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A FAMILY OF FOURIER INTEGRAL OPERATORS AND THE FUNDAMENTAL SOLUTION FOR A SCHRÖDINGER EQUATION*

HITOSHI KITADA AND HITOSHI KUMANO-GO

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Introduction

In this paper we shall study the theory of Fourier integral operators on R^n depending on a parameter $h\in(0,1)$ with non-homogeneous phase functions and certain symbols in sections 1–4, and apply this theory to the construction of the fundamental solutions for the Cauchy problem of a pseudo-differential equation of Schrödinger's type in sections 5 and 6.

In section 1 we shall study a calculus of a family of pseudo-differential operators $P_h=p_h(X,D_x)$ with C^∞-symbols $p_h(x,\xi)$ depending on a parameter $h\in(0,1)$, which is defined by

$$(1)\qquad P_hu(x)=\int e^{ix\cdot\xi}p_h(x,\xi)\hat{u}(\xi)d\xi,\quad u\in\mathcal{S},$$

where $d\xi=(2\pi)^{-n}d\xi$, $\hat{u}(\xi)$ denotes the Fourier transform of u, and $\mathcal{S}$ denotes the Schwartz space of rapidly decreasing functions on R^n. Let $\mathcal{B}(R^{2n})$ be the space of C^∞-functions in R^{2n} whose derivatives of any order are all bounded in R^{2n}. Then, the symbols $p_h(x,\xi)$ are defined as those functions which satisfy

$$(2)\qquad \text{“}\{h^{-m-\rho|\alpha|+\delta|\beta|}D_x^\beta\partial_\xi^\alpha p_h(x,\xi)\}_{0<h<1}\text{ is bounded in }\mathcal{B}(R^{2n})\text{”}$$

for any α, β with some $-\infty<m<\infty$ and $0\leqq\delta\leqq\rho\leqq1$, and we denote this symbol class by $B^m_{\rho,\delta}(h)$.

In section 2 we shall first define a class $P(\tau,l)$ of phase functions with $0\leqq\tau<1$ and an integer $l\geqq0$ as the class of C^∞-functions such that $J(x,\xi)\equiv\phi(x,\xi)-x\cdot\xi$ satisfies

$$(3)\qquad \begin{aligned}|J|_l\equiv&\sum_{|\alpha+\beta|\leqq1}\sup_{x,\xi}\{|D_x^\beta\partial_\xi^\alpha J(x,\xi)|/\langle x;\xi\rangle^{2-|\alpha+\beta|}\}\\&+\sum_{2\leqq|\alpha+\beta|\leqq2+l}\sup_{x,\xi}\{|D_x^\beta\partial_\xi^\alpha J(x,\xi)|\}\leqq\tau\\&\qquad(\langle x;\xi\rangle=(1+|x|^2+|\xi|^2)^{1/2})\end{aligned}$$

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in the analogy to the class $\mathscr{P}(\tau, l)$ defined in Kumano-go [9]. The class $P_{\rho\,\delta}(\tau, l; h)$ $(0<h<l)$ will then be defined as the class of functions $\phi_h(x, \xi)$ such that $\tilde{\phi}_h(x,\xi)$ defined by

$$(4) \qquad \tilde{\phi}_h(x, \xi) = h^{\rho-\delta}\phi_h(h^\delta x, h^{-\rho}\xi)$$

belongs to $P(\tau, l)$ and for $\tilde{J}_h(x,\xi) = \tilde{\phi}_h(x,\xi)-x\cdot\xi$

$$(5) \qquad \text{"}\{D_x^\beta\partial_\xi^\alpha \tilde{J}_h(x, \xi)\}_{\substack{|\alpha+\beta|=2\\ 0<h<1}} \text{ is bounded in } \mathscr{B}(R^{2n})\text{."}$$

Let $\phi_j(x,\xi)\in P(\tau_j, 0), j=1,2,\cdots,$ with $\bar{\tau}_\infty\equiv\sum_{j=1}^{\infty}\tau_j\leqq 1/4$. Then, according to Kumono-go, Taniguchi and Tozaki [10] and Kumano-go and Taniguchi [11] we define the $\#-(\nu+1)$ product $\Phi_{\nu+1}=\phi_1\#\cdots\#\phi_{\nu+1}$ of $\phi_1,\cdots,\phi_{\nu+1}$ for any ν and prove that for a constant $c_0\geqq 1$

$$(6) \qquad \Phi_{\nu+1}(x, \xi)\in P(c_0\bar{\tau}_{\nu+1}, 0) \text{ with } \bar{\tau}_{\nu+1} = \tau_1+\cdots+\tau_{\nu+1}\,.$$

This result is the fundamental one of section 2. All the properties for the $\#-(\nu+1)$ product $\Phi_{\nu+1,h}=\phi_{1,h}\#\cdots\#\phi_{\nu+1,h}$ of $\phi_{1,h},\cdots,\phi_{\nu+1,h}$ for $\phi_{j,h}\in P_{\rho,\delta}(\tau,0;h)$, $j=1,2,\cdots,$ with $\bar{\tau}_\infty\leqq 1/4$, can be derived from those for $\tilde{\Phi}_{\nu+1,h}=\tilde{\phi}_{1,h}\#\cdots\#\tilde{\phi}_{\nu+1,h}$ with $\tilde{\phi}_{j,h}$ defined by (4).

In section 3 we shall define Fourier integral operators $P_h(\phi_h)=p_h(\phi_h; X, D_x)$ of class $\boldsymbol{B}^m_{\rho,\delta}(\phi_h)$ with phase function $\phi_h(x,\xi)\in P_{\rho,\delta}(\tau,0;h)$ and symbols $p_h(x,\xi)\in B^m_{\rho,\delta}(h)$ by

$$(7) \qquad P_h(\phi_h)u(x) = \int e^{i\phi_h(x,\xi)}p_h(x, \xi)\hat{u}(\xi)d\!\!{}^-\xi,\ u\in\mathscr{S}\,,$$

and study an elementary claculus of Fourier integral operators of this class. Section 4 is devoted to the proof of the representation formulae for the $(\nu+1)$ multi-product $P_{1,h}(\phi_{1,h})\cdots P_{\nu+1,h}(\phi_{\nu+1,h})$ of $P_{j,h}(\phi_{j,h})\in\boldsymbol{B}^{m_j}_{\rho,\delta}(\phi_{j,h}), j=1,2,\cdots,$ with $\bar{\tau}_\infty\leqq 1/4$. The multi-product $P_{1,h}(\phi_{1,h})\cdots P_{\nu+1,h}(\phi_{\nu+1,h})$ can be represented as a Fourier integral operator with phase function $\Phi_{\nu+1,h}=\phi_{1,h}\#\cdots\#\phi_{\nu+1,h}$ and some symbol $r_{\nu+1,h}\in B^{\bar{m}_{\nu+1}}_{\rho,\delta}(h)$ with $\bar{m}_{\nu+1}=m_1+\cdots+m_{\nu+1}$.

Sections 5 and 6 will be devoted to the construction of the fundamental solution $U_h(t,s)$ for the Cauchy problem of an equation of Schrödinger's type.

Let $H(t,x,\xi)$ be a real-valued function on $[0,T]\times R^{2n}$ with $0<T\leqq 1$ such that continuous derivative $D_x^\beta\partial_\xi^\alpha H(t,x,\xi)$ exists on $[0,T]\times R^{2n}$ for any α, β, and satisfies

$$(8) \qquad |D_x^\beta\partial_\xi^\alpha H(t, x, \xi)|\leqq\begin{cases} C_{\alpha,\beta}\langle x;\xi\rangle^{2-|\alpha+\beta|} & (|\alpha+\beta|\leqq 1), \\ C_{\alpha,\beta} & (|\alpha+\beta|\geqq 2), \end{cases} \quad \text{on } [0, T]\times R^{2n}\,,$$

and $\tilde{H}_h(t,x,\xi)$ be a complex valued function of class $\mathscr{B}^0([0,T];\ B^0_{\rho,\delta}(h))$ such that

$$|D_x^\beta \partial_\xi^\alpha \tilde{H}_h(t, x, \xi)| \leqq C'_{\alpha,\beta} h^{\rho|\alpha|-\delta|\beta|} \quad \text{on } [0, T]\times R^{2n}\,. \tag{9}$$

Set

$$\begin{cases} H_h(t, x, \xi) = h^{\delta-\rho} H(t, h^{-\delta}x, h^\rho \xi)\,, \\ K_h(t, x, \xi) = H_h(t, x, \xi) + \tilde{H}_h(t, x, \xi)\,. \end{cases} \tag{10}$$

(See (5.3) and Remark after (5.3).) Then, setting

$$L_h = D_t + K_h(t, X, D_x)\,, \tag{11}$$

we shall consider the Cauchy problem of a pseudo-differential equation of Schrödinger's type

$$\begin{cases} L_h u \equiv (D_t + K_h(t, X, D_x))u = 0 \quad \text{on } [0, T_0]\,, \\ \qquad u|_{t=s} = v \in \mathscr{S} \qquad\qquad (0 \leqq s \leqq T_0) \end{cases} \tag{12}$$

for a small $0 < T_0 \leqq T$.

In section 5 we shall construct two kinds of the approximate fundamental solution for the problem (12). Let $\phi(t,s;x,\xi)$ be the solution of the Hamilton-Jacobi quation

$$\begin{cases} \partial_t \phi(t, s; x, \xi) + H(t, x, \nabla_x \phi(t, s; x, \xi)) = 0 \quad \text{on } [0, T_0]^2 \times R^{2n}\,, \\ \phi(s, s; x, \xi) = x \cdot \xi \quad \text{on } [0, T_0] \times R^{2n}\,, \end{cases} \tag{13}$$

and set

$$\phi_h(t, s; x, \xi) = h^{\delta-\rho} \phi(t, s; h^{-\delta}x, h^\rho \xi)\,. \tag{14}$$

Then it is proved that $\phi_h(t,s) \in P_{\rho,\delta}(c_l|t-s|, l; h)$ for $t,s \in [0, \tilde{T}_l]$ with constants $c_l \geqq 1$ and $0 < \tilde{T}_l \leqq T_0$ such that $c_l \tilde{T}_l < 1$ for any l. Let $I(\phi_h(t,s))$ be the Fourier integral operator with phase function $\phi_h(t,s)$ and symbol 1. Then, we shall first prove that $I(\phi_h(t,s))$ is the approximate fundamental solution of order zero in the sense

$$\begin{cases} \sigma(L_h I(\phi_h(t, s))) \in \mathscr{B}^0([0, T_0]^2; B^0_{0,\delta}(h))\,, \\ I(\phi_h(s, s)) = I\,, \end{cases} \tag{15}$$

where $\sigma(L_h I(\phi_h(t,s)))$ denotes the symbol of $L_h I(\phi_h(t,s))$.

Next for the special case $0 \leqq \delta < \rho \leqq 1$, solving transport equations we shall find the symbol $e_h(t,s;x,\xi) \in \mathscr{B}^1([0, T_0]^2; B^0_{\rho,\delta}(h))$ such that the Fourier integral operator $E_h(\phi_h(t,s)) = e_h(\phi_h(t,s); t,s; X, D_x)$ with symbol $e_h(t,s;x,\xi)$ is the approximate fundamental solution of order infinity in the sense

$$\begin{cases} \sigma(L_h E_h(\phi_h(t, s))) \in \mathscr{B}^0([0, T_0]^2; B^{\infty}_{\rho,\delta}(h))\,, \\ \qquad E_h(\phi_h(s, s)) = I\,. \end{cases} \tag{16}$$

In section 6, using the approximate fundamental solutions constructed in section 5, we shall by Levi method construct the fundamental solution $U_h(t,s)$ of the problem (12), that is,

$$\begin{cases} L_h U_h(t,s) = 0 & \text{on } [0, T_0], \\ U_h(s,s) = I & (0 \leqq s \leqq T_0), \end{cases} \tag{17}$$

and investigate the properties of $U_h(t,s)$ together with its L^2-properties. Finally for $\tilde{L}_h = D_t + H_h(t,X,D_x)$ defined by

$$H_h(t,x,\xi) = h^{\delta-\rho} H(t, h^{-\delta}x, h^{\rho}\xi) \tag{18}$$

we shall investigate the convergence of the iterated integral of Feynman's type as in Fujiwara [2]-[5] and Kitada [6].

We note that, recently, Fujiwara in [4] and [5] has proved the pointwise convergence of the iterated integral of Feynman's type for the operator $\tilde{L}_h$ when $H_h(t,X,D_x)$ has the form $H_h(t,X,D_x) = -h\Delta + h^{-1}V(t,x)$. But it should be noted that, in the present paper, the convergence of the iterated integral of Feynman's type is proved in the symbol class $B^0_{\rho,\delta}(h)$ in case $0 \leqq \delta \leqq \rho \leqq 1$ and $B^{\infty}_{\rho,\delta}(h)$ in case $0 \leqq \delta < \rho \leqq 1$. We also should note the following facts: i) When H_h and $\tilde{H}_h$, and hence K_h, do not depend on h, $L_h = L \equiv D_t + K(t,X,D_x)$ is included in the case $\delta = \rho = 0$ and $L_h = D_t + h^{-1}H(t,X,hD_x) + \tilde{H}(t,X,hD_x)$ (the usual Schrödinger operator) is in the case $\delta = 0$, $\rho = 1$. Furthermore, in the general case $0 \leqq \delta \leqq \rho \leqq 1$ the symbol $u_h(t,s;x,\xi)$ of the fundamental solution $U_h(t,s)$ is uniformly bounded in the class $B^0_{\rho,\delta}(h)$ on $\{(t,s;h;\rho,\delta) | 0 \leqq s,\ t \leqq T_0,\ 0 < h < 1,\ 0 \leqq \delta \leqq \rho \leqq 1\}$. ii) Let $H^w_h(t,X,D_x)$ be the Weyl operator for $H(t,x,\xi)$ defined by

$$H^w_h(t,x,\xi) = h^{\delta-\rho} \mathrm{O}_s\!-\!\iint e^{-iy\cdot\eta} H\Big(t, h^{-\delta}\Big(x + \frac{y}{2}\Big), h^{\rho}(\xi+\eta)\Big) d\eta dy\,. \tag{19}$$

Then it is easy to see that $H^w_h(t,X,D_x)$ is symmetric on $\mathcal{S}$ and $H^w_h(t,x,\xi)$ has the form (10) with some $\tilde{H}^w_h(t,x,\xi)$ satisfying (9). So we can construct the fundamental solution $U^w_h(t,s)$ for $L^w_h = D_t + H^w_h(t,X,D_x)$, although the convergence of the iterated integral of Feynman's type for L^w_h is not proved generally. iii) From the symmetry with respect to x and ξ we can construct the fundamental solution $U'_h(t,s)$ for the operator $L'_h = D_t + H'_h(t,X,D_x) + \tilde{H}'_h(t,X,D_x)$, where $H'_h(t,x,\xi) = h^{\delta-\rho}H(t, h^{\rho}x, h^{-\delta}\xi)$ with $0 \leqq \delta \leqq \rho \leqq 1$ and $\tilde{H}'_h(t,x,\xi)$ is a function satisfying

$$|D^{\alpha}_x \partial^{\beta}_{\xi} \tilde{H}'_h(t,x,\xi)| \leqq C'_{\alpha,\beta} h^{\rho|\alpha| - \delta|\beta|} \quad \text{on } [0,T] \times R^{2n}\,. \tag{9$'$}$$

During the preparation of our present paper we have received a mimeographed paper [12] by Chazarain which is closely related to our paper, where he uses an approximate fundamental solution to the operator of the form $L_h =$

$D_t-\frac{1}{2}h\Delta+h^{-1}V(x)$ without constructing the fundamental solution.

1. A family of pseudo-differential operators

Let $x=(x_1,\cdots,x_n)$ denote a point of R^n, and let $\alpha=(\alpha_1,\cdots,\alpha_n)$ be a multi-index whose components α_j are non-negative integers. Then, we use the usual notation:

$$|\alpha| = \alpha_1+\cdots+\alpha_n,\ \alpha! = \alpha_1!\cdots\alpha_n!,\ x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n},$$
$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n},\ D_x^\alpha = D_{x_1}^{\alpha_1}\cdots D_{x_n}^{\alpha_n},$$
$$\partial_{x_j} = \frac{\partial}{\partial x_j},\ D_{x_j} = \frac{1}{i}\frac{\partial}{\partial x_j}\quad (j=1,\cdots,n),$$
$$\langle x\rangle = (1+|x|^2)^{1/2},\ \langle x;x'\rangle = (1+|x|^2+|x'|^2)^{1/2}.$$

For an open set Ω of R^n let $\mathscr{B}(\Omega)$ denote the set of all C^∞-functions defined in Ω whose derivatives of any order are all bounded in Ω. We often write $\mathscr{B}=\mathscr{B}(R^n)$ simply. Let $\mathscr{S}$ denote the Schwartz space on R^n of rapidly decreasing functions.

For $u\in\mathscr{S}_x$ the Fourier transform $\hat{u}(\xi)=\mathscr{F}[u](\xi)$ is defined by

$$\mathscr{F}[u](\xi) = \int e^{-ix\cdot\xi}u(x)dx,\ x\cdot\xi = x_1\xi_1+\cdots+x_n\xi_n.$$

Then the inverse Fourier transform $\bar{\mathscr{F}}[v](x)$ for $v(\xi)\in\mathscr{S}_\xi$ is defined by

$$\bar{\mathscr{F}}[v](x) = \int e^{ix\cdot\xi}v(\xi)đ\xi,\ đ\xi = (2\pi)^{-n}d\xi.$$

Definition 1.1 i) We say that a function $p=p(x,\xi,x',\xi',x'')\in\mathscr{B}(R^{5n})$ belongs to the symbol class $\mathscr{B}$ if p satisfies

$$(1.1)\qquad |p_{(\beta,\beta',\beta'')}^{(\alpha,\alpha')}(x,\xi,x',\xi',x'')| \leqq C_{\alpha,\alpha'\,\beta,\beta',\beta''},$$

where $p_{(\beta,\beta',\beta'')}^{(\alpha,\alpha')}=\partial_\xi^\alpha\partial_{\xi'}^{\alpha'}D_x^\beta D_{x'}^{\beta'}D_{x''}^{\beta''}p$.

ii) We say that a family $\{p_h\}_{0<h<1}$ of functions $p_h(x,\xi,x',\xi',x'')\in\mathscr{B}(R^{5n})$ belongs to the class $\{B_{\rho,\delta}^m(h)\}_{0<h<1}(m\in\boldsymbol{R},\ 0\leqq\delta\leqq\rho\leqq1)$ if $p_h(0<h<1)$ satisfy

$$(1.2)\qquad \begin{aligned}&|p_{h(\beta,\beta',\beta'')}^{(\alpha,\alpha')}(x,\xi,x',\xi',x'')|\\ &\leqq C'_{\alpha,\alpha',\beta,\beta',\beta''}h^{m+\rho|\alpha+\alpha'|-\delta|\beta+\beta'+\beta''|}\end{aligned}$$

for constants $C'_{\alpha,\alpha',\beta,\beta',\beta''}$ independent of $0<h<1$ and x,ξ,x',ξ',x'', and we write

$$\{p_h\}_{0<h<1}\in\{B_{\rho,\delta}^m(h)\}_{0<h<1}$$

or simply $p_h\in B_{\rho,\delta}^m(h)$.

