Lemma.** *Let $T$ be an irreducible representation of $G$ over $F$, $H$ a normal subgroup of $G$ of index $n$ and $T_0$ be an irreducible component of $T_H$, $T_H$ the restriction of $T$ to $H$. Then we have the following.*

(i) *If $T_H = T_0$, then $d_T = d_{T_0}$ and $s_T \mid ns_{T_0}$.*

(ii) *If $T = T_0^G$, then $d_T = nd_{T_0}$ and $s_T \mid 2s_{T_0}$. Furthermore if $2 \mid d_{T_0}s_{T_0}$ or $s_T$ is odd, then $s_T \mid s_{T_0}$.*

Proof. (i) is clear. We prove (ii). Clearly $d_T = nd_{T_0}$. We set $s_T = s$, $d_{T_0} = d_0$ and $s_{T_0} = s_0$. Let $x_1, \cdots, x_n$ be a complete set of coset representatives of $H$ in $G$. We extend $T_0$ to all elements of $G$ by setting $T_0(x) = 0$ for all $x \notin H$. We may assume that $T(x)$ is a $n \times n$ matrix of blocks whose $(i, j)$-th entry is the $d_0 \times d_0$ matrix $T_0(x_i^{-1}xx_j)$:

$$T(x) = \begin{pmatrix} T_0(x_1^{-1}xx_1) \cdots\cdots\cdots T_0(x_1^{-1}xx_n) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ T_0(x_n^{-l}xx_1) \cdots\cdots\cdots T_0(x_n^{-1}xx_n) \end{pmatrix} \quad (x \in G).$$

Hence for each $x \in G$, *we have* $|T(x)| = (-1)^{d_0 m} |T_0(y_1)| \cdots |T_0(y_n)|$, where $y_i \in H$, $i = 1, \cdots, n$, and $m$ is an integer. Therefore, for each $x \in G$, $|T(x)|^{2s_0} = 1$ and hence $s \mid 2s_0$. If $s$ is odd then $s \mid s_0$, and if $d_0$ or $s_0$ is even then $|T(x)|^{s_0} = 1$ for each $x \in G$ and hence $s \mid s_0$.

Proof of Theorem 2. We prove (i) by induction on $|G|$. Put $d_T = d$ and $s_T = s$. Since $ds \nmid |G|$, by Theorem 1, (ii) $2 \mid d$ and $2 \mid s$. In particular $P \neq 1$ and $2 \mid |G: G'|$, where $G'$ is the commutator subgroup of $G$. Let $H$ be a normal subgroup of $G$ of index 2 and $T_0$ be an irreducible component of $T_H$. By Clifford's theorem, $T_H = T_0$ or $T = T_0^G$. Put $d_{T_0} = d_0$ and $s_{T_0} = s_0$.

(a) Suppose $T_H = T_0$. Since $ds \nmid |G|$, by Lemma, (i) $d_0 s_0 \nmid |H|$ and hence both $d_0$ and $s_0$ are even. Therefore by the induction hypothesis, $P \cap H$ is cyclic. Suppose $P$ is not cyclic. For each $x \in P$, $\langle x^2 \rangle \neq P \cap H$ and $|T(x)|^2 = |T_0(x^2)|$. Hence $|T_0(x^2)|^{s_0/2} = 1$ and $|T(x)|^{s_0} = 1$. On the other hand, for each 2-regular element $x \in G$, $|T(x)|^{s_0} = 1$, because $s \mid 2s_0$. Therefore for each $x \in G$, $|T(x)|^{s_0} = 1$ and hence $s \mid s_0$. Thus $ds \mid d_0 s_0$ and $d_0 s_0 \mid 2|H| = |G|$, which is a contradiction. Therefore $P$ is cyclic.

(b) Suppose $T = T_0^G$. We may assume $4 \mid |G|$. Suppose $d_0 s_0$ is odd.

Then since $d=2d_0$, $s$ is even and $s \mid 2s_0$ we have $ds=4r$ with $r$ odd. By Theorem 1, (i) $r \mid |G|$ and hence $ds \mid |G|$, which is a contradiction. Thus $2 \mid d_0s_0$ and by Lemma, (ii) $s \mid s_0$. Then $ds \mid 2d_0s_0$ and hence $d_0s_0 \nmid |H|$. By the induction hypothesis, $P \cap H$ is cyclic. Suppose $P$ is not cyclic. For each $x \in P \cap H$, $|T(x)| = |T_0(x)| \, |T_0(y^{-1}xy)| = |T_0(xy^{-1}xy)|$, where $y$ is an element of $P$ which does not belong to $P \cap H$. Since $P \cap H$ is a cyclic 2-group and $x$ and $y^{-1}xy$ are of the same order, $xy^{-1}xy$ does not generate $P \cap H$. Hence $|T_0(xy^{-1}xy)|^{s_0/2}=1$ and $|T(x)|^{s_0/2}=1$. For each $x \in P$ which does not belong to $P \cap H$, $|T(x)| = |T_0(x^2)|$, because $d_0$ is even. Since $P$ is not cyclic, $|T_0(x^2)|^{s_0/2}=1$, hence $|T(x)|^{s_0/2}=1$. Therefore for each $x \in P$, $|T_0(x)|^{s_0/2}=1$. On the other hand, for each 2-regular element $x \in G$, $|T_0(x)|^{s_0/2}=1$ because $s \mid s_0$. Hence $s \mid s_0/2$ and $ds \mid 2d_0 \cdot s_0/2 = d_0s_0$. By Theorem 1, (i) we have $ds \mid |G|$. This is a contradiction. Therefore $P$ is cyclic. By Burnside's theorem $G$ has the normal 2-complement $K$. Thus (i) is proved.