REMARK 1°. For $p\in B$ and $p_h\in B_{\rho,\delta}^m(h)$, we define semi-norms $|p|_l$ and

$|p_h|_l^{(m)}$, $l=0,1,2,\cdots$, respectively by

$$|p|_l=\max_{|\alpha+\alpha'+\beta+\beta'+\beta''|\leq l}\inf\{C_{\alpha,\alpha',\beta,\beta',\beta''}\text{ of (1.1)}\} \tag{1.3}$$

and

$$\begin{aligned}&|\{p_h\}_{0<h<1}|_l^{(m)}\ (\text{or simply } |p_h|_l^{(m)})\\ &=\max_{|\alpha+\alpha'+\beta+\beta'+\beta''|\leq l}\inf\{C'_{\alpha,\alpha',\beta,\beta',\beta''}\text{ of (1.2)}\}\,.\end{aligned} \tag{1.4}$$

Then B and $\{B^m_{\rho,\delta}(h)\}_{0<h<1}$ are Fréchet spaces provided with these semi-norms, respectively.

2°. Symbols $p(x,\xi)$, $p_h(x,\xi)$ (resp. $p(\xi,x')$, $p_h(\xi,x')$) independent of x', ξ', x'' (resp. x, ξ', x'') are often called single symbols.

3°. For a symbol $p(x,\xi,x',\xi',x'')\in B$, if we put $p_h(x,\xi,x',\xi',x'')=p(x,\xi,x',\xi',x'')(0<h<1)$, then p_h belongs to $B^0_{0,0}(h)$. In this sense we can write $B\subset B^0_{0,0}(h)$. Hence, all the statements concerning the symbol class $B^0_{0,0}(h)$ hold for B as a special case.

4°. For $p_h\in B^m_{\rho,\delta}(h)$ define $\tilde{p}_h$ by

$$\tilde{p}_h(x,\xi,x',\xi',x'')=p_h(h^\delta x, h^{-\rho}\xi, h^\delta x', h^{-\rho}\xi', h^\delta x'')\,. \tag{1.5}$$

Then, we have that $\tilde{p}_h\in B^m_{0,0}(h)$ and

$$|p_h|_l^{(m)}\ (\text{in } B^m_{\rho,\delta}(h)) = |\tilde{p}_h|_l^{(m)}\ (\text{in } B^m_{0,0}(h))\,. \tag{1.6}$$

For $a(\eta,y)\in C^\infty(R^n_\eta\times R^n_y)$ satisfying

$$|\partial_\eta^\alpha D_x^\beta a(\eta,y)|\leq C_{\alpha,\beta}\langle\eta;y\rangle^{\tau+\sigma|\alpha+\beta|} \tag{1.7}$$

for some $\tau\in\boldsymbol{R}$ and $0\leq\sigma<1$ we define the oscillatory integral $\mathrm{O_s}[e^{-iy\cdot\eta}a(\eta,y)]=\mathrm{O_s}-\iint e^{-iy\cdot\eta}a(\eta,y)d\eta dy$ by

$$\mathrm{O_s}-\iint e^{-iy\cdot\eta}a(\eta,y)d\eta dy=\lim_{\varepsilon\to 0}\iint e^{-iy\cdot\eta}\chi(\varepsilon\eta,\varepsilon y)a(\eta,y)d\eta dy\,, \tag{1.8}$$

where $\chi(\eta,y)\in\mathcal{S}(R^n_\eta\times R^n_y)$ such that $\chi(0,0)=1$. (It is shown in [7] that the limit in the right hand side of (1.8) exists and is independent of any particular choice of $\chi(\eta,y)$.)

DEFINITION 1.2. For a $p_h\in B^m_{\rho,\delta}(h)$ we define a family $\{P_h\}_{0<h<1}$ of pseudo-differential operators $P_h=p_h(X,D_x,X',D_{x'},X'')$ $(0<h<1)$ by

$$\begin{aligned}P_hu(x)&=\mathrm{O_s}-\iiiint e^{-i(y^1\cdot\eta^1+y^2\cdot\eta^2)}p_h(x,\eta^1,x+y^1,\eta^2,x+y^1+y^2)\\ &\quad\times u(x+y^1+y^2)d\eta^1 d\eta^2 dy^1 dy^2\,,\\ &\quad u\in\mathcal{B}(R^n)\,,\end{aligned} \tag{1.9}$$

and write $\{P_h\}_{0<h<1}\in\{\boldsymbol{B}^m_{\rho,\delta}(h)\}_{0<h<1}$, or simply $P_h\in\boldsymbol{B}^m_{\rho,\delta}(h)$.

REMARK. For symbols $p_h(x,\xi,x')$, $p_h(x,\xi)$, $p_h(\xi,x')$, we have from (1.9) the representation formulae:

$$\begin{aligned} &p_h(X, D_x, X')u(x) \\ (1.9)' \quad &= \mathrm{O_s}-\iint e^{-iy\cdot\eta} p_h(x, \eta, x+y)u(x+y)đ\eta dy \\ &= \mathrm{O_s}-\iint e^{i(x-x')\cdot\xi} p_h(x, \xi, x')u(x')đ\xi dx', \ u\in\mathcal{B}(R^n), \end{aligned}$$

$$(1.10) \qquad p_h(X, D_x)u(x) = \int e^{ix\cdot\xi} p_h(x, \xi)\hat{u}(\xi)đ\xi, \ u\in\mathcal{S},$$

$$(1.10)' \qquad \widehat{p_h(D_x, X')u(\xi)} = \int e^{-ix'\cdot\xi} p_h(\xi, x')u(x')dx', \ u\in\mathcal{S}.$$

Now we state several fundamental theorems for a family of pseudo-differential operators.

Theorem 1.3. *Let $p_{j,h}(x,\xi,x',\xi',x'')\in B^{m_j}_{\rho,\delta}(h)$, $j=0,1,2,\cdots$, such that $m_0\leqq m_1\leqq\cdots\leqq m_j\leqq\cdots\to\infty$. Then, there exists $p_h(x,\xi,x',\xi',x'')\in B^{m_0}_{\rho,\delta}(h)$ such that*

$$(1.11) \qquad p_h\sim\sum_{j=0}^{\infty} p_{j,h}$$

in the sense that for any $N\geqq 1$

$$(1.12) \qquad p_h(x, \xi, x', \xi', x'')-\sum_{j=0}^{N-1} p_{j,h}(x, \xi, x', \xi', x'')\in B^{m_N}_{\rho,\delta}(h).$$

Furthermore such a $p_h\in B^{m_0}_{\rho,\delta}(h)$ exists uniquely modulo $B^{\infty}(h)\equiv\bigcap_{m\in R} B^m_{0,0}(h)$ $(=\bigcap_{m\in R} B^m_{\rho,\delta}(h))$.

Proof. Let $\chi(\theta)$ be a C^∞-function on $[0,\infty]$ such that

$$(1.13) \qquad \begin{cases} 0\leqq\chi(\theta)\leqq 1 \ \text{ on } [0, \infty], \\ \chi(\theta) = 1 (0\leqq\theta\leqq 1/2), \ = 0 \ (\theta\geqq 1). \end{cases}$$

Then, for any fixed $\varepsilon>0$ we have

$$(1.14) \qquad 1-\chi(\varepsilon^{-1}h)\in B^{\infty}(h).$$

Now we assume that

$$(1.15) \quad |p_{j,h(\beta,\beta',\beta'')}^{(\alpha,\alpha')}(x, \xi, x', \xi', x'')| \leqq C_{j,\alpha,\alpha',\beta,\beta',\beta''} h^{m_j+\rho|\alpha+\alpha'|-\delta|\beta+\beta'+\beta''|}$$

and set

$$(1.16) \qquad C_j = \max_{|\alpha+\alpha'+\beta+\beta'+\beta''|\leqq j} \{C_{j,\alpha,\alpha',\beta,\beta',\beta''}\}.$$

Choose $0=k_0<k_1<\cdots<k_l<\cdots\to\infty$ such that

$$(1.17) \qquad m_{k_0}<m_{k_1}<\cdots<m_{k_l}<\cdots\to\infty,$$

and choose $1\geqq\varepsilon_0>\varepsilon_1>\cdots>\varepsilon_j>\cdots\to 0$ such that

$$C_j\varepsilon_j^{m_{k_l}-m_{k_l-1}}\leqq\frac{1}{2^j}\quad\text{for } k_l\leqq j<k_{l+1}\,.$$

Then, setting

$$p_h(x,\xi,x',\xi',x'')=\sum_{j=0}^{\infty}\chi(\varepsilon_j^{-1}h)p_{j,h}(x,\xi,x',\xi',x'')\,, \tag{1.18}$$

we see in a usual way that p_h is the desired one (cf. [7]). Q.E.D.

Theorem 1.4. *For $p_h(x,\xi,x',\xi',x'')\in B^m_{\rho,\delta}(h)$, define $p_{h,L}(x,\xi,x')$ and $p_{h,R}(x,\xi,x')$, respectively, by*

$$\begin{aligned} &p_{h,L}(x,\xi',x'')\\ &=\mathrm{O}_s-\iint e^{-iy\cdot\eta}p_h(x,\xi'+\eta,x+y,\xi',x'')d\eta dy \end{aligned} \tag{1.19}$$

and

$$\begin{aligned} &p_{h,R}(x,\xi,x'')\\ &=\mathrm{O}_s-\iint e^{-iy\cdot\eta}p_h(x,\xi,x''+y,\xi-\eta,x'')d\eta dy\,. \end{aligned} \tag{1.20}$$

Then we have

$$p_{h,L}(x,\xi,x'),\ p_{h,R}(x,\xi,x')\in B^m_{\rho,\delta}(h) \tag{1.21}$$

and for $P_h=p_h(X,D_x,X',D_x,X'')$

$$P_h=p_{h,L}(X,D_x,X')=p_{h,R}(X,D_x,X')\,. \tag{1.22}$$

Furthermore, the mappings: $B^m_{\rho,\delta}(h)\ni p_h\mapsto p_{h,L},p_{h,R}\in B^m_{\rho,\delta}(h)$ are continuous, and for a fixed even integer $n_0(>n)$ and any l there exists a constant C_l such that

$$|p_{h,L}|_l^{(m)},\ |p_{h,R}|_l^{(m)}\leqq C_l|p_h|_{l+2n_0}^{(m)}\,. \tag{1.23}$$

If, in particular, $0\leqq\delta<\rho\leqq 1$, we have the asymptotic expansion formulae

$$\begin{cases} \text{i)}\ \ p_{h,L}(x,\xi,x')\sim\displaystyle\sum_\alpha\frac{1}{\alpha!}p_{h(0,\alpha,0)}^{(\alpha,0)}(x,\xi,x,\xi,x')\,,\\ \text{ii)}\ \ p_{h,R}(x,\xi,x')\sim\displaystyle\sum_\alpha\frac{(-1)^{|\alpha|}}{\alpha!}p_{h(0,\alpha,0)}^{(0,\alpha)}(x,\xi,x',\xi,x')\,. \end{cases} \tag{1.24}$$

REMARK. Since $p_{h(0,\alpha,0)}^{(\alpha,0)}(x,\xi,x',\xi,x'')$, $p_{h(0,\alpha,0)}^{(0,\alpha)}(x,\xi,x',\xi',x'')\in B^{m+(\rho-\delta)|\alpha|}_{\rho,\delta}(h)$, and $m+(\rho-\delta)|\alpha|\to\infty$ as $|\alpha|\to\infty$ when $0\leqq\delta<\rho\leqq 1$, the formulae (1.24) have the definite meaning.

Proof. By the usual method we have (1.22). Consider

$$p_{h,\theta}(x, \xi', x'') \tag{1.25}$$
$$= \mathrm{O_s}-\iint e^{-iy\cdot\eta} p_h(x, \xi'+\theta\eta, x+y, \xi', x'')d\eta dy \quad (0\leqq\theta\leqq1)\,.$$

For a fixed even $n_0(>n)$ we have by integration by parts

$$\begin{aligned} &p_{h,\theta}(x, \xi', x'') \\ &= \mathrm{O_s}-\iint e^{-iy\cdot\eta}(1+h^{n_0\delta}|\eta|^{n_0})^{-1}(1+h^{n_0\delta}(-\Delta_y)^{n_0/2}) \\ &\quad\times \{(1+h^{-n_0\delta}|y|^{n_0})^{-1}(1+h^{-n_0\delta}(-\Delta_\eta)^{n_0/2}) \\ &\quad\times p_h(x, \xi'+\theta\eta, x+y, \xi', x'')\}\, d\eta dy\,. \end{aligned}$$

Then, noting $\delta\leqq\rho$, we have for a constant $C>0$ (independent of $0\leqq\theta\leqq1$)

$$|p_{h,\theta}(x, \xi', x'')|\leqq C|p_h|_{2n_0}^{(m)}h^m \quad (0\leqq\theta\leqq1)\,. \tag{1.26}$$

Differentiating the both sides of (1.25), we have also

$$\begin{aligned} &|p_{h,\theta}{}_{(\beta,\beta')}^{(\alpha)}(x, \xi', x'')| \\ &\leqq C_{\alpha,\beta,\beta'}|p_h|_{2n_0+|\alpha+\beta+\beta'|}^{(m)}h^{m+\rho|\alpha|-\delta|\beta+\beta'|} \quad (0\leqq\theta\leqq1)\,. \end{aligned} \tag{1.27}$$

Then, setting $\theta=1$, we get (1.21) for $p_{h,L}(x,\xi,x')$.

In the case $0\leqq\delta<\rho\leqq1$ we write

$$\begin{aligned} &p_h(x, \xi'+\eta, x+y, \xi', x'') \\ &= \sum_{|\alpha|<N}\frac{\eta^\alpha}{\alpha!}p_h{}^{(\alpha,0)}(x, \xi', x+y, \xi', x'') \\ &\quad+N\sum_{|\gamma|=N}\frac{\eta^\gamma}{\gamma!}\int_0^1(1-\theta)^{N-1}p_h{}^{(\gamma,0)}(x, \xi'+\theta\eta, x+y, \xi', x'')d\theta\,. \end{aligned} \tag{1.28}$$

Then, in the definition (1.19) we have from (1.28)

$$\begin{aligned} &\mathrm{O_s}-\iint e^{-iy\cdot\eta}\eta^\alpha p_h{}^{(\alpha,0)}(x, \xi', x+y, \xi', x'')d\eta dy \\ &= p_h{}_{(0,\alpha,0)}^{(\alpha,0)}(x, \xi', x, \xi', x'')\,, \end{aligned} \tag{1.29}$$

and

$$\begin{aligned} &\mathrm{O_s}-\iint e^{-iy\cdot\eta}\eta^\gamma p_h{}^{(\gamma,0)}(x, \xi'+\theta\eta, x+y, \xi', x'')d\eta dy \\ &= \mathrm{O_s}-\iint e^{-iy\cdot\eta}p_h{}_{(0,\gamma,0)}^{(\gamma,0)}(x, \xi'+\theta\eta, x+y, \xi', x'')d\eta dy\,. \end{aligned} \tag{1.30}$$

Hence, replacing p_h of (1.25) by $p_h{}_{(0,\gamma,0)}^{(\gamma,0)}$ and using (1.26), (1.27), we have from (1.28)–(1.30) the formula (1.24)–i). Similarly we get (2.21) and (1.24)–ii) for $p_{h,R}(x,\xi,x')$. Q.E.D.

As the special cases of Theorem 1.4 we get the following Theorems 1.5–1.7.

Theorem 1.5. *For* $p_h(x,\xi,x')\in B^m_{\rho,\delta}(h)$ *set*

$$p_{h,L}(x,\xi)=\mathrm{O_s}-\iint e^{-iy\cdot\eta}p_h(x,\xi+\eta,x+y)đ\eta dy \tag{1.31}$$

and

$$p_{h,R}(\xi,x')=\mathrm{O_s}-\iint e^{-iy\cdot\eta}p_h(x'+y,\xi-\eta,x')đ\eta dy\,. \tag{1.32}$$

Then we have

$$p_{h,L}(x,\xi),\ p_{h,R}(\xi,x')\in B^m_{\rho,\delta}(h) \tag{1.33}$$

and for $P_h=p_h(X,D_x,X')$

$$P_h=p_{h,L}(X,D_x)=p_{h,R}(D_x,X')\,. \tag{1.34}$$

Furthermore, the mappings: $B^m_{\rho,\delta}(h)\ni p_h\mapsto p_{h,L},\ p_{h,R}\in B^m_{\rho,\delta}(h)$ *are continuous, and for a fixed even* $n_0(>n)$ *and any* l *there exists a constant* C_l *such that*

$$|p_{h,L}|^{(m)}_l,\ |p_{h,R}|^{(m)}_l\leqq C_l|p_h|^{(m)}_{l+2n_0}\,. \tag{1.35}$$

If, in particular, $0\leqq\delta<\rho\leqq 1$, *we have the asymptotic formulae*

$$\begin{cases}\text{i)}\ \ p_{h,L}(x,\xi)\sim\sum_\alpha\dfrac{1}{\alpha!}p_{h(0,\alpha)}^{(\alpha)}(x,\xi,x)\,,\\ \text{ii)}\ \ p_{h,R}(\xi,x')\sim\sum_\alpha\dfrac{(-1)^{|\alpha|}}{\alpha!}p_{h(\alpha,0)}^{(\alpha)}(x',\xi,x')\,.\end{cases} \tag{1.36}$$

Corollary. *For* $P_h=p_h(X,D_x,X',D_{x'},X'')\in \boldsymbol{B}^m_{\rho,\delta}(h)$ *we define single symbols* $P_{h\,LL}(x,\xi)=(p_{h,L})_L(x,\xi), p_{h,RR}(\xi,x')=(p_{h,R})_R(\xi,x')\in B^m_{\rho,\delta}(h)$, *respectively, by* (1.19) *and* (1.20). *Then we get*

$$P_h=p_{h,LL}(X,D_x)=p_{h,RR}(D_x,X')\,. \tag{1.37}$$

Theorem 1.6. *For* $P_{j,h}=p_{j,h}(X,D_x)\in\boldsymbol{B}^{m_j}_{\rho,\delta}(h)$ $(j=1,2)$ *define* $p_h(x,\xi)$ *by*

$$p_h(x,\xi)=\mathrm{O_s}-\iint e^{-iy\cdot\eta}p_{1,h}(x,\xi+\eta)p_{2,h}(x+y,\xi)đ\eta dy\,. \tag{1.38}$$

Then, we have $p_h(x,\xi)\in B^{m_1+m_2}_{\rho,\delta}(h)$ *and* $p_h(X,D_x)=P_{1,h}P_{2,h}$.

Furthermore, the mapping: $B^{m_1}_{\rho,\delta}(h)\times B^{m_2}_{\rho,\delta}(h)\ni(p_{1,h},p_{2\,h})\mapsto p_h\in B^{m_1+m_2}_{\rho,\delta}(h)$ *is continuous, and for a fixed even* $n_0(>n)$ *and any* l *there exists a constant* C_l *such that*

$$|p_h|^{(m_1+m_2)}_l\leqq C_l|P_{1,h}|^{(m_1)}_{l+2n_0}|p_{2,h}|^{(m_2)}_{l+2n_0}\,. \tag{1.39}$$

If, in parricular, $0\leqq\delta<\rho\leqq 1$ *we have the expansion formula*

$$p_h(x,\xi)\sim\sum_\alpha\frac{1}{\alpha!}p_{1,h}^{(\alpha)}(x,\xi)p_{2,h(\alpha)}(x,\xi)\,. \tag{1.40}$$

Theorem 1.7. *For $Q_{j,h}=q_{j,h}(D_x,X')\in B^{m_j}_{\rho,\delta}(h)$ $(j=1,2)$ define $q_h(\xi,x')$ by*

$$q_h(\xi,x') = \mathrm{O}_s-\iint e^{-iy\cdot\eta}q_{1,h}(\xi,x'+y)q_{2,h}(\xi-\eta,x')d\eta dy\,. \tag{1.41}$$

Then, we have $q_h(\xi,x')\in B^{m_1+m_2}_{\rho,\delta}(h)$ and $q_h(D_x,X')=Q_{1,h}Q_{2,h}$.

Furthermore, the mapping: $B^{m_1}_{\rho,\delta}(h)\times B^{m_2}_{\rho,\delta}(h)\ni(q_{1,h},q_{2,h})\mapsto q_h\in B^{m_1+m_2}_{\rho,\delta}(h)$ is continuous, and for a fixed even $n_0(>n)$ and any l there exists a constatn C_l such that

$$|q_h|^{(m_1+m_2)}_l\leqq C_l|q_{1,h}|^{(m_1)}_{l+2n_0}|q_{2,h}|^{(m_2)}_{l+2n_0}\,. \tag{1.42}$$

If, in particular, $0\leqq\delta<\rho\leqq 1$, we have the expansion formula

$$q_h(\xi,x')\sim\sum_\alpha\frac{(-1)^{|\alpha|}}{\alpha!}q_{1,h(\alpha)}(\xi,x')q_{2,h}{}^{(\alpha)}(\xi,x')\,. \tag{1.43}$$

The next theorem concerns the multiproduct of pseudodifferential operators and will play an important role also in considering the multi-product of Fourier integral operators (see Theorem 4.3).

Theorem 1.8. *For $P_{j,h}=p_{j,h}(X,D_x,X')\in B^{m_j}_{\rho,\delta}(h)$ $(h=1,2,\cdots,\nu+1,\nu\geqq 1)$ define $q_{\nu+1,h}(x,\xi,x')$ by*

$$\begin{aligned} q_{\nu+1,h}&(x,\xi,x')\\ &=\mathrm{O}_s-\overbrace{\int\cdots\int}^{2\nu}\exp\Big(-i\sum_{j=1}^{\nu}y^j\cdot\eta^j\Big)\\ &\quad\times\prod_{j=1}^{\nu}p_{j,h}(x+\bar{y}^{j-1},\xi+\eta^j,x+\bar{y}^j)p_{\nu+1,h}(x+\bar{y}^\nu,\xi,x')d\boldsymbol{\eta}^\nu d\boldsymbol{y}^\nu\,, \end{aligned} \tag{1.44}$$

where $\bar{y}^0=0$, $\bar{y}^j=y^1+\cdots+y^j$ $(j=1,\cdots,\nu)$, $d\boldsymbol{\eta}^\nu=d\eta^1\cdots d\eta^\nu$, $d\boldsymbol{y}^\nu=dy^1\cdots dy^\nu$.

Then, we have

$$q_{\nu+1,h}(x,\xi,x')\in B^{\bar{m}_{\nu+1}}_{\rho,\delta}(h)\ (\bar{m}_{\nu+1}=m_1+\cdots+m_{\nu+1})\,, \tag{1.45}$$

and for $Q_{\nu+1,h}=q_{\nu+1,h}(X,D_x,X')$

$$Q_{\nu+1,h}=P_{1,h}\cdots P_{\nu+1,h}\,. \tag{1.46}$$

Furthermore, there exists a constant $C_0>0$ such that for a fixed even $n_0(>n)$

$$|q_{\nu+1,h}|^{(\bar{m}_{\nu+1})}_l\leqq C_0^{\nu+1}\sum_{l_1+\cdots+l_{\nu+1}\leqq l}\ \prod_{j=1}^{\nu+1}|p_{j,h}|^{(m_j)}_{3n_0+l_j}\,. \tag{1.47}$$

Proof. By the usual method we have (1.46) (cf. [7]). By integration by parts we write

$$
\begin{aligned}
&q_{\nu+1,h}(x, \xi, x') \\
(1.48) \qquad &= \mathrm{O}_s-\overbrace{\int\cdots\int}^{2\nu} \exp\left(-i\sum_{j=1}^{\nu} y^j\cdot\eta^j\right) \\
&\times \prod_{j'=1}^{\nu}(1+h^{-n_0\delta}|y^{j'}|^{n_0})^{-1}(1+h^{-n_0\delta}(-\Delta_{\eta^{j'}})^{n_0/2}) \\
&\times \prod_{j=1}^{\nu} p_{j,h}(x+\bar{y}^{j-1}, \xi+\eta^j, x+\bar{y}^j)p_{\nu+1,h}(x+\bar{y}^\nu, \xi, x')\boldsymbol{d\eta}^\nu \boldsymbol{dy}^\nu\,.
\end{aligned}
$$

Now we make a change of variables:

$$z^j = y^1+\cdots+y^j \ (\Leftrightarrow y^j = z^j - z^{j-1},\ z^0 = 0)$$

for $j=1, \cdots, \nu$. Then, noting

$$\sum_{j=1}^{\nu} y^j\cdot\eta^j = \sum_{k=1}^{\nu} z^k\cdot(\eta^k-\eta^{k+1}),\quad \eta^{\nu+1}=0$$

we make again the integration by parts. Then, we have from (1.48)

$$
\begin{aligned}
&q_{\nu+1,h}(x, \xi, x') \\
&= \overbrace{\int\cdots\int}^{2\nu} \exp\left(-i\sum_{k=1}^{\nu} z^k\cdot(\eta^k-\eta^{k+1})\right) \\
(1.49) \qquad &\times \prod_{k=1}^{\gamma}(1+h^{n_0\delta}|\eta^k-\eta^{k+1}|^{n_0})^{-1}(1+h^{n_0\delta}(-\Delta_{z^k})^{n_0/2}) \\
&\times \Big\{\prod_{j'=1}^{\nu}(1+h^{-n_0\delta}|z^{j'}-z^{j'-1}|^{n_0})^{-1}(1+h^{-n_0\delta}(-\Delta_{\eta^{j'}})^{n_0/2}) \\
&\times \prod_{j=1}^{\nu} p_{j,h}(x+z^{j-1}, \xi+\eta^j, x+z^j)p_{\nu+1,h}(x+z^\nu, \xi, x')\Big\}\boldsymbol{d\eta}^\nu \boldsymbol{dz}^\nu\,.
\end{aligned}
$$

Hence, noting $\delta\leqq\rho$ we have for a constant $C_1>0$

$$
\begin{aligned}
&|q_{\lambda+1,h}(x, \xi, x')| \\
&\leqq C_1^{\nu+1}\prod_{j=1}^{\nu+1}|p_{j,h}|_{3n_0}^{(m_j)}h^{\bar{m}_{\nu+1}} \\
(1.50) \qquad &\times \overbrace{\int\cdots\int}^{2\nu} \prod_{k=1}^{\nu}(1+h^{n_0\delta}|\eta^k-\eta^{k+1}|^{n_0})^{-1} \\
&\times \prod_{j'=1}^{\nu}(1+h^{-n_0\delta}|z^{j'}-z^{j'-1}|^{n_0})^{-1}\boldsymbol{d\eta}_\nu \boldsymbol{dz}^\nu\,.
\end{aligned}
$$

So for another constant $C_2>0$ we have

$$(1.51) \qquad |q_{\nu+1,h}(x, \xi, x')| \leqq C_2^{\nu+1}\prod_{j=1}^{\nu+1}|p_{j,h}|_{3n_0}^{(m_j)}h^{\bar{m}_{\nu+1}}\,.$$

In order to get the estimate for $q_{\nu+1,h}{}_{(\beta,\beta')}^{(\alpha)}(x, \xi, x')$ we difierentiate the both sides of (1.44) and apply (1.51). Then, we get (1.47) and (1.45). Q.E.D.

The following theorem is also a key theorem in considering the multi-product of Fourier integral operators (see Theorems 3.8 and 4.2).

Theorem 1.9. *Let $n_0(>n)$ be a fixed even integer. Then there exists a constant $c_0>0$ such that for any $P_h\equiv p_h(X,D_x,X')\in \boldsymbol{B}^0_{\rho,\delta}(h)$ with $|p_h|^{(0)}_{3n_0}\leqq c_0$ the operator $I-P_h$ has the inverse $(I-P_h)^{-1}$ in $\boldsymbol{B}^0_{\rho,\delta}(h)$.*

Proof. For $\nu\geqq 1$ we define $p_{\nu+1,h}(x,\xi,x')\in B^0_{\rho,\delta}(h)$ by (1.44) for $p_{j,h}=p_h$ $(j=1,\cdots,\nu+1)$. Then, by Theorem 1.8 we have

$$P_h^{\nu+1}=p_{\nu+1,h}(X,D_x,X') \tag{1.52}$$

and the estimate

$$|p_{\nu+1,h}|^{(0)}_l\leqq C_0^{\nu+1}\sum_{l_1+\cdots+l_{\nu+1}\leqq l}\ \prod_{j=1}^{\nu+1}|p_h|^{(0)}_{3n_0+l_j}\,. \tag{1.53}$$

Hence, when $\nu+1\geqq l$, we have

$$\begin{aligned}|p_{\nu+1,h}|^{(0)}_l&\leqq C_0^{\nu+1}(|p_h|^{(0)}_{3n_0})^{\nu+1-l}\sum_{l_1+\cdots+l_{\nu+1}\leqq l}(|p_h|^{(0)}_{3n_0+l})^l\\&\leqq(C_0c_0)^{\nu+1-l}\,(C_0|p_h|^{(0)}_{3n_0+l})^l\,C_{\nu,l}\,,\end{aligned} \tag{1.54}$$

where $C_{\nu,l}=\sum_{j=0}^{l}\binom{\nu+j}{j}$. We note that $C_{\nu,l}\leqq C_l\nu^l\,(\nu=1,2,\cdots)$ for a constant $C_l>0$. Then, we see that, if we choose $c_0>0$ such that $C_0\,c_0<1$, the series for $\sigma(P_h^{\nu+1})=p_{\nu+1,h}(x,\xi,x')$

$$1+\sigma(P_h)+\sigma(P_h^2)+\cdots+\sigma(P_h^{\nu+1})+\cdots$$

converges in $B^0_{\rho,\delta}(h)$ which means that

$$(I-P_h)^{-1}=I+P_h+P_h^2+\cdots+P_h^{\nu+1}+\cdots$$

exists in $\boldsymbol{B}^0_{\rho,\delta}(h)$. Q.E.D.

Proposition 1.10. *For any fixed $0<h<1$ the operator $P_h\in\boldsymbol{B}^m_{\rho,\delta}(h)$ defines continuous mappings $P_h\colon\mathcal{B}\to\mathcal{B}$ and $P_h\colon\mathcal{S}\to\mathcal{S}$.*

Proof. By the corollary of Theorem 1.5 we may only consider the case $P_h=p_h(X,D_x)\in\boldsymbol{B}^m_{\rho,\delta}(h)$. Then we have

$$P_hu(x)=\mathrm{O}_s\text{-}\iint e^{-iy\cdot\eta}p_h(x,\eta)u(x+y)\,đ\eta dy$$

for $u\in\mathcal{B}$, and

$$P_hu(x)=\int e^{ix\cdot\xi}p_h(x,\xi)\hat{u}(\xi)\,đ\xi$$

for $u\in\mathcal{S}$. Thus the proof is clear. Q.E.D.

Proposition 1.11. *For any fixed* $0<h<1$ *and the operator* $P_h=p_h(X, D_x, X', D_{x'}, X'')\in \boldsymbol{B}^m_{\rho,\delta}(h)$ *we have*

$$(P_h u, v)=(u, P_h^* v),\quad u, v\in\mathcal{S}, \tag{1.55}$$

where $P_h^*=p_h^*(X, D_x, X', D_x, X'')\in \boldsymbol{B}^m_{\rho,\delta}(h)$ *is defined by*

$$p_h^*(x, \xi, x', \xi', x'')=\overline{p_h(x'', \xi', x', \xi, x)}. \tag{1.56}$$

Furthermore P_h: $\mathcal{S}\to\mathcal{S}$ *is extended to the operator* P_h: $\mathcal{S}'\to\mathcal{S}'$ *uniquely by*

$$(P_h u, v)=(u, P_h^* v),\ u\in\mathcal{S}',\ v\in\mathcal{S}, \tag{1.57}$$

where $\mathcal{S}'$ *denotes the dual space of* $\mathcal{S}$.

Proof is clear by Proposition 1.10.

REMARK. If $P_h=p_h(X, D_x, X')\in \boldsymbol{B}^m_{\rho,\delta}(h)$, then we have $P_h^*=p_h^*(X', D_x, X'')$. But from the definition (1.9) we can get easily $P_h^*=p_h^*(X, D_x, X')$.

Theorem 1.12. *Let* $M=2\left(\left[\frac{n}{2}\right]+\left[\frac{5n}{4}\right]+2\right)$. *Then, there exists a constant* C *such that for any* $P_h=p_h(X, D_x, X')\in \boldsymbol{B}^m_{\rho,\delta}(h)$ *we have*

$$\|P_h u\|_{L^2}\leqq C|p_h|^{(m)}_M h^m\|u\|_{L^2}\quad (u\in L^2(R^n)). \tag{1.58}$$

Proof. Set $r_h(x, \xi, x')=h^{-m}p_h(h^\delta x, h^{-\delta}\xi, h^\delta x')$. Then noting $\delta\leqq\rho$ we have for $|\alpha+\beta+\beta'|\leqq M$

$$|r_{h(\beta,\beta')}^{(\alpha)}(x, \xi, x')|\leqq|p_h|^{(m)}_M.$$

By a change of variables $x=h^\delta\tilde{x}$, $\xi=h^{-\delta}\tilde{\xi}$, $x'=h^\delta\tilde{x}'$, we have

$$P_h u(h^\delta\tilde{x})=h^m \mathrm{O}_s\text{-}\iint e^{i(\tilde{x}-\tilde{x}')\cdot\tilde{\xi}} r_h(\tilde{x}, \tilde{\xi}, \tilde{x}')$$
$$\times u(h^\delta\tilde{x}')\,d\tilde{\xi}\,d\tilde{x}'.$$

Then, setting $v_h(\tilde{x})=u(h^\delta\tilde{x})$ and $w_h(\tilde{x})=P_h u(h^\delta\tilde{x})$ we have by the Calderón-Vaillancourt theorem ([1])

$$\|w_h\|_{L^2}\leqq C|p_h|^{(m)}_M h^m\|v_h\|_{L^2} \tag{1.59}$$

for a constant C independent of $0<h<1$. Q.E.D.

2. A family of phase functions

Let $\mathcal{B}^{m,\infty}(R^{2n})$ denote the set of C^∞-functions f in $R^{2n}=R^n_x\times R^n_\xi$ whose derivatives $\partial^\alpha_\xi D^\beta_x f(x, \xi)$ are bounded on R^{2n} for $|\alpha+\beta|\geqq m$. Then, we define the classes of phase functions as follows.

DEFINITION 2.1. i) For $0\leqq\tau<1$ and integer $l\geqq 0$ we say that a real valued function $\phi(x,\xi)$ in R^{2n} belongs to the class $\tilde{P}(\tau,l)$ of phase functions, when $\phi(x,\xi)$ is of class C^{l+2} and satisfies for $J(x,\xi)\equiv\phi(x,\xi)-x\cdot\xi$

$$(2.1)\qquad \begin{aligned} |J|_l\equiv &\sum_{|\alpha+\beta|\leqq 1}\sup_{x,\xi}\{|J^{(\alpha)}_{(\beta)}(x,\xi)|/\langle x;\xi\rangle^{2-|\alpha+\beta|}\} \\ &+\sum_{2\leqq|\alpha+\beta|\leqq l+2}\sup_{x,\xi}\{|J^{(\alpha)}_{(\beta)}(x,\xi)|\}\leqq\tau\,. \end{aligned}$$

ii) We say that a phase function $\phi(x,\xi)$ $(\in\tilde{P}(\tau,l))$ belongs to the class $P(\tau,l)$, when $\phi(x,\xi)$, moreover, belongs to $\mathscr{B}^{2,\infty}(R^{2n})$.

iii) We say that a family $\{\phi_h(x,\xi)\}_{0<h<1}$ of C^∞-functions $\phi_h(x,\xi)$ in R^{2n} belongs to the class $\{P_{\rho,\delta}(\tau,l;h)\}_{0<h<1}$ with $0\leqq\delta\leqq\rho\leqq 1$, when the functions

$$(2.2)\qquad \begin{cases}\tilde{\phi}_h(x,\xi)\equiv h^{\rho-\delta}\phi_h(h^\delta x,h^{-\rho}\xi)\,,\\ \tilde{J}_h(x,\xi)\equiv h^{\rho-\delta}J_h(h^\delta x,h^{-\rho}\xi)\,,\end{cases}$$

satisfy

$$(2.3)\qquad \tilde{\phi}_h(x,\xi)\in P(\tau,l)\ \text{ for any } h\in(0,1)$$

and

$$(2.4)\qquad \sup_{h,x,\xi}|\tilde{J}_{h(\beta)}^{(\alpha)}(x,\xi)|<\infty\quad\text{for}\quad|\alpha+\beta|\geqq 2\,.$$

We write this as $\{\phi_h(x,\xi)\}_{0<h<1}\in\{P_{\rho,\delta}(\tau,l;h)\}_{0<h<1}$ or simply as $\phi_h(x,\xi)\in P_{\rho\,\delta}(\tau,l;h)$.