Now we show (iii). Let $T_1$ be an irreducible component of $T_K$, $\tilde{K}$ the inertial group of $T_1$ in $G$ and let $\tilde{T}$ be an irreducible representation of $\tilde{K}$ such that $T=\tilde{T}^G$ and that $T_1$ is an irreducible component of $\tilde{T}_K$. Put $d_{T_1}=d_1$, $s_{T_1}=s_1$, $d_{\tilde{T}}=\tilde{d}$ and $s_{\tilde{T}}=\tilde{s}$. Since $\tilde{K}/K$ is cyclic, $\tilde{T}_K=T_1$ and $\tilde{d}=d_1$ (see the proof of [1, (9.12)]). As $\tilde{d}$ is odd, $\tilde{d}\tilde{s} \mid |\tilde{K}|$ by Theorem 1, (ii). If $2 \mid \tilde{d}\tilde{s}$, then $s \mid \tilde{s}$ by Lemma, (ii). Hence $ds \mid |G\colon \tilde{K}|\tilde{d}\tilde{s} \mid |G\colon \tilde{K}|\,|\tilde{K}| = |G|$. This yields a contradiction. Hence $2 \nmid \tilde{d}\tilde{s}$. Since $|\tilde{K}\colon K|$ is a power of 2, by Theorem 1, (i) $\tilde{d}\tilde{s} \mid |K|$. Therefore $ds \mid 2|G\colon \tilde{K}|\tilde{d}\tilde{s} \mid 2|G\colon \tilde{K}|\,|K|$. Thus we see $\tilde{K}=K$. This completes the proof of (iii).

Finally we prove (ii). From (iii), $|P| \mid d$. From (i), $C_P(K)$ is a central subgroup of $G$. Hence $d \mid |G\colon C_P(K)|$. Therefore $C_P(K)=1$. This completes the proof of the theorem.

Proof of Theorem 3. We set $|P|=2^a$, $P=\langle x \rangle$ and $y=x^{2^{a-1}}$. Since $C_P(K)=1$, $y$ induces a non-identity automorphism of $K$. By [3, Satz 108], there is a conjugate class of $K$ which is not fixed by $y$. Hence $y$ does not fix some irreducible representation of $K$ over $F$, say $T_0$. Since $\langle yK \rangle$ is the unique minimal subgroup of $G/K$, $K$ is the inertial group of $T_0$ in $G$. Hence $T_0^G$ is an irreducible representation of $G$. We set $T=T_0^G$. Then $2^a \mid d_T$ and we see $|T(x)|=-1$. Hence $d_Ts_T \nmid |G|$.

Proof of Theorem 4. We prove by induction on $|G|$. If $G$ is abelian, then the theorem is trivial. We assume that the theorem is true for any proper subgroup of $G$. First we prove $ds \leqslant |G|$. Suppose $ds > |G|$. By Theorem 1, (i) $ds=2|G|$. Since $d \mid |G|$, $s$ is even and $2 \mid |G\colon G'|$. Let $H$ be a normal subgroup of $G$ of index 2 and $T_0$ be an irreducible component of $T_H$. By the induction hypothesis and Lemma, $T=T_0^G$ and $d_{T_0}s_{T_0}=|H|$. By the induction hypothesis, $H$ is cyclic, $d=2$ and $s=|G|$. Hence $G$ is abelian, which contradicts $d=2$. Thus we have proved $ds\leqslant|G|$. Next we prove the remaining part of the theorem. Suppose $ds=|G|$. We may assume $G\neq 1$. Since $d<|G|$, $s\neq 1$ and hence $G\neq G'$. Let $L$ be a normal subgroup of $G$ of prime index $p$ and $T_1$ be an irreducible component of $T_L$. We prove $L$ is cyclic. By the induction hypothesis and Lemma, if $T_L=T_1$ or if $T=T_1^G$ and $d_{T_1}s_{T_1}$ is even, we see easily $L$ is cyclic. Put $d_{T_1}=d_1$ and $s_{T_1}=s_1$. If $T=T_1^G$ and $d_1s_1$ is odd, then we see $|L|=d_1s_1$ or $|L|=2d_1s_1$. By the induction hypothesis, $|L|=d_1s_1$ implies $L$ is cyclic. In the case $|L|=2d_1s_1$, let $U$ be the normal 2-complement of $L$. As $d_1s_1$ is odd, by Clifford's theorem and Lemma, (ii) we see that $(T_1)_U$ is irreducible and $s_1$ is the order of det $(T_1)_U$ and that $d_1s_1=|U|$. Hence by the induction hypothesis, $U$ is cyclic. Hence $d_1=1$ and $s_1=|U|$ and hence $|L'|\,|\,2$. On the other hand $L'\subset U$. Therefore $L'=1$ and $L$ is cyclic. Thus we have proved that $L$ is cyclic. If $T_L=T_1$, then $d=1$ and $s=|G|$, hence $G$ is cyclic. Suppose $T=T_1^G$, then $d=p$, $|G'|=p$ and $s=|G:G'|$. Let $M$ be any normal subgroup of $G$ of prime index. By the argument applied to $L$ and by $d=p$, $|G:M|=p$. Hence $G/G'$ is a $p$-group and hence $G$ is a $p$-group. Since $|G'|=p$, $G'$ is a central subgroup of $G$. By $s=|G:G'|$, $G/G'$ is cyclic. Therefore $G$ is abelian. This contradicts $d=p$, and this completes the proof.

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