REMARK 1°. If $\phi_h(x,\xi)=\phi(x,\xi)\in P(\tau,l)$ (independent of $0<h<1$), then $\phi_h(x,\xi)\in P_{0,0}(\tau,l;h)$. So we can write $P(\tau,l)\subset P_{0,0}(\tau,l;h)$.

2°. By the definition, $P(\tau,l)\subset\tilde{P}(\tau,l)$.

3°. $\tilde{P}(\tau,l)\subset\tilde{P}(\tau',l')$, $P(\tau,l)\subset P(\tau',l')$, if $\tau\leqq\tau'$ and $l\geqq l'$.

4°. For $\phi(x,\xi)\in P(\tau,l)$ set $\phi_h(x,\xi)=h^{\delta-\rho}\phi(h^{-\delta}x,h^\rho\xi)$. Then, $\phi_h(x,\xi)\in P(\tau,l;h)$, since $\tilde{\phi}_h(x,\xi)$ defined by (2.2) is equal to $\phi(x,\xi)$.

5°. In sections 3 and 4, for $\phi_h(x,\xi)\in P_{\rho,\delta}(\tau,0;h)$ we often use the important semi-norm $|J_h|_{2,\sigma}$ ($\sigma\geqq 0$, integer) with $J_h(x,\xi)=\phi_h(x,\xi)-x\cdot\xi$ defined by

$$(2.1)'\qquad |J_h|_{2,\sigma}=\sum_{2\leqq|\alpha+\beta|\leqq 2+\sigma}\sup_{\substack{x,\xi\\ 0<h<0}}\{|\tilde{J}_{h(\beta)}^{(\alpha)}(x,\xi)|\}\,,$$

where $\tilde{J}_h(x,\xi)=\tilde{\phi}_h(x,\xi)-x\cdot\xi\,(\in P_{0\,0}(\tau,0;h))$. Then, since $\tilde{\tilde{J}}_h=\tilde{J}_h$, we have $|J_h|_{2,\sigma}=|\tilde{J}_h|_{2,\sigma}$ and (2.1)′ is rewritten as

$$(2.1)''\qquad |J_h|_{2,\sigma}=\sum_{2\leqq|\alpha+\beta|\leqq 2+\sigma}\sup_{\substack{x,\xi\\ 0<h<1}}\{h^{\delta(|\beta|-1)-\rho(|\alpha|-1)}|J_{h(\beta)}^{(\alpha)}(x,\xi)|\}$$

by virtue of (2.2). For $\phi(x,\xi)\in P(\tau,0)\subset P_{0,0}(\tau,0;h)$ we have

$$(2.1)'''\qquad |J_h|_{2,\sigma}=\sum_{2\leqq|\alpha+\beta|\leqq 2+\sigma}\sup_{x,\xi}\{|J^{(\alpha)}_{(\beta)}(x,\xi)|\}\,.$$

Proposition 2.2. *Let $\phi_j(x,\xi)\in\tilde{P}(\tau,0)$, $j=1,2,\cdots$, and let $\tau_\infty\equiv\sum_{j=1}^{\infty}\tau_j\leqq\tau_0$ for some $0<\tau_0<1/2$. Then, for any $\nu\geqq 1$ and $(x,\xi)\in R^{2n}$ the solution $\{X_\nu^j,\Xi_\nu^j\}_{j=1}^{\nu}(x,\xi)$ $(\in R^{2\nu n})$ of the equation*

$$(2.5)\qquad \begin{cases} \text{i)} & X_\nu^j=\nabla_\xi\phi_j(X_\nu^{j-1},\,\Xi_\nu^j)\,, \\ \text{ii)} & \Xi_\nu^j=\nabla_x\phi_{j+1}(X_\nu^j,\,\Xi_\nu^{j+1})\,, \end{cases}\quad (j=1,\cdots,\nu)$$

exists uniquely, where $X_\nu^0=x$, $\Xi_\nu^{\nu+1}=\xi$.

Proof. Set

$$(2.6)\qquad y_\nu^j=X_\nu^j-X_\nu^{j-1},\ \eta_\nu^j=\Xi_\nu^j-\Xi_\nu^{j+1},\ (j=1,\cdots,\nu)\,.$$

Then

$$(2.7)\qquad \begin{cases} X_\nu^j=x+\bar{y}_\nu^j\ (\bar{y}_\nu^j=y_\nu^1+\cdots+y_\nu^j,\ \bar{y}_\nu^0=0,\ j=0,\cdots,\nu)\,, \\ \Xi_\nu^j=\bar{\bar{\eta}}_\nu^j+\xi\ (\bar{\bar{\eta}}_\nu^j=\eta_\nu^j+\cdots+\eta_\nu^\nu,\ \bar{\bar{\eta}}_\nu^{\nu+1}=0,\ j=1,\cdots,\nu+1)\,, \end{cases}$$

and the equation (2.5) is equivalent to

$$(2.8)\qquad \begin{cases} \text{i)} & y_\nu^j=\nabla_\xi J_j(x+\bar{y}_\nu^{j-1},\bar{\bar{\eta}}+\xi)\,, \\ \text{ii)} & \eta_\nu^j=\nabla_x J_{j+1}(x+\bar{y}_\nu^j,\,\bar{\bar{\eta}}_\nu^{j+1}+\xi)\,, \end{cases}\quad (j=1,\cdots,\nu)\,.$$

Now we define a mapping $\boldsymbol{T}_\nu$: $R^{2\nu n}\ni(\boldsymbol{y}_\nu,\boldsymbol{\eta}_\nu)=(y_\nu^1,\cdots,y_\nu^\nu,\eta_\nu^1,\cdots,\eta_\nu^\nu)\mapsto(\tilde{\boldsymbol{y}}_\nu,\tilde{\boldsymbol{\eta}}_\nu)=(\tilde{y}_\nu^1,\cdots,\tilde{y}_\nu^\nu,\tilde{\eta}_\nu^1,\cdots,\tilde{\eta}_\nu^\nu)=\boldsymbol{T}_\nu(\boldsymbol{y}_\nu,\boldsymbol{\eta}_\nu)\in R^{2\nu n}$ by

$$(2.9)\qquad \begin{cases} \text{i)} & \tilde{y}_\nu^j=\nabla_\xi J_j(x+\bar{y}_\nu^{j-1},\,\bar{\bar{\eta}}_\nu^j+\xi)\,, \\ \text{ii)} & \tilde{\eta}_\nu^j=\nabla_x J_{j+1}(x+\bar{y}_\nu^j,\,\bar{\bar{\eta}}_\nu^{j+1}+\xi), \end{cases}\quad (j=1,\cdots,\nu)\,.$$

Then, using the norm

$$\|(\boldsymbol{y}_\nu,\,\boldsymbol{\eta}_\nu)\|=\sum_{j=1}^{\nu}(|y_\nu^j|+|\eta_\nu^j|)\,,$$

we have for $(\tilde{\boldsymbol{y}}'_\nu,\tilde{\boldsymbol{\eta}}'_\nu)=\boldsymbol{T}_\nu(\boldsymbol{y}'_\nu,\boldsymbol{\eta}'_\nu)$

$$\begin{aligned} &|\tilde{y}'^j_\nu-\tilde{y}_\nu^j| \\ &=|\nabla_\xi J_j(x+\bar{y}'^{j-1}_\nu,\,\bar{\bar{\eta}}'^j_\nu+\xi)-\nabla_\xi J_j(x+\bar{y}^{j-1},\,\bar{\bar{\eta}}_\nu^j+\xi)| \\ &\leqq\int_0^1|\vec{\nabla}_x\nabla_\xi J_j(x+\bar{y}_\nu^{j-1}+\theta(\bar{y}'^{j-1}_\nu-\bar{y}_\nu^{j-1}),\,\bar{\bar{\eta}}_\nu^j+\xi+\theta(\bar{\bar{\eta}}'^j_\nu-\bar{\bar{\eta}}_\nu^j))|\,d\theta\cdot|\bar{y}'^{j-1}_\nu-\bar{y}_\nu^{j-1}| \\ &+\int_0^1|\vec{\nabla}_\xi\nabla_\xi J_j(x+\bar{y}_\nu^{j-1}+\theta(\bar{y}'^{j-1}_\nu-\bar{y}_\nu^{j-1}),\,\bar{\bar{\eta}}_\nu^j+\xi+\theta(\bar{\bar{\eta}}'^j_\nu-\bar{\bar{\eta}}_\nu^j))|\,d\theta\cdot|\bar{\bar{\eta}}'^j_\nu-\bar{\bar{\eta}}_\nu^j| \\ &\leqq\tau_j\sum_{k=1}^{\nu}(|y'^k_\nu-y_\nu^k|+|\eta'^k_\nu-\eta_\nu^k|)\,. \end{aligned}$$

Hence we get

$$\sum_{j=1}^{\nu} |\tilde{y}_\nu'^j - \tilde{y}_\nu^j| \leqq \bar{\tau}_{\nu+1} \|(\boldsymbol{y}_\nu', \boldsymbol{\eta}_\nu') - (\boldsymbol{y}_\nu, \boldsymbol{\eta}_\nu)\|$$
$$(\bar{\tau}_{\nu+1} = \tau_1 + \cdots + \tau_{\nu+1}),$$

and similarly get

$$\sum_{j=1}^{\nu} |\tilde{\eta}_\nu'^j - \tilde{\eta}_\nu^j| \leqq \bar{\tau}_{\nu+1} \|(\boldsymbol{y}_\nu', \boldsymbol{\eta}_\nu') - (\boldsymbol{y}_\nu, \boldsymbol{\eta}_\nu)\| .$$

Consequently we get

$$\|(\tilde{\boldsymbol{y}}_\nu', \tilde{\boldsymbol{\eta}}_\nu') - (\tilde{\boldsymbol{y}}_\nu, \tilde{\boldsymbol{\eta}}_\nu)\| \leqq 2\tau_0 \|(\boldsymbol{y}_\nu', \boldsymbol{\eta}_\nu') - (\boldsymbol{y}_\nu, \boldsymbol{\eta}_\nu)\| ,$$

which means that the mapping $\boldsymbol{T}_\nu$ is contractive. Hence, we see that the equation (2.8) has a unique solution $\{\boldsymbol{y}_\nu, \boldsymbol{\eta}_\nu\}(x, \xi)$. Q.E.D.

Proposition 2.3. *Let* $\phi_j(x,\xi) \in \tilde{P}(\tau_j, l)$ *(resp.* $P(\tau_j, l)$*),* $j=1,2,\cdots,$ *and let* $\bar{\tau}_\infty \leqq \tau_0$ *for some* $0<\tau_0<1/2$. *Then we have that the solution* $\{X_\nu^j, \Xi_\nu^j\}_{j=1}^{\nu}(x,\xi)$ *of* (2.5) *is of class* C^{l+1} *(resp.* C^∞*).*

Proof. Consider the function $\{\boldsymbol{f}_\nu, \boldsymbol{g}_\nu\}(\boldsymbol{z}_\nu, \boldsymbol{\gamma}_\nu; x, \xi) = \{f_\nu^1, \cdots, f_\nu^\nu, g_\nu^1, \cdots, g_\nu^\nu\}(\boldsymbol{z}_\nu, \boldsymbol{\gamma}_\nu; x, \xi)$ defined by

$$(2.10) \qquad \begin{cases} \text{i)} \;\; f_\nu^j = z_\nu^j - \nabla_\xi J_j(x + \bar{z}_\nu^{j-1}, \bar{\bar{\gamma}}_\nu^j + \xi), \\ \text{ii)} \;\; g_\nu^j = \gamma_\nu^j - \nabla_x J_{j+1}(x + \bar{z}_\nu^j, \bar{\bar{\gamma}}_\nu^{j+1} + \xi), \end{cases}$$
$$(j = 1, \cdots, \nu, \; \bar{z}_\nu^0 = 0, \; \bar{\bar{\gamma}}_\nu^{\nu+1} = 0).$$

Then, we have the Jacobian

$$\frac{D(\boldsymbol{f}_\nu, \boldsymbol{g}_\nu)}{D(\boldsymbol{z}_\nu, \boldsymbol{\gamma}_\nu)} = \det \left[I - \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \right],$$

where I is the unit matrix and

$$H_{11} = [h_{11,jk} = \vec{\nabla}_x \nabla_\xi J_j \; (k<j), = 0 \quad (k \geqq j)],$$
$$H_{12} = [h_{12,jk} = \vec{\nabla}_\xi \nabla_\xi J_j \; (k \geqq j), = 0 \quad (k<j)],$$
$$H_{21} = [h_{21,jk} = \vec{\nabla}_x \nabla_x J_{j+1} \; (k \leqq j), = 0 \quad (k>j)],$$
$$H_{22} = [h_{22,jk} = \vec{\nabla}_\xi \nabla_x J_{j+1} \; (k>j), = 0 \quad (k \leqq j)].$$

Hence, we have $\dfrac{D(\boldsymbol{f}_\nu, \boldsymbol{g}_\nu)}{D(\boldsymbol{z}_\nu, \boldsymbol{\gamma}_\nu)} \neq 0$, since

$$\sum_{j=1}^{\nu} (\|h_{11,jk}\| + \|h_{21,jk}\|) \leqq \sum_{j=1}^{\nu} (\tau_j + \tau_{j+1}) \leqq 2\tau_0 < 1 ,$$

and

$$\sum_{j=1}^{\nu} (\|h_{12,jk}\| + \|h_{22,jk}\|) \leqq \sum_{j=1}^{\nu} (\tau_j + \tau_{j+1}) \leqq 2\tau_0 < 1 .$$

Then by the implicit function thorem we see that the solutions of (2.8) and also (2.5) are of class C^{l+1} (resp. C^∞) with respect to (x,ξ). Q.E.D.

Proposition 2.4. *Let* $\phi_j(x,\xi)\in\tilde{P}(\tau_j,l)$, $j=1,2,\cdots$, *and let* $\bar{\tau}_\infty\leqq\tau_0$ *with* $0<\tau_0\leqq 1/4$. *Then we have*

i) *There exists a constant* $c_l>0$ *such that*

(2.11)
$$|(X_\nu^j-X_\nu^{j-1},\ \Xi_\nu^{j-1}-\Xi_\nu^j)|\leqq 4\tau_j\langle x;\xi\rangle \quad (\nu\geqq 1,\ j=1,\cdots,\nu+1)$$

and

(2.12)
$$|\partial_\xi^\alpha\partial_x^\beta(X_\nu^j-X_\nu^{j-1},\ \Xi_\nu^{j-1}-\Xi_\nu^j)|\leqq c_l\tau_j \quad (\nu\geqq 1,\ j=1,\cdots,\nu+1,\ 1\leqq|\alpha+\beta|\leqq l+1)\,.$$

ii) *Furthermore, assume that* $\phi_j(x,\xi)\in P(\tau_j,l)$, $j=1,2,\cdots$, *and, setting*

$$J_j(x,\xi)=\phi_j(x,\xi)-x\cdot\xi,\ \nabla=(\nabla_x,\nabla_\xi)\,,$$

assume that

(2.13)
$$\text{“}\{\vec{\nabla}\nabla J_j(x,\xi)/\tau_j\}_{j=1}^\infty \text{ is bounded in } \mathcal{B}(R^{2n})\text{”}.$$

Then we have

(2,.14)
$$\text{“}\{\nabla(X_\nu^j-X_\nu^{j-1},\ \Xi_\nu^{j-1}-\Xi_\nu^j)/\tau_j\}_{\substack{\nu=1,2,\cdots,\\ j=1,\cdots,\nu+1}} \text{ is bounded in } \mathcal{B}(R^{2n})\text{.”}$$

Proof. Since $y_\nu^j=X_\nu^j-X_\nu^{j-1}$, $\eta_\nu^j=\Xi_\nu^j-\Xi_\nu^{j+1}$ $(j=1,\cdots,\nu,\ X_0^\nu=x,\ \Xi_\nu^{\nu+1}=\xi)$ are the solution of (2.8), we have

(2.15)
$$\begin{cases} |y_\nu^j|\leqq\tau_j\langle x+\bar{y}_\nu^{j-1};\ \bar{\bar{\eta}}_\nu^j+\xi\rangle \\ \qquad\leqq\tau_j\{\sum_{k=1}^{\nu}(|y_\nu^k|+|\eta_\nu^k|)+\langle x;\xi\rangle\}\,, \\ |\eta_\nu^j|\leqq\tau_{j+1}\langle x+\bar{y}_\nu^j;\ \bar{\bar{\eta}}_\nu^{j+1}+\xi\rangle \\ \qquad\leqq\tau_{j+1}\{\sum_{k=1}^{\nu}(|y_\nu^k|+|\eta_\nu^k|)+\langle x;\xi\rangle\} \\ \qquad\qquad (j=1,\cdots,\nu)\,. \end{cases}$$

Here, we used the inequality

(2.16)
$$\langle x+y;\ \xi+\eta\rangle\leqq|y|+|\eta|+\langle x;\xi\rangle,\quad (x,\xi),(y,\eta)\in R^{2n}\,.$$

Then, from (2.15) we have

$$\sum_{j=1}^{\nu}(|y_\nu^j|+|\eta_\nu^j|)\leqq 2\bar{\tau}_\infty\{\sum_{k=1}^{\nu}(|y_\nu^k|+|\eta_\nu^k|)+\langle x;\xi\rangle\}\,,$$

and noting $2\bar{\tau}_\infty(1-2\bar{\tau}_\infty)^{-1}\leqq 1$ by $\bar{\tau}_\infty\leqq 1/4$ we have

(2.17)
$$\sum_{j=1}^{\nu}(|y_\nu^j|+|\eta_\nu^j|)\leqq\langle x;\xi\rangle\,.$$

Applying (2.17) to the right sides of (2.15), we get

$$|y_\nu^j|\leqq 2\tau_j\langle x;\xi\rangle,\ |\eta_\nu^j|\leqq 2\tau_{j+1}\langle x;\xi\rangle\ (\nu\geqq 1, j=1,\cdots,\nu),$$

which means that (2.11) holds. If we differentiate the both sides of (2.8), by induction we get (2.12) by the similar way. Also (2.14) can be obtained similarly. Q.E.D.

Summarizing Propositions 2.2–2.4, we get

Theorem 2.5. *Let $\phi_j(x,\xi)\in P(\tau_j, l)$, $j=1,2,\cdots$, and let $\bar{\tau}_\infty\leqq\tau_0$ with $0<\tau_0\leqq 1/4$. Assume, further, that $\{\vec{\nabla}\nabla J_j(x,\xi)/\tau_j\}_{j=1}^\infty$ is bounded in $\mathcal{B}(R^{2n})$. Then, there exists a constant $c_l>0$ such that the solution $\{X_\nu^j, \Xi_\nu^j\}_{j=1}^\nu(x,\xi)$ of (2.5) exists uniquely and satisfies*

$$(2.18)\qquad \begin{cases}|(X_\nu^j-X_\nu^{j-1}, \Xi_\nu^{j-1}-\Xi_\nu^j)|\leqq 4\tau_j\langle x;\xi\rangle,\\ |(X_\nu^j-x, \Xi_\nu^j-\xi)|\leqq 4\bar{\tau}_{\nu+1}\langle x;\xi\rangle\\ \qquad\qquad (\nu\geqq 1, j=1,\cdots,\nu),\end{cases}$$

and

$$(2.19)\qquad \begin{cases}|\partial_\xi^\alpha\partial_x^\beta(X_\nu^j-X_\nu^{j-1}, \Xi_\nu^{j-1}-\Xi_\nu^j)|\leqq c_l\tau_j,\\ |\partial_\xi^\alpha\partial_x^\beta(X_\nu^j-x, \Xi_\nu^j-\xi)|\leqq c_l\bar{\tau}_{\nu+1}\\ \qquad (\nu\geqq 1, j=1,\cdots,\nu, 1\leqq|\alpha+\beta|\leqq l+1).\end{cases}$$

Furthermore we have

$$(2.20)\qquad \begin{aligned}&\text{“}\{\nabla(X_\nu^j-X_\nu^{j-1}, \Xi_\nu^{j-1}-\Xi_\nu^j)/\tau_j\}_{j=1,\cdots,\nu}^{\nu=1,2,\cdots}\\ &\textit{and}\ \ \{\nabla(X_\nu^j-x, \Xi_\nu^j-\xi)/\bar{\tau}_{\nu+1}\}_{j=1,\cdots,\nu}^{\nu=1,2,\cdots}\\ &\textit{are bounded in}\ \mathcal{B}(R^{2n})\text{”}.\end{aligned}$$

DEFINITION 2.6. Let $\phi_j(x,\xi)\in\tilde{P}(\tau_j,0)$, $j=1,2,\cdots$, and let $\bar{\tau}_\infty\equiv\sum_{j=1}^\infty\tau_j\leqq\tau_0$ with $0<\tau_0<1/2$. Then using the solution $\{X_\nu^j, \Xi_\nu^j\}_{j=1}^\nu(x,\xi)$ of (2.5) (from Proposition 2.2), we define the $\sharp-(\nu+1)$ product $\Phi_{\nu+1}=\phi_1\sharp\cdots\sharp\phi_{\nu+1}$ of $\phi_1,\cdots,\phi_{\nu+1}$ by

$$(2.21)\qquad \Phi_{\nu+1}(x,\xi)=\sum_{j=1}^\nu(\phi_j(X_\nu^{j-1}, \Xi_j^\nu)-X_\nu^j\cdot\Xi_\nu^j)+\phi_{\nu+1}(X_\nu^\nu,\xi)$$

with $X_\nu^0=x$.

Theorem 2.7. *Let $\phi_j(x,\xi)\in\tilde{P}(\tau_j;0)$, $j=1,2,\cdots$, and let $\bar{\tau}_\infty\leqq\tau_0$ with a small constant $0<\tau_0\leqq 1/4$. Then, we have the following:*

i) *There exists a constant $c_0\geqq 1$ with $c_0\tau_0<1$ such that*

$$(2.22)\qquad \Phi_{\nu+1}(x,\xi)\in\tilde{P}(c_0\bar{\tau}_{\nu+1}, 0)\quad(\nu\geqq 1),$$

and we have

$$(2.23)\qquad \begin{cases} \text{i)} & \nabla_x \Phi_{\nu+1}(x,\xi) = \nabla_x \phi_1(x, \Xi_\nu^1)\,, \\ \text{ii)} & \nabla_\xi \Phi_{\nu+1}(x,\xi) = \nabla_\xi \phi_{\nu+1}(X_\nu^\nu, \xi) \end{cases}$$

and, setting $\boldsymbol{J}_{\nu+1}(x,\xi)=\Phi_{\nu+1}(x,\xi)-x\cdot\xi$,

$$(2.24)\qquad \begin{cases} \text{i)} & \nabla_x \boldsymbol{J}_{\nu+1}(x,\xi) = \nabla_x J_1(x, \Xi_\nu^1)+\sum_{k=1}^{\nu}(\Xi_\nu^k-\Xi_\nu^{k+1})\,, \\ \text{ii)} & \nabla_\xi \boldsymbol{J}_{\nu+1}(x,\xi) = \sum_{k=1}^{\nu}(X_\nu^k-X_\nu^{k-1})+\nabla_\xi J_{\nu+1}(X_\nu^\nu, \xi)\,. \end{cases}$$

ii) *We have the associative law*:

$$(2.25)\qquad \begin{aligned} \Phi_{\nu+1} &= (\phi_1\#\cdots\#\phi_\nu)\#\phi_{\nu+1} \\ &= \phi_1\#(\phi_2\#\cdots\#\phi_{\nu+1})\,. \end{aligned}$$

iii) *Furthermore, assume that* $\phi_j(x,\xi)\in P(\tau_j, l)$, $j=1,2,\cdots$, *and let* $\bar{\tau}_\infty\leqq\tau_{0,l}$ *with a small constant* $0<\tau_{0,l}\leqq\tau_0$. *Then, there exists a constant* $c_{0,l}(\geqq c_0)$ *with* $c_{0,l}\tau_{0,l}<1$ *such that*

$$(2.26)\qquad \Phi_{\nu+1}(x,\xi)\in P(c_{0,l}\bar{\tau}_{\nu+1}, l)$$

and, if $\{\vec{\nabla}\nabla J_j(x,\xi)/\tau_j\}_{j=1}^\infty$ *is bounded in* $\mathcal{B}(R^{2n})$, *we have*

$$(2.27)\qquad \text{“}\{\vec{\nabla}\nabla \boldsymbol{J}_{\nu+1}(x,\xi)/\bar{\tau}_{\nu+1}\}_{\nu=1}^\infty \textit{ is bounded in } \mathcal{B}(R^{2n})\text{”}\,.$$

Proof. i) Using the definition (2.21) we can write

$$\begin{aligned} \boldsymbol{J}_{\nu+1}(x,\xi) &= \sum_{j=1}^{\nu}\{J_j(X_\nu^{j-1}, \Xi_\nu^j)+(X_\nu^{j-1}-X_\nu^j)\cdot\Xi_\nu^j\} \\ &\quad +J_{\nu+1}(X_\nu^\nu,\xi)+\sum_{k=1}^{\nu}(X_\nu^k-X_\nu^{k-1})\cdot\xi \\ &= \sum_{j=1}^{\nu+1}\{J_j(X_\nu^{j-1}, \Xi_\nu^j)+(X_\nu^{j-1}-X_\nu^j)\cdot(\Xi_\nu^j-\xi)\}\,. \end{aligned}$$

Then, by Proposition 2.4 and (2.11) we have

$$|\boldsymbol{J}_{\nu+1}(x,\xi)|\leqq\sum_{j=1}^{\nu+1}\{\tau_j\langle X_\nu^{j-1};\Xi_\nu^j\rangle^2+4\tau_j\cdot4\bar{\tau}_{\nu+1}\langle x;\xi\rangle^2\}\,,$$

and, writing $X_\nu^{j-1}=\sum_{k=1}^{j-1}(X_\nu^k-X_\nu^{k-1})+x$, $\Xi_\nu^j=\sum_{k=j}^{\nu}(\Xi_\nu^k-\Xi_\nu^{k+1})+\xi$, we get by (2.11)

$$(2.28)\qquad |\boldsymbol{J}_{\nu+1}(x,\xi)|\leqq C_1\bar{\tau}_{\nu+1}\langle x;\xi\rangle^2$$

for a constant $C_1>0$.

From the definition (2.21) and Proposition 2.3 we see that $\Phi_{\nu+1}(x,\xi)$ is C^1-class. Then, differentiating the both sides of (2.21) we have

$$\begin{aligned} \nabla_x\Phi_{\nu+1}(x,\xi) &= \sum_{j=1}^{\nu}\{\vec{\nabla}_x X_\nu^{j-1}\nabla_x\phi_j+\vec{\nabla}_x\Xi_\nu^j\nabla_\xi\phi_j \\ &\quad -\nabla_x\vec{X}_\nu^j\Xi_\nu^j-\nabla_x\vec{\Xi}_\nu^j X_\nu^j\}+\vec{\nabla}_x X_\nu^\nu\nabla_x\phi_{\nu+1}\,, \end{aligned}$$

and using (2.5) we have

$$\begin{aligned}\nabla_x\Phi_{\nu+1}(x,\xi) &= \nabla_x\phi_1+\sum_{j=2}^{\nu}\vec{\nabla}_x X_\nu^{j-1}\Xi_\nu^{j-1} \\ &+\sum_{j=1}^{\nu}\{\vec{\nabla}_x\Xi_\nu^j X_\nu^j-\nabla_x\vec{X}_\nu^j\Xi_\nu^j-\nabla_x\vec{\Xi}_\nu^j X_\nu^j\} \\ &+\vec{\nabla}_x X_\nu^\nu\Xi_\nu^\nu = \nabla_x\phi_1(x,\Xi_\nu^1)\,.\end{aligned}$$

Hence, we get (2.23)-i). From (2.23)-i) we write

$$\begin{aligned}\nabla_x \boldsymbol{J}_{\nu+1}(x,\xi) &= \nabla_x\Phi_{\nu+1}(x,\xi)-\xi \\ &= \nabla_x J_1(x,\Xi_\nu^1)+(\Xi_\nu^1-\xi) \\ &= \nabla_x J_1(x,\Xi_\nu^1)+\sum_{k=1}^{\nu}(\Xi_\nu^k-\Xi_\nu^{k+1})\,,\end{aligned}$$

and get (2.24)-i). Similarly we get (2.23)-ii) and (2.24)-ii).

From Proposition 2.4 and (2.24) we have for a constant $C_2>0$

$$|\nabla \boldsymbol{J}_{\nu+1}(x,\xi)|\leqq C_2\bar{\tau}_{\nu+1}\langle x;\xi\rangle\,. \tag{2.29}$$

Similarly, if we differentiate the both sides of (2.24), we have for a constant $C_3>0$

$$|\vec{\nabla}\nabla \boldsymbol{J}_{\nu+1}(x,\xi)|\leqq C_3\bar{\tau}_{\nu+1}\,. \tag{2.30}$$

Hence, setting $c_0=C_1+C_2+C_3$ and choosing $0<\tau_0\leqq 1/4$ such that $c_0\tau_0<1$, from (2.28)–(2.30) we get (2.22).

ii) Let $\Phi_\nu(x,\xi)=(\phi_1\#\cdots\#\phi_\nu)(x,\xi)$ and $\tilde{\Phi}_{\nu+1}(x,\xi)=(\Phi_\nu\#\phi_{\nu+1})(x,\xi)$. Let $\{\tilde{X}_\nu,\tilde{\Xi}_\nu\}(x,\xi)$ be the solution of

$$\begin{cases}\text{i)} & \tilde{X}_\nu = \nabla_\xi\Phi_\nu(x,\tilde{\Xi}_\nu)\,,\\ \text{ii)} & \tilde{\Xi}_\nu = \nabla_x\phi_{\nu+1}(\tilde{X}_\nu,\xi)\,.\end{cases} \tag{2.31}$$

Then we have

$$\tilde{\Phi}_{\nu+1}(x,\xi) = \Phi_\nu(x,\tilde{\Xi}_\nu)-\tilde{X}_\nu\cdot\tilde{\Xi}_\nu+\phi_{\nu+1}(\tilde{X}_\nu,\xi)\,. \tag{2.32}$$

On the other hand, by the definition of $\Phi_\nu(x,\xi)$ we have

$$\begin{aligned}\Phi_\nu(x,\tilde{\Xi}_\nu) &= \sum_{j=1}^{\nu-1}\{\phi_j(X_{\nu-1}^{j-1}(x,\tilde{\Xi}_\nu),\Xi_{\nu-1}^j(x,\tilde{\Xi}_\nu)) \\ &\quad -X_{\nu-1}^j(x,\Xi_\nu)\cdot\Xi_{\nu-1}^j(x,\tilde{\Xi}_\nu)\} \\ &\quad +\phi_\nu(X_{\nu-1}^{\nu-1}(x,\tilde{\Xi}_\nu),\tilde{\Xi}_\nu)\,,\end{aligned} \tag{2.33}$$

and for $\{X_{\nu-1}^j,\Xi_{\nu-1}^j\}_{j=1}^{\nu-1}(x,\Xi_\nu)$ we have

$$\begin{cases}X_{\nu-1}^j(x,\tilde{\Xi}_\nu) = \nabla_\xi\phi_j(X_{\nu-1}^{j-1}(x,\tilde{\Xi}_\nu),\Xi_{\nu-1}^j(x,\tilde{\Xi}_\nu))\,,\\ \Xi_{\nu-1}^j(x,\tilde{\Xi}_\nu) = \nabla_x\phi_{j+1}(X_{\nu-1}^j(x,\tilde{\Xi}_\nu),\Xi_{\nu-1}^{j+1}(x,\tilde{\Xi}_\nu)) \\ (j=1,\cdots,\nu-1,\ X_{\nu-1}^0(x,\tilde{\Xi}_\nu)=x,\ \Xi_{\nu-1}^\nu(x,\tilde{\Xi}_\nu)=\tilde{\Xi}_\nu)\,.\end{cases} \tag{2.34}$$

Hence, if we set

$$
(2.35)\qquad \begin{cases} \tilde{X}_\nu^j(x,\xi) = X_{\nu-1}^j(x,\tilde{\Xi}_\nu(x,\xi))\,, \\ \tilde{\Xi}_\nu^j(x,\xi) = \Xi_{\nu-1}^j(x,\tilde{\Xi}_\nu(x,\xi)), j=1,\cdots,\nu-1\,, \\ \tilde{X}_\nu^\nu(x,\xi) = \tilde{X}_\nu(x,\xi),\ \tilde{\Xi}_\nu^\nu(x,\xi) = \tilde{\Xi}_\nu(x,\xi), \end{cases}
$$

we have by (2.34)

$$
(2.36)\qquad \begin{cases} \tilde{X}_\nu^j = \nabla_\xi \phi_j(\tilde{X}_\nu^{j-1}, \tilde{\Xi}_\nu^j)\,, \\ \tilde{\Xi}_\nu^j = \nabla_x \phi_{j+1}(\tilde{X}_\nu^j, \tilde{\Xi}_\nu^{j+1}), j=1,\cdots,\nu-1\,. \end{cases}
$$

From (2.31)-ii) we have

$$
(2.37)\qquad \tilde{\Xi}_\nu^\nu = \nabla_x \phi_{\nu+1}(\tilde{X}_\nu^\nu, \xi)\,,
$$

and applying (2.23)-ii) to (2.31)-i) by replacing $\nu+1$ by ν, we have

$$
(2.38)\qquad \begin{aligned} \tilde{X}_\nu^\nu = \tilde{X}_\nu &= \nabla_\xi \phi_\nu(X_{\nu-1}^{\nu-1}(x,\tilde{\Xi}_\nu), \tilde{\Xi}_\nu) \\ &= \nabla_\xi \phi_\nu(\tilde{X}_\nu^{\nu-1}, \tilde{\Xi}_\nu^\nu)\,. \end{aligned}
$$

Hence, from (2.36)-(2.38) we see that $\{\tilde{X}_\nu^j, \tilde{\Xi}_\nu^j\}_{j=1}^\nu(x,\xi)$ is the solution of (2.5), and by the uniqueness, is equal to $\{X_\nu^j, \Xi_\nu^j\}_{j=1}^\nu(x,\xi)$. Then from (2.32) and (2.33) we have $\Phi_{\nu+1}(x,\xi)=\tilde{\Phi}_{\nu+1}(x,\xi)=((\phi_1\#\cdots\#\phi_\nu)\#\phi_{\nu+1})(x,\xi)$. Similarly we get $\Phi_{\nu+1}=\phi_1\#(\phi_2\#\cdots\#\phi_{\nu+1})$.

iii) If we differentiate the both sides of (2.24), then from Theorem 2.5 we get (2.26) and (2.27) by induction. Q.E.D.

Theorem 2.8. *Let* $\{\phi_{j,h}(x,\xi)\}_{0<h<1}, j=1,2,\cdots,$ *belong to* $\{P_{\rho,\delta}(\tau_j,0;h)\}_{0<h<1}$, *and let* $\bar{\tau}_\infty \leqq \bar{\tau}_0$ *with* $0<\tau_0<1/2$. *For* $\phi_{j,h}(x,\xi)$ *we define* $\tilde{\phi}_{j,h}(x,\xi)$ *by*

$$
(2.39)\qquad \tilde{\phi}_{j,h}(x,\xi) = h^{\rho-\delta}\phi_{j,h}(h^\delta x, h^{-\rho}\xi)\,.
$$

Then we have the following:

i) *Let* $\{X_{\nu,h}^j, \Xi_{\nu,h}^j\}_{j=1}^\nu(x,\xi)$ *and* $\{\tilde{X}_{\nu,h}^j, \tilde{\Xi}_{\nu,h}^j\}_{j=1}^\nu(x,\xi)$ *be the solution of the equation* (2.5) *for* $\{\phi_{j,h}\}_{j=1}^{\nu+1}$ *and* $\{\tilde{\phi}_{j,h}\}_{j=1}^{\nu+1}$, *respectively. Then they are uniquely defined as* C^∞*-functions on* $R_x^n\times R_\xi^n$, *and satisfy the relation*

$$
(2.40)\qquad \begin{cases} \tilde{X}_{\nu,h}^j(x,\xi) = h^{-\delta}X_{\nu,h}^j(h^\delta x, h^{-\rho}\xi)\,, \\ \tilde{\Xi}_{\nu,h}^j(x,\xi) = h^{\rho}\Xi_{\nu,h}^j(h^\delta x, h^{-\rho}\xi) \\ \qquad\qquad (\nu\geqq 1, j=1,\cdots,\nu)\,. \end{cases}
$$

ii) *Let* $\Phi_{\nu+1,h}(x,\xi)$ *and* $\tilde{\Phi}_{\nu+1,h}(x,\xi)$ *be defined by* (2.21) *for* $\{\phi_{j,h}\}_{j=1}^{\nu+1}$ *and* $\{\tilde{\phi}_{j,h}\}_{j=1}^{\nu+1}$, *respectively. Then we have the relation*

$$
(2.41)\qquad \begin{cases} \text{i)}\quad \tilde{\Phi}_{\nu+1,h}(x,\xi) = h^{\rho-\delta}\Phi_{\nu+1,h}(h^\delta x, h^{-\rho}\xi)\,, \\ \text{ii)}\quad \Phi_{\nu+1,h}(x,\xi) = h^{\delta-\rho}\tilde{\Phi}_{\nu+1,h}(h^{-\delta}x, h^{\rho}\xi)\,. \end{cases}
$$

Proof. i) From (2.5) for $\{\tilde{\phi}_{j,h}\}_{j=1}^{\nu+1}$ and (2.39) we have

$$
(2.42) \qquad \begin{cases} \tilde{X}_{\nu,h}^{j}(x,\xi) = h^{-\delta}\nabla_\xi \phi_{j,h}(h^\delta \tilde{X}_{\nu,h}^{j-1}(x,\xi),\ h^{-\rho}\tilde{\Xi}_{\nu,h}^{j}(x,\xi))\,, \\ \tilde{\Xi}_{\nu,h}^{j}(x,\xi) = h^{\rho}\nabla_x \phi_{j+1,h}(h^\delta \tilde{X}_{\nu,h}^{j}(x,\xi),\ h^{-\rho}\tilde{\Xi}_{\nu,h}^{j+1}(x,\xi)) \\ \qquad\qquad (\tilde{X}_{\nu,h}^{0}(x,\xi) = x,\ \tilde{\Xi}_{\nu,h}^{\nu+1}(x,\xi) = \xi)\,. \end{cases}
$$

Then from (2.5) for $\{\phi_{j,h}\}_{j=1}^{\nu+1}$ we get easily (2.40), since the solution of (2.5) is unique. By Proposition 2.3, the solution is C^∞.

ii) If we use (2.39) and (2.42), then by the definition (2.21) for $\tilde{\Phi}_{\nu+1,h}(x,\xi)$ we have

$$
(2.43) \qquad \begin{aligned} &\tilde{\Phi}_{\nu+1,h}(x,\xi) \\ &= \sum_{j=1}^{\nu} \{h^{\rho-\delta}\phi_{j,h}(X_{\nu,h}^{j-1}(h^\delta x, h^{-\rho}\xi),\ \Xi_{\nu,h}^{j}(h^\delta x, h^{-\rho}\xi)) \\ &\quad -h^{-\delta}X_{\nu,h}^{j}(h^\delta x, h^{-\rho}\xi)\cdot h^{\rho}\Xi_{\nu,h}^{j}(h^\delta x, h^{-\rho}\xi)\} \\ &\quad +h^{\rho-\delta}\phi_{\nu+1,h}(X_{\nu,h}^{\nu}(h^\delta x, h^{-\rho}\xi),\ h^{-\rho}\xi) \\ &= h^{\rho-\delta}\Phi_{\nu+1,h}(h^\delta x, h^{-\rho}\xi)\,, \end{aligned}
$$

which proves (2.41)-i) together with (2.41)-ii). Q.E.D.

Summing up, we have the following

Theorem 2.9. *Let $\{\phi_{j,h}(x,\xi)\}_{0<h<1}, j=1,2,\cdots$, belong to $\{P_{\rho,\delta}(\tau_j, l; h)\}_{0<h<1}$, and let $\bar{\tau}_\infty \leqq \tau_0 \leqq 1/4$. Let $\{\tilde{X}_{\nu,h}^{j},\ \tilde{\Xi}_{\nu,h}^{j}\}_{j=1}^{\nu}(x,\xi)$ and $\tilde{\Phi}_{\nu+1\,h}(x,\xi)$ be defined by* (2.5) *and* (2.21) *for $\{\tilde{\phi}_{j,h}\}_{j=1}^{\nu+1}$ of* (2.39)*, respectively. Assume, further, that $\{\vec{\nabla}\nabla \tilde{J}_{j,h}(x,\xi)/\tau_j\}_{\substack{j=1,2,\cdots \\ 0<h<1}}$ is bounded in $\mathcal{B}(R^{2n})$ for $\tilde{J}_{j,h}(x,\xi)=\tilde{\phi}_{j,h}(x,\xi)-x\cdot\xi$. Then, for $\{\tilde{X}_{\nu,h}^{j},\ \tilde{\Xi}_{\nu,h}^{j}\}_{j=1}^{\nu}(x,\xi)$ $(\nu\geqq 1,\ 0<h<1)$ and $\tilde{\Phi}_{\nu+1,h}(x,\xi)$ $(\nu\geqq 1,\ 0<h<1)$,* Theorem 2.5 *and* Theorem 2.7 *hold, respectively, and $\{\vec{\nabla}\nabla \boldsymbol{J}_{\nu+1,h}(x,\xi)/\bar{\tau}_{\nu+1}\}_{\substack{\nu=1,2,\cdots \\ 0<h<1}}$ is bounded in $\mathcal{B}(R^{2n})$.*

3. A family of Fourier integral operators

We define a family of Fourier integral operators.

DEFINITION 3.1. Let $\{\phi_h(x,\xi)\}_{0<h<1} \in \{P_{\rho,\delta}(\tau, 0; h)\}_{0<h<1}$ and $\{p_h(x,\xi)\}_{0<h<1}$, $\{q_h(\xi,x')\}_{0<h<1} \in \{B_{\rho,\delta}^{m}(h)\}_{0<h<1} (0\leqq\delta\leqq\rho\leqq 1)$. Then, the associated family of Fourier, and conjugate Fourier, integral operators $P_h(\phi_h)=p_h(\phi_h; X, D_x)$ and $Q_h(\phi_h^*)=q_h(\phi_h^*; D_x, X')$ are defined, respectively, by

$$
(3.1) \qquad \begin{aligned} &P_h(\phi_h)u(x) \\ &= \mathrm{O_s}-\iint e^{i(\phi_h(x,\xi)-x'\cdot\xi)} p_h(x,\xi)u(x')\,đ\xi dx' \end{aligned}
$$

and

$$Q_h(\phi_h^*)u(x)$$

(3.2)
$$= \mathrm{O_s}-\iint e^{i(x\cdot\xi-\phi_h(x',\xi))}q_h(\xi, x')u(x')d\xi dx'$$

for $u\in\mathcal{S}$. We write these as

$$\{P_h(\phi_h)\}_{0<h<1}\in\{\boldsymbol{B}^m_{\rho,\delta}(\phi_h)\}_{0<h<1}\,,$$
$$\{Q_h(\phi_h^*)\}_{0<h<1}\in\{\boldsymbol{B}^m_{\rho,\delta}(\phi_h^*)\}_{0<h<1}\,,$$

or simply

$$P_h(\phi_h)\in\boldsymbol{B}^m_{\rho,\delta}(\phi_h),\ Q_h(\phi_h^*)\in\boldsymbol{B}^m_{\rho,\delta}(\phi_h^*)\,.$$

REMARK. By the following proposition we can write also for $u\in\mathcal{S}$

(3.1)′
$$P_h(\phi_h)u(x) = \int e^{i\phi_h(x,\xi)}p_h(x, \xi)\hat{u}(\xi)d\xi\,,$$

(3.2)′
$$\widehat{Q_h(\phi_h^*)u}(\xi) = \int e^{-i\phi_h(x',\xi)}q_h(\xi, x')u(x')dx'\,.$$

The following proposition justifies the above definition.

Proposition 3.2. *Let* $\phi_h(x,\xi)\in P_{\rho,\delta}(\tau,0;h)$. *Then for any fixed* $0<h<1$, $P_h(\phi_h)\in\boldsymbol{B}^m_{\rho,\delta}(\phi_h)$ *and* $Q_h(\phi_h^*)\in\boldsymbol{B}^m_{\rho,\delta}(\phi_h^*)$ *define continuous maps*: $P_h(\phi_h)$, $Q_h(\phi_h^*)$: $\mathcal{S}\to\mathcal{S}$.

REMARK. For $P_h(\phi_h)=p_h(\phi_h;X,D_x)\in\boldsymbol{B}^m_{\rho,\delta}(\phi_h)$ if we define $Q'_h(\phi_h^*)=q'_h(\phi_h^*;D_x, X')\in\boldsymbol{B}^m_{\rho,\delta}(\phi_h^*)$ by $q'_h(\xi,x')=p_h(x',\xi)$, then we have $(P_h(\phi_h)u,v)=(u,Q'_h(\phi_h^*)v)$ for $u, v\in\mathcal{S}$. Hence, by this relation we can extend $P_h(\phi_h)$: $\mathcal{S}\to\mathcal{S}$ to $\mathcal{S}'\to\mathcal{S}'$, uniquely. The same thing holds for $Q_h(\phi_h^*)\in\boldsymbol{B}^m_{\rho,\delta}(\phi_h^*)$.

Proof. We first give the proof for $Q_h(\phi_h^*)=q_h(\phi_h^*;D_x, X')\in\boldsymbol{B}^m_{\rho,\delta}(\phi_h^*)$.
Set $\psi_h(x,\xi,x')=x\cdot\xi-\phi_h(x',\xi)=(x-x')\cdot\xi-J_h(x',\xi)$. Then, we have from (2.2)

(3.3)
$$\begin{aligned}|\nabla_{x'}\psi_h| &\geqq |\xi|-\tau h^{-\rho}\langle h^{-\delta}x';h^\rho\xi\rangle\\ &\geqq(1-\tau)|\xi|-\tau h^{-\rho}\langle h^{-\delta}x'\rangle\,.\end{aligned}$$

Hence we have $|\nabla_{x'}\psi_h|\geqq(1-\tau)|\xi|/2$ if $(1-\tau)|\xi|/2\geqq h^{-\rho}\langle h^{-\delta}x'\rangle$ and have $2\geqq(1-\tau)|\xi|/(h^{-\rho}\langle h^{-\delta}x'\rangle)$ if $(1-\tau)|\xi|/2\leqq h^{-\rho}\langle h^{-\delta}x'\rangle$. So there exists a constant $C_{\tau,h}>0$ such that

(3.4)
$$\langle\nabla_{x'}\psi_h\rangle\geqq C_{\tau,h}\langle\xi\rangle/\langle x'\rangle\,.$$

On the the other hand using the inequality

(3.5)
$$\langle\xi\rangle^2/\langle\xi-\eta\rangle^2\leqq2\langle\eta\rangle^2\,,$$

we have for some constants c, $C'_{\tau,h}>0$,

$$\begin{aligned}&\langle x\rangle^2/\langle\nabla_\xi\psi_h\rangle^2=\langle x\rangle^2/\langle x-\nabla_\xi\phi_h\rangle^2\\ &\leqq c\langle\nabla_\xi\phi_h(x',\xi)\rangle^2=c\langle\nabla_\xi J_h(x',\xi)+x'\rangle^2\\ &\leqq C'_{\tau,h}\langle x';\xi\rangle^2.\end{aligned}$$

Hence we get for some $\tilde{C}_{\tau,h}>0$

$$\langle\nabla_\xi\psi_h\rangle\geqq\tilde{C}_{\tau,h}\langle x\rangle/\langle x';\xi\rangle. \tag{3.6}$$

Now setting

$$\begin{cases}L_1=\langle\nabla_{x'}\psi_h\rangle^{-2}\{1-i\nabla_{x'}\psi_h\cdot\nabla_{x'}\}\,,\\ L_2=\langle\nabla_\xi\psi_h\rangle^{-2}\{1-i\nabla_\xi\psi_h\cdot\nabla_\xi\}\,,\end{cases}$$

we write

$$\begin{aligned}&Q_h(\phi_h^*)u(x)\\ &=\iint e^{i\psi_h}({}^tL_1)^{l_1}({}^tL_2)^{l_2}\{q_h(\xi,x')u(x')\}\,d\xi dx'\,,\end{aligned}$$

where ${}^tL_j$ $(j=1,2)$ denote the transposed operators of L_j. Then noting $u(x')\in\mathcal{S}$ and choosing large integers $l_2>n$ and $l_1>l_2+n$, we see from (3.4) and (3.6) that $Q_h(\phi_h^*)\colon\mathcal{S}\to\mathcal{S}$ is continuous.

For $P_h(\phi_h)\in \boldsymbol{B}^m_{\rho,\delta}(\phi_h)$, consider $\gamma_h(x,\xi,x')=\phi_h(x,\xi)-x'\cdot\xi$. Then $\nabla_{x'}\gamma_h=-\xi$ and $\nabla_\xi\gamma_h=\nabla_\xi\phi_h(x,\xi)-x'$. Hence noting

$$\begin{aligned}\langle\nabla_\xi\phi_h(x,\xi);\xi\rangle&\geqq c(1+|\nabla_\xi J_h(x,\xi)+x|+|\xi|)\\ &\geqq c'(1+(1-\tau)\langle x;\xi\rangle)\geqq c''\langle x\rangle\end{aligned}$$

for constants c, c', $c''>0$ and again using inequality (3.5), we obtain

$$\langle x\rangle^2/\langle\nabla_\xi\gamma_h\rangle^2\leqq C_{\tau,h}\langle x';\xi\rangle^2 \tag{3.7}$$

for some constant $C_{\tau,h}>0$. Hence we see that $P_h(\phi_h)\colon\mathcal{S}\to\mathcal{S}$ is continuous in a way similar to the proof for $Q_h(\phi_h^*)$. Q.E.D.

Corollary. *Let $\phi_h(x,\xi)\in P_{\rho,\delta}(\tau,0;h)$. Let $p_{j,h}(x,\xi)$ and $q_{j,h}(\xi,x')\in B^m_{\rho,\delta}(h)$ converge to some $p_h(x,\xi)$ and $q_h(\xi,x')\in B^m_{\rho,\delta}(h)$ as $j\to\infty$ tn $B^m_{\rho,\delta}(h)$, respectively. Then for any $u\in\mathcal{S}$, $P_{j,h}(\phi_h)u$ and $Q_{j,h}(\phi_h^*)u$ converge to $P_h(\phi_h)u$ and $Q_h(\phi_h^*)u$ in $\mathcal{S}$ as $j\to\infty$, respectively.*

Proposition 3.3. *For $\phi(x,\xi)\in P(\tau,l)$ set*

$$\begin{cases}\tilde{\nabla}_x\phi(x,\xi,x')=\int_0^1\nabla_x\phi(x'+\theta(x-x'),\xi)d\theta\\ \qquad(=\tilde{\nabla}_x\phi(x',\xi,x))\,,\\ \tilde{\nabla}_\xi\phi(\xi,x',\xi')=\int_0^1\nabla_\xi\phi(x',\xi'+\theta(\xi-\xi'))d\theta\\ \qquad(=\tilde{\nabla}_\xi\phi(\xi',x'\,\xi))\,.\end{cases} \tag{3.8}$$

Then, the inverses

$$\begin{cases} \xi = \xi(x, \eta, x') = \tilde{\nabla}_x \phi^{-1}(x, \eta, x'), \\ x' = x'(\xi, w', \xi') = \tilde{\nabla}_\xi \phi^{-1}(\xi, w', \xi') \end{cases} \tag{3.9}$$

of the mappings: $\xi \mapsto \eta = \tilde{\nabla}_x \phi(x, \xi, x')$ *and* $x' \mapsto w' = \tilde{\nabla}_\xi \phi(\xi, x', \xi')$, *respectively, exist uniquely, and satisfy*

$$\left\| \frac{\partial \xi}{\partial \eta} - I \right\|, \ \left\| \frac{\partial x'}{\partial w'} - I \right\| \leqq \frac{\tau}{1-\tau} \tag{3.10}$$

and for constants C_l, $C_{\beta,\alpha,\beta'}$, $C_{\alpha,\beta',\alpha'}$,

$$\begin{aligned} &\left| \partial_x^\beta \partial_\eta^\alpha \partial_{x'}^{\beta'} \left(\frac{\partial \xi}{\partial \eta} \right) \right| \leqq \begin{cases} C_l \tau & (|\beta+\alpha+\beta'| \leqq l), \\ C_{\beta,\alpha,\beta'} |J|_{2,\sigma} (1+|J|_{2,\sigma})^{\sigma-1}, \end{cases} \\ &\left| \partial_\xi^\alpha \partial_{w'}^{\beta'} \partial_{\xi'}^{\alpha'} \left(\frac{\partial x'}{\partial w'} \right) \right| \leqq \begin{cases} C_l \tau & (|\alpha+\beta'+\alpha'| \leqq l), \\ C_{\alpha,\beta',\beta'} |J|_{2,\sigma'} (1+|J|_{2,\sigma'})^{\sigma'-1}, \end{cases} \end{aligned} \tag{3.11}$$

where $\sigma = |\beta+\alpha+\beta'| \geqq 1$, $\sigma' = |\alpha+\beta'+\alpha'| \geqq 1$, *and* $|J|_{2,\sigma}$ *is defined by* (2.1)$'$.

Proof. Set

$$\begin{cases} \tilde{\nabla}_x J(x, \xi, x') = \int_0^1 \nabla_x J(x'+\theta(x-x'), \xi) d\theta, \\ \tilde{\nabla}_\xi J(\xi, x', \xi') = \int_0^1 \nabla_\xi J(x', \xi'+\theta(\xi-\xi')) d\theta. \end{cases} \tag{3.12}$$

According to [8] consider the mapping $\gamma = F_\eta(\xi)$: $R^n \ni \xi \mapsto \gamma \in R^n$ defined by

$$F_\eta(\xi) = \eta - \tilde{\nabla}_x J(x, \xi, x'). \tag{3.13}$$

Then we see that $\xi = \tilde{\nabla}_x \phi^{-1}(x, \eta, x')$ is determined as the fixed point of this mapping. Since $\|\vec{\nabla}_\xi \tilde{\nabla}_x J\| \leqq \tau < 1$, it is easy to see that the map F_η is contractive. Hence, we get the uniquely determined fixed point ξ of F_η satisfying $\eta = \tilde{\nabla}_x \phi(x, \xi, x')$.

Using the relation

$$\frac{\partial \eta}{\partial \xi} = I + W(\xi), \ W(\xi) = \int_0^1 \vec{\nabla}_\xi \nabla_x J(x'+\theta(x-x'), \xi) d\theta,$$

we get

$$\frac{\partial \xi}{\partial \eta} = \left(\frac{\partial \eta}{\partial \xi} \right)^{-1} = I + \sum_{k=1}^\infty (-W(\xi))^k \tag{3.14}$$

and

$$\left\| \frac{\partial \xi}{\partial \eta} - I \right\| \leqq \sum_{k=1}^\infty \tau^k = \frac{\tau}{1-\tau}.$$

Hence, we get the first part of (3.10), and similarly get the second part. To get (3.11) we differentiate the both sides of (3.14). Then, by induction we have (3.11). Q.E.D.

Theorem 3.4. *For* $\phi_h(x,\xi)\in P_{\rho,\delta}(\tau,l;h)$ *set*

$$\tilde{\phi}_h(x,\xi)=h^{\rho-\delta}\phi_h(h^{\delta}x,h^{-\rho}\xi)\ (\in P(\tau,l),\ 0<h<1) \tag{3.15}$$

and define $\tilde{\nabla}_x\phi_h$, $\tilde{\nabla}_\xi\phi_h$ *and* $\tilde{\nabla}_x\tilde{\phi}_h$, $\tilde{\nabla}_\xi\tilde{\phi}_h$, *respectively, by* (3.8). *Then, we have*

$$\begin{cases}\tilde{\nabla}_x\phi_h(x,\xi,x')=h^{-\rho}\tilde{\nabla}_x\tilde{\phi}_h(h^{-\delta}x,h^{\rho}\xi,h^{-\delta}x')\,,\\ \tilde{\nabla}_\xi\phi_h(\xi,x',\xi')=h^{\delta}\tilde{\nabla}_\xi\tilde{\phi}_h(h^{\rho}\xi,h^{-\delta}x',h^{\rho}\xi')\,,\end{cases} \tag{3.16}$$

and, the inverses

$$\begin{cases}\xi=\tilde{\nabla}_x\phi_h^{-1}(x,\eta,x'),\ \xi=\tilde{\nabla}_x\tilde{\phi}_h^{-1}(x,\eta,x')\,,\\ x'=\tilde{\nabla}_\xi\phi_h^{-1}(\xi,w',\xi'),\ x'=\tilde{\nabla}_\xi\tilde{\phi}_h^{-1}(\xi,w',\xi')\end{cases} \tag{3.17}$$

of the mappings $\eta=\tilde{\nabla}_x\phi_h(x,\xi,x')$, $\eta=\tilde{\nabla}_x\tilde{\phi}_h(x,\xi,x')$, $w'=\nabla_\xi\phi_h(\xi,x',\xi')$, $w'=\tilde{\nabla}_\xi\tilde{\phi}_h(\xi,x',\xi')$ *exist uniquely and satisfy the following*:

$$\begin{cases}\tilde{\nabla}_x\phi_h^{-1}(x,\eta,x')=h^{-\rho}\tilde{\nabla}_x\tilde{\phi}_h^{-1}(h^{-\delta}x,h^{\rho}\eta,h^{-\delta}x')\,,\\ \tilde{\nabla}_\xi\phi_h^{-1}(\xi,w',\xi')=h^{\delta}\tilde{\nabla}_\xi\tilde{\phi}_h^{-1}(h^{\rho}\xi,h^{-\delta}w',h^{\rho}\xi')\,,\end{cases} \tag{3.18}$$

$$\vec{\nabla}_\eta(\tilde{\nabla}_x\phi_h^{-1})(x,\eta,x'),\ \vec{\nabla}_{w'}(\tilde{\nabla}_\xi\phi_h^{-1})(\xi,w',\xi')\in B^0_{\rho,\delta}(h)\,, \tag{3.19}$$

$$\left\|\frac{\partial(\tilde{\nabla}_x\phi_h^{-1})}{\partial\eta}-I\right\|,\ \left\|\frac{\partial(\tilde{\nabla}_\xi\phi_h^{-1})}{\partial w'}-I\right\|\leqq\frac{\tau}{1-\tau}\,. \tag{3.20}$$

REMARK. Since $\tilde{\phi}_h(x,\xi)\in P(\tau,l)$ $(0<h<1)$, we see from (3.16) that

$$\vec{\nabla}_\xi\tilde{\nabla}_x\phi_h(x,\xi,x'),\ \vec{\nabla}_{x'}\tilde{\nabla}_\xi\phi_h(\xi,x',\xi)\in B^0_{\rho,\delta}(h)\,. \tag{3.21}$$

Proof. (3.16) is clear. The existence of $\tilde{\nabla}_x\phi_h^{-1}$, $\tilde{\nabla}_\xi\phi_h^{-1}$ and the relation (3.18) are clear from (3.16). Since $\tilde{\phi}_h(x,\xi)\in P(\tau,l)$ $(0<h<1)$, we can apply proposition 3.3 to $\phi_h(x,\xi)$. Then we have (3.10) for $\tilde{\nabla}_x\tilde{\phi}_h^{-1}$ and $\tilde{\nabla}_\xi\tilde{\phi}_h^{-1}$. Thus (3.20) follows from (3.18) and (3.10). Moreover, since $\{J_{h(\beta)}^{(\alpha)}(x,\xi)\}_{0<h<1}(|\alpha+\beta|=2)$ is bounded in $\mathscr{B}(R^{2n})$ by the definition of $\{P_{\rho,\delta}(\tau,l;h)\}_{0<h<1}$, we have (3.11) for $\tilde{\nabla}_x\tilde{\phi}_h^{-1}$ and $\tilde{\nabla}_\xi\tilde{\phi}_h^{-1}$ for constants independent of $0<h<1$. Then (3.19) follows from (3.18) and (3.11). Q.E.D.

Under these preparations, we begin to study the calculus of Fourier integral operators.

Therem 3.5. *Let* $P_h(\phi_h)=p_h(\phi_h;X,D_x)\in \boldsymbol{B}^m_{\rho,\delta}(\phi_h)$ *and* $Q_h(\phi_h^*)=q_h(\phi_h^*;D_x,X')\in \boldsymbol{B}^{m'}_{\rho,\delta}(\phi_h^*)$ *for* $\phi_h(x,\xi)\in P_{\rho,\delta}(\tau,0;h)$. *Then we have the following*:

i) *Setting*

$$\begin{aligned}(3.22)\qquad & s_h(x,\xi,x') \\ &= p_h(x,\tilde{\nabla}_x\phi_h^{-1}(x,\xi,x'))q_h(\tilde{\nabla}_x\phi_h^{-1}(x,\xi,x'),x') \\ &\times\left|\frac{D(\tilde{\nabla}_x\phi_h^{-1})}{D(\xi)}(x,\xi,x')\right| \quad (\in B^{m+m'}_{\rho,\delta}(h)),\end{aligned}$$

we define $r_h(x,\xi)$ *by*

$$(3.23)\qquad r_h(x,\xi') = \mathrm{O_s}\text{-}\iint e^{-iy\cdot\eta}s_h(x,\xi'+\eta,x+y)đ\eta dy\,.$$

Then, we have $r_h(x,\xi)\in B^{m+m'}_{\rho,\delta}(h)$ *and*

$$(3.24)\qquad P_h(\phi_h)Q_h(\phi_h^*) = r_h(X,D_x)\,.$$

Moreover, we have the estimate

$$(3.25)\qquad |r_h|_l^{(m+m')}\leqq C_l\exp(|J_h|_{2,l+2n_0})|p_h|^{(m)}_{l+2n_0}|q_h|^{(m')}_{l+2n_0}\quad (n_0>n,\ \text{even}),$$

where $|J_h|_{2,l}$ *is defined by* (2.1)′.

ii) *Setting*

$$\begin{aligned}(3.26)\qquad & s_h(\xi,x',\xi') \\ &= q_h(\xi,\tilde{\nabla}_\xi\phi_h^{-1}(\xi,x',\xi'))p_h(\tilde{\nabla}_\xi\phi_h^{-1}(\xi,x',\xi'),\xi') \\ &\times\left|\frac{D(\tilde{\nabla}_\xi\phi_h^{-1})}{D(x')}(\xi,x',\xi')\right| \quad (\in B^{m+m'}_{\rho,\delta}(h)),\end{aligned}$$

we define $r_h(x,\xi)$ *by*

$$(3.27)\qquad r_h(x,\xi') = \mathrm{O_s}\text{-}\iint e^{-iy\cdot\eta}s_h(\xi'+\eta,x+y,\xi')đ\eta dy\,.$$

Then, we have $r_h(x,\xi)\in B^{m+m'}_{\rho,\delta}(h)$ *and*

$$(3.28)\qquad Q_h(\phi_h^*)P_h(\phi_h) = r_h(X,D_x)\,.$$

Moreover, we have the estimate

$$(3.29)\qquad |r_h|_l^{(m+m')}\leqq C_l\exp(|J_h|_{2,l+2n_0})|p_h|^{(m)}_{l+2n_0}|q_h|^{(m')}_{l+2n_0}\,.$$

REMARK. Strictly speaking, in (3.25) and (3.29) we can replace $\exp(|J_h|_{2,l+2n_0})$ by $(1+|J_h|_{2,l+2n_0})^{l'}$ for some l'. The situation will be the same in all the statements in what follows.

Proof. i) From (3.1)′, (3.2)′ and Proposition 3.2, we write for $u\in\mathcal{S}$

$$\begin{aligned}K_hu(x)&\equiv P_h(\phi_h)Q_h(\phi_h^*)u(x)\\ &= O_s\text{-}\iint e^{i(\phi_h(x,\zeta)-\phi_h(x',\zeta))}p_h(x,\zeta)q_h(\zeta,x')u(x')đ\zeta dx'\,.\end{aligned}$$

Using $\phi_h(x',\zeta)-\phi_h(x,\zeta)=(x-x')\cdot\tilde{\nabla}_x\phi_h(x,\zeta,x')$, by Theorem 3.4 we make the change of variable $\xi=\tilde{\nabla}_x\phi_h(x,\zeta,x')$. Then, we have

$$(3.30)\qquad \begin{aligned} &K_h u(x)\\ &= \mathrm{O}_s-\iint e^{i(x-x')\cdot\xi} s_h(x,\xi,x')u(x')d\xi dx'\,. \end{aligned}$$

Now set

$$(3.31)\qquad \begin{cases} \gamma_h(x,\zeta,x') = p_h(x,\zeta)q_h(\zeta,x')\,,\\ \tilde{\gamma}_h(x,\zeta,x') = \gamma_h(h^\delta x, h^{-\rho}\zeta, h^\delta x')\,,\\ \tilde{s}_h(x,\xi,x') = s_h(h^\delta x, h^{-\rho}\xi, h^\delta x')\,. \end{cases}$$

Then, by the definition we see that

$$\tilde{\gamma}_h(x,\zeta,x')\in B^{m+m'}_{0,0}(h)\,,$$

and, noting (3.18), see that

$$(3.32)\qquad \begin{aligned} \tilde{s}_h(x,\xi,x') &= \tilde{\gamma}_h(x, \tilde{\nabla}_x\tilde{\phi}_h^{-1}(x,\xi,x'), x')\\ &\times\left|\frac{D(\tilde{\nabla}_x\tilde{\phi}_h^{-1})}{D(\xi)}(x,\xi,x')\right| \quad (\in B^{m+m'}_{0,0}(h))\,. \end{aligned}$$

Hence, noting (3.11) of Proposition 3.3, we have

$$(3.33)\qquad |\tilde{s}_h|_l^{(m+m')}\leqq C_l(1+|\tilde{J}_h|_{2,l})^{l(l+1)}|p_h|_l^{(m)}|q_h|_l^{(m')}\,.$$

On the other hand by Theorem 1.4 we see that $r_h(x,\xi)=s_{h,L}(x,\xi)$. So we have (3.24), and by (1.23) have

$$(3.34)\qquad |r_h|_l^{(m+m')}\leqq C_l|s_h|_{l+2n_0}^{(m+m')}\ (n_0>n,\ \text{even}).$$

Hence, noting $|s_h|_l^{(m+m')}$ (in $B^{m+m'}_{\rho,\delta}(h)$)$=|\tilde{s}_h|_l^{(m+m')}$ (in $B^{m+m'}_{0,0}(h)$) and $|\tilde{J}_h|_{2,l}=|J_h|_{2,l}$, from (3.33) and (3.34) we have (3.25).

ii) We write for $u\in\mathcal{S}$

$$\begin{aligned} \widehat{K_h u(\xi)} &= \widehat{Q_h(\phi_h^*)P_h(\phi_h)u(\xi)}\\ &= \mathrm{O}_s-\iint e^{-i(\phi_h(z',\xi)-\phi_h(z',\xi'))}q_h(\xi,z')p_h(z',\xi')\\ &\qquad\times\hat{u}(\xi')d\xi'dz'\,. \end{aligned}$$

Using $\phi_h(z,\xi)-\phi_h(z,\xi')=(\xi-\xi')\cdot\tilde{\nabla}_\xi\phi_h(\xi,z,\xi')$, we make a change of variable $x'=\tilde{\nabla}_\xi\phi_h(\xi,z,\xi')$. Then, we can prove ii) in a way similar to i). Q.E.D.

Theorem 3.6. *Let $P_h(\phi_h)=p_h(\phi_h;X,D_x)\in \boldsymbol{B}^m_{\rho,\delta}(\phi_h)$ and $Q_h(\phi_h^*)=q_h(\phi_h^*;D_x,X')\in\boldsymbol{B}^m_{\rho,\delta}(\phi_h^*)$ for $\phi_h(x,\xi)\in P_{\rho,\delta}(\tau,l;h)$. Then, $P_h(\phi_h)$, $Q_h(\phi_h^*):L^2(R^n)\to L^2(R^n)$ are continuous and we have for $u\in\mathcal{S}$*

$$(3.35)\qquad \begin{cases} \|P_h(\phi_h)u\|_{L^2}\leqq C\exp(|J_h|_{2,M+2n_0})h^m|p_h|_{M+2n_0}^{(m)}\|u\|_{L^2}\\ \|Q_h(\phi_h^*)u\|_{L^2}\leqq C\exp(|J_h|_{2,M+2n_0})h^m|q_h|_{M+2n_0}^{(m)}\|u\|_{L^2}\,, \end{cases}$$

where $M=2\left(\left[\frac{n}{2}\right]+\left[\frac{5n}{4}\right]+2\right)$.

Proof. Since $||P_h(\phi_h)u||_{L^2}^2 \leqq ||P_h(\phi_h)^* P_h(\phi_h)u||_{L^2}||u||_{L^2}$ and $||Q_h(\phi_h^*)u||_{L^2}^2 \leqq ||Q_h(\phi_h^*)^* Q_h(\phi_h)u||_{L^2}||u||_{L^2}$ for $u\in\mathcal{S}$, noting the remark of Proposition 3.2 we get (3.35) from Theorem 1.12 and Theorem 3.5. Q.E.D.

Theorem 3.7. *Let* $P_h=p_h(X,D_x)\in \boldsymbol{B}^m_{\rho,\delta}(h)$ *and* $Q_h(\phi_h)=q_h(\phi_h;X,D_x)\in \boldsymbol{B}^{m'}_{\rho,\delta}(\phi_h)$ *for* $\phi_h(x,\xi)\in P_{\rho,\delta}(\tau,0;h)$. *Then, we have the following*:

i) *Setting*

$$(3.36)\qquad \begin{aligned} &s_h(x,\xi,x',\xi') \\ &= p_h(x,\xi+\tilde{\nabla}_x J_h(x,\xi',x'))q_h(x',\xi') \quad (\in B^{m+m'}_{\rho,\delta}(h)), \end{aligned}$$

we define $r_h(x,\xi)$ *by*

$$(3.37)\qquad r_h(x,\xi') = \mathrm{O_s}-\iint e^{-iy\cdot\eta}s_h(x,\xi'+\eta,x+y,\xi')d\!\!\!{}^-\eta dy\,.$$

Then $r_h(x,\xi)\in B^{m+m'}_{\rho,\delta}(h)$ *and* $R_h(\phi_h)\equiv r_h(\phi_h;X,D_x)=P_hQ_h(\phi_h)$. *Moreover, we have*

$$(3.38)\qquad |r_h|_l^{(m+m')}\leqq C_l \exp(|J_h|_{2,l+2n_0-1})|p_h|^{(m)}_{l+2n_0}|q_h|^{(m')}_{l+2n_0}\ (n_0>n,\ \text{even}).$$

In the case: $0\leqq\delta<\rho\leqq 1$ *we have the asymptotic expansion formula*

$$(3.39)\qquad \begin{aligned} &r_h(x,\xi') \\ &\sim \sum_\alpha \frac{1}{\alpha!}D^\alpha_{x'}\{p_h^{(\alpha)}(x,\tilde{\nabla}_x\phi_h(x,\xi',x'))q_h(x',\xi')\}|_{x'=x}\,. \end{aligned}$$

ii) *Setting*

$$(3.40)\qquad \begin{aligned} &s_h(x,\xi,x',\xi') \\ &= q_h(x,\xi)p_h(x'+\tilde{\nabla}_\xi J_h(\xi,x,\xi'),\xi') \quad (\in B^{m+m'}_{\rho,\delta}(h)), \end{aligned}$$

we define $r_h(x,\xi)$ *by*

$$(3.41)\qquad r_h(x,\xi') = \mathrm{O_s}-\iint e^{-iy\cdot\eta}s_h(x,\xi'+\eta,x+y,\xi')d\!\!\!{}^-\eta dy\,.$$

Then, we have $r_h(x,\xi)\in B^{m+m'}_{\rho,\delta}(h)$, $R_h(\phi_h)\equiv r_h(\phi_h;X,D_x)=Q_h(\phi_h)P_h$, *and the estimate of the form* (3.38).

In the case: $0\leqq\delta<\rho\leqq 1$ *we have the expansion formula*

$$(3.42)\qquad \begin{aligned} &r_h(x,\xi') \\ &\sim \sum_\alpha \frac{1}{\alpha!}\partial^\alpha_\xi\{q_h(x,\xi)p_{h(\alpha)}(\tilde{\nabla}_\xi\phi_h(\xi,x,\xi'),\xi')\}|_{\xi=\xi'}\,. \end{aligned}$$

REMARK. For $P_h=p_h(D_x,X')\in \boldsymbol{B}^m_{\rho,\delta}(h)$ and $Q_h(\phi_h^*)=q_h(\phi_h^*;D_x,X')\in \boldsymbol{B}^{m'}_{\rho,\delta}(\phi_h^*)$, consider $(P_hQ_h(\phi_h^*))^*$ and $(Q_h(\phi_h^*)P_h)^*$. Then, from Theorem 3.7 we have a similar theorem for $P_hQ_h(\phi_h^*)$ and $Q_h(\phi_h^*)P_h$.

Proof. i) We have formally for $u\in\mathcal{S}$

$$P_hQ_h(\phi_h)u(x)$$
$$=\int e^{i\phi_h(x,\xi')}\{\iint e^{i\psi}p_h(x,\xi)q_h(x',\xi')d\xi dx'\}\hat{u}(\xi')d\xi',$$

where

$$\psi=\phi_h(x',\xi')-\phi_h(x,\xi')+(x-x')\cdot\xi$$
$$=-(x'-x)\cdot(\xi-\xi'-\tilde{\nabla}_xJ(x,\xi',x'))\,.$$

Then, by the change of variable $\tilde{\xi}=\xi-\tilde{\nabla}_xJ(x,\xi',x')$, we see that for

$$r_h(x,\xi)=\mathrm{O}_s-\iint e^{-i(x'-x)\cdot(\tilde{\xi}-\xi')}s_h(x,\tilde{\xi},x',\xi')d\tilde{\xi}dx'$$

we have $R_h(\phi_h)=P_hQ_h(\phi_h)$. Again by the change of variable: $x'-x=y$, $\tilde{\xi}-\xi'=\eta$ we get (3.37).

Now set

$$\tilde{s}_h(x,\xi,x',\xi')=s_h(h^\delta x,h^{-\rho}\xi,h^\delta x',h^{-\rho}\xi')\,.$$

Then using (3.16) we have

$$\tilde{s}_h(x,\xi,x',\xi')$$
$$=p_h(h^\delta x,h^{-\rho}(\xi+\tilde{\nabla}_x\tilde{J}_h(x,\xi',x')))q_h(h^\delta x',h^{-\rho}\xi) \tag{3.43}$$
$$\in B_{0,0}^{m+m'}(h)\,.$$

So we see that $s_h(x,\xi,x',\xi')\in B_{\rho,\delta}^{m+m'}(h)$. Since $r_h(x,\xi)=s_{h,L}(x,\xi)$ in Theorem 1.4, we see that $r_h(x,\xi)\in B_{\rho,\delta}^{m+m'}(h)$ and satisfies (3.38).

In the case $0\leqq\delta<\rho\leqq 1$, again by Theorem 1.4,

$$r_h(x,\xi')\sim\sum_\alpha\frac{1}{\alpha!}s_{h(0,\alpha)}^{(\alpha,0)}(x,\xi',x,\xi')$$
$$\sim\sum_\alpha\frac{1}{\alpha!}D_{x'}^\alpha\{p_h^{(\alpha)}(x,\xi'+\tilde{\nabla}_xJ_h(x,\xi,x'))q_h(x',\xi')\}_{|x'=x}\,.$$

Then noting $\xi'+\tilde{\nabla}_xJ_h(x,\xi',x')=\tilde{\nabla}_x\phi_h(x,\xi',x')$, we get (3.39). Similarly we can prove ii). Q.E.D.

Theorem 3.8. *Let $\tilde{l}=5n_0$ with an even $n_0>n$ and take a small $0<\tilde{\tau}<1$. Then, for any $\phi_h(x,\xi)\in P_{\rho,\delta}(\tilde{\tau},\tilde{l};h)$ we can find $q_h(\xi,x')$ and $r_h(x,\xi)\in B_{\rho,\delta}^0(h)$ such that for $Q_h(\phi_h^*)=q_h(\phi_h^*;D_x,X')$ and $R_h(\phi_h)=r_h(\phi_h;X,D_x)$ we have*

$$\begin{cases}\text{i)}\;\; I(\phi_h)Q_h(\phi_h^*)=Q_h(\phi_h^*)I(\phi_h)=I\,,\\ \text{ii)}\;\; I(\phi_h^*)R_h(\phi_h)=R_h(\phi_h)I(\phi_h^*)=I\,,\end{cases} \tag{3.44}$$

where $I(\phi_h)$ *and* $I(\phi_h^*)$ *denote the Fourier, and conjugate Fourier, integral operators with symbols* 1, *respectively. Moreover we have*

$$|q_h|_l^{(0)},\ |r_h|_l^{(0)} \leqq C_l \exp(2|J_h|_{2,l+7n_0})\,. \tag{3.45}$$

Proof. By Theorem 3.5 if we set

$$\begin{aligned} &s_h(x,\xi') \\ &= \mathrm{O}_s\text{-}\iint e^{-iy\cdot\eta}\left|\frac{D(\tilde{\nabla}_x\phi_h^{-1})}{D(\xi)}(x,\xi'+\eta,x+y)\right| d\eta dy\,, \end{aligned}$$

then we have

$$I(\phi_h)I(\phi_h^*) = s_h(X, D_x)\,.$$

Define $s_{0,h}(x,\xi)$ *by*

$$\begin{aligned} s_{0,h}(x,\xi') &= s_h(x,\xi')-1 \\ &= \mathrm{O}_s\text{-}\iint e^{-iy\cdot\eta}\left\{\left|\frac{D(\tilde{\nabla}_x\phi_h^{-1})}{D(\xi)}(x,\xi'+\eta,x+y)\right|-1\right\} d\eta dy\,. \end{aligned} \tag{3.46}$$

Then we have for $S_{0,h}=s_{0,h}(X,D_x)$

$$I(\phi_h)I(\phi_h^*) = I+S_{0,h}\,. \tag{3.47}$$

Since

$$t_{0,h}(x,\xi,x') = \left|\frac{D(\tilde{\nabla}_x\phi_h^{-1})}{D(\xi)}(x,\xi,x')\right|-1 \in B^0_{\rho,\delta}(h)\,, \tag{3.48}$$

by Theorem 1.4 we have for a constant $C_l>0$

$$|s_{0,h}|_l^{(0)} \leqq C_l |t_{0,h}|_{l+2n_0}^{(0)} \quad ((n_0>n,\ \text{even})). \tag{3.49}$$

Hence, by Theorem 1.9, if $C_{3n_0}|t_{0,h}|_{5n_0}^{(0)} \leqq c_0$ for a constant c_0 of Theorem 1.9, the inverse $(I+S_{0,h})^{-1}$ exists in $\boldsymbol{B}^0_{\rho,\delta}(h)$. Then, setting

$$Q_h(\phi_h^*) = I(\phi_h^*)\,(I+S_{0,h})^{-1}\,, \tag{3.50}$$

we get the required equality (i) of (3.44).

Since

$$\sigma((I+S_{0,h})^{-1}) = 1+\sum_{\nu=1}^{\infty}(-1)^\nu\sigma(S_{0,h}^\nu)\,, \tag{3.51}$$

by Theorem 1.8 we have, using the constant C_0 of Theorem 1.8

$$\begin{aligned} &|\sigma(S_{0,h}^{\nu+1})|_l^{(0)} \\ &\leqq C_0^{\nu+1} \sum_{l_1+\cdots+l_{\nu+1}\leqq l}\ \prod_{j=1}^{\nu+1} |s_{0,h}|_{3n_0+l_j}^{(0)}\,. \end{aligned} \tag{3.52}$$

Hence, we have for a constant M_l

$$|\sigma(S_{0,h}^{\nu+1})|_l^{(0)} \leqq M_l C_0^{\nu+1}(|s_{0,h}|_{3n_0+l}^{(0)})^{\nu+1} \tag{3.53}$$

when $\nu+1 \leqq l$, and

$$\begin{aligned} |\sigma(S_{0,h}^{\nu+1})|_l^{(0)} \leqq M_l \nu^l C_0^{\nu+1}(|s_{0,h}|_{3n_0}^{(0)})^{\nu+1-l} \\ \times (|s_{0,h}|_{3n_0+l}^{(0)})^l \end{aligned} \tag{3.54}$$

when $\nu+1 > l$. From (3.11) and (3.48) we see that for a constant $M_l' > 0$

$$|t_{0,h}|_l^{(0)} \leqq M_l' |J_h|_{2,l}(1+|J_h|_{2,l})^{ln} . \tag{3.55}$$

Hence, by (3.49) we have

$$|s_{0,h}|_{3n_0}^{(0)} \leqq C_{3n_0} M_{5n_0}' |J_h|_{2,5n_0} 2^{5n_0 \cdot n} \tag{3.56}$$

if $|J_h|_{2,5n_0} \leqq 1$. Hence, if we set $\tilde{l} = 5n_0$ and choose $0 < \tilde{\tau} < 1$ such that

$$C_0 C_{3n_0} M_{5n_0}' \tilde{\tau} 2^{5n_0 \cdot n} \leqq 1/2 ,$$

then from (3.49), (3.51)-(3.56) we see that

$$|\sigma((I+S_{0,h})^{-1})|_l^{(0)} \leqq \tilde{M}_l \exp(|J_h|_{2,l+5n_0}) . \tag{3.57}$$

Finally, applying Theorem 3.7 to (3.50), we get (3.45) for q_h, and similary get (3.45) for r_h. Q.E.D.

4. Multi-products of Fourier integral operators

The following theorem is the basic one for the calculus of Fourier integral operators.

Theorem 4.1. *Let $\phi_{j,h}(x,\xi) \in P_{\rho,\delta}(\tau_j, 0; h)$, $j=1,2$, with $\tau_1+\tau_2 \leqq 1/4$, and define $q_h(x,\xi)$ by*

$$q_h(x, \xi') = \mathrm{Os}-\iint e^{i\psi_h(x,\xi,x',\xi')} đ\xi dx' , \tag{4.1}$$

where

$$\begin{aligned} &\psi_h(x, \xi, x', \xi') \\ &= \phi_{1,h}(x, \xi) - x' \cdot \xi + \phi_{2,h}(x', \xi') - (\phi_{1,h} \# \phi_{2,h})(x, \xi') . \end{aligned} \tag{4.2}$$

Then, $q_h(x,\xi) \in B_{\rho,\delta}^0(h)$, and for $Q_h(\phi_{1,h} \# \phi_{2,h}) = q_h(\phi_{1,h} \# \phi_{2,h}; X, D_x)$ we have

$$I(\phi_{1,h}) I(\phi_{2,h}) = Q_h(\phi_{1,h} \# \phi_{2,h}) . \tag{4.3}$$

Moreover, there exist constants $C_l > 0$ such that

$$|q_h|_l^{(0)} \leqq C_l \exp\left(\sum_{j=1}^{2} |J_{j,h}|_{2,2l+2n+1}\right) \tag{4.4}$$

and

$$(4.4)' \qquad |q_h-1|_l^{(0)}\leqq C_l\Big(\sum_{j=1}^{2}|J_{j,h}|_{2,2l+2n+2}\Big)\exp\Big(\sum_{j=1}^{2}|J_{j,h}|_{2,2l+2n+2}\Big).$$

Proof. I) Set $\Phi_h(x,\xi)=(\phi_{1,h}\#\phi_{2,h})(x,\xi)$. Since we can write formally for $u\in\mathcal{S}$

$$I(\phi_{1,h})I(\phi_{2,h})u(x) = \int e^{i\Phi_h(x,\xi')}\Big\{\iint e^{i\psi_h(x,\xi,x',\xi')}d\!\!{}^-\xi dx'\Big\}\hat{u}(\xi')d\!\!{}^-\xi',$$

we get (4.3) by limit process if we show $q_h(x,\xi)\in B^0_{\rho,\delta}(h)$. So, setting

$$(4.5) \qquad \tilde{q}_h(x,\xi)=q_h(h^\delta x, h^{-\rho}\xi),$$

we shall show that

$$(4.6) \qquad \tilde{q}_h(x,\xi)\in B^0_{0,0}(h).$$

This will be done through several steps.

Noting (2.39) of Theorem 2.8, set

$$(4.7) \qquad \begin{cases}\tilde{\phi}_{j,h}(v,\xi)=h^{\rho-\delta}\phi_{j,h}(h^\delta x, h^{-\rho}\xi), \quad j=1,2,\\ \tilde{\Phi}_h(x,\xi)=h^{\rho-\delta}\Phi_h(h^\delta x, h^{-\rho}\xi),\\ \tilde{\psi}_h(x,\xi,x',\xi')=h^{\rho-\delta}\psi_h(h^\delta x, h^{-\rho}\xi, h^\delta x', h^{-\rho}\xi').\end{cases}$$

Then, from the definition of $\tilde{\phi}_{j,h}$, Theorem 2.8 and Theorem 2.9, we have

$$(4.8) \qquad \begin{cases}\text{i)} \ \ \tilde{\phi}_{j,h}(x,\xi)\in P_{0,0}(\tau_j, 0;h), \quad j=1,2,\\ \text{ii)} \ \ \tilde{\Phi}_h(x,\xi)\in P_{0,0}(c_0\bar{\tau}, 0;h)\\ \qquad \text{with some } c_0\geqq 1 \text{ and } \bar{\tau}=\tau_1+\tau_2.\end{cases}$$

We also have

$$(4.9) \qquad \tilde{q}_h(x,\xi') = h^{-2n\sigma}\mathrm{O}_s\text{-}\iint e^{ih^{-2\sigma}\tilde{\psi}_h(x,\xi,x',\xi')}d\!\!{}^-\xi dx',$$

where $\sigma=(\rho-\delta)/2$.

II) Let $\{\tilde{X}_h,\tilde{\Xi}_h\}(x,\xi)$ be the solution of

$$(4.10) \qquad \tilde{X}_h=\nabla_\xi\tilde{\phi}_{1,h}(x,\tilde{\Xi}_h),\ \tilde{\Xi}_h=\nabla_x\tilde{\phi}_{2,h}(\tilde{X}_h,\xi).$$

Then we have

$$(4.11) \qquad \tilde{\phi}_h(x,\xi)=\tilde{\phi}_{1,h}(x,\tilde{\Xi}_h)-\tilde{X}_h\cdot\tilde{\Xi}_h+\tilde{\phi}_{2,h}(\tilde{X}_h,\xi)$$

and for $|\alpha+\beta|\geqq 1$

$$(4.12)\qquad |\partial_\xi^\alpha \partial_x^\beta(\tilde{X}_h - x, \tilde{\Xi}_h - \xi)| \leqq C_{\alpha,\beta}(|J_1|_{2,|\alpha+\beta|-1} + |J_2|_{2,|\alpha+\beta|-1})^{|\alpha+\beta|},$$

which is proved by induction by using (4.10).

Now we make a change of variables: $x' = \tilde{X}_h(x, \xi') + y$, $\xi = \tilde{\Xi}_h(x, \xi') + \eta$. Then, setting

$$(4.13)\qquad \begin{aligned} \tilde{\varphi}_h(y, \eta; x, \xi) = \tilde{\phi}_{1,h}(x, \tilde{\Xi}_h(x, \xi) + \eta) - (\tilde{X}_h(x, \xi) + y)\cdot(\tilde{\Xi}_h(x, \xi) + \eta) \\ + \tilde{\phi}_{2,h}(\tilde{X}_h(x, \xi) + y, \xi) - \tilde{\Phi}_h(x, \xi), \end{aligned}$$

we can write

$$(4.14)\qquad \tilde{q}_h(x, \xi) = h^{-2\sigma n} \mathrm{O_s}-\iint e^{ih^{-2\sigma}\tilde{\varphi}_h(y,\eta;x,\xi)} đ\eta dy.$$

From (4.13) we have

$$(4.15)\qquad \begin{cases} \nabla_y \tilde{\varphi}_h = -(\tilde{\Xi}_h + \eta) + \nabla_x \tilde{\phi}_{2,h}(\tilde{X}_h + y, \xi), \\ \nabla_\eta \tilde{\varphi}_h = -(\tilde{X}_h + y) + \nabla_\xi \tilde{\phi}_{1,h}(x, \tilde{\Xi}_h + \eta). \end{cases}$$

Hence, using (4.10) we have

$$(4.16)\qquad \begin{cases} \nabla_y \tilde{\varphi}_h = -\eta + \vec{\tilde{\nabla}}_x \nabla_x \tilde{J}_{2,h}(\tilde{X}_h, \xi, \tilde{X}_h + y) y, \\ \nabla_\eta \tilde{\varphi}_h = -y + \vec{\tilde{\nabla}}_\xi \nabla_\xi \tilde{J}_{1,h}(\tilde{\Xi}_h, x, \tilde{\Xi}_h + \eta)\eta, \end{cases}$$

where

$$(4.17)\qquad \begin{cases} \tilde{\nabla}_x \tilde{J}_{2,h}(x, \xi, x') = \int_0^1 \nabla_x \tilde{J}_{2,h}(x + \theta(x' - x), \xi) d\theta, \\ \tilde{\nabla}_\xi \tilde{J}_{1,h}(\xi, x, \xi') = \int_0^1 \nabla_\xi \tilde{J}_{1,h}(x, \xi + \theta(\xi' - \xi)) d\theta. \end{cases}$$

So from (4.16) and (4.12) we have

$$(4.18)\qquad (1 - \tau)(|y| + |\eta|) \leqq |\nabla_y \tilde{\varphi}_h| + |\nabla_\eta \tilde{\varphi}_h| \leqq (1 + \tau)(|y| + |\eta|),$$

and

$$(4.19)\qquad \begin{cases} \text{i)}\quad |\partial_\xi^\alpha \partial_x^\beta(\nabla_y \tilde{\varphi}_h, \nabla_\eta \tilde{\varphi}_h)| \leqq C_{\alpha,\beta}(1 + \sum_{k=1}^2 |J_{k,h}|_{2,|\alpha+\beta|})^{|\alpha+\beta|+1}(|y| + |\eta|), \\ \text{ii)}\quad |\partial_\xi^\alpha \partial_x^\beta \partial_\eta^{\alpha'} \partial_y^{\beta'}(\nabla_y \tilde{\varphi}_h, \nabla_\eta \tilde{\varphi}_h)| \\ \qquad \leqq C_{\alpha,\beta,\alpha',\beta'}(1 + \sum_{k=1}^2 |J_{k,h}|_{2,|\alpha+\beta+\alpha'+\beta'|})^{|\alpha+\beta|+1}\langle y; \eta\rangle \\ \qquad\qquad (|\alpha' + \beta'| \geqq 1). \end{cases}$$

On the other hand, using (4.10), (4.11) and (4.13) we can write

$$(4.20)\qquad \begin{aligned} \tilde{\varphi}_h = -y\cdot\eta + (\tilde{\nabla}_x \tilde{J}_{2,h}(\tilde{X}_h, \xi, \tilde{X}_h + y) - \nabla_x \tilde{J}_{2,h}(\tilde{X}_h, \xi))y \\ + (\tilde{\nabla}_\xi \tilde{J}_{1,h}(\tilde{\Xi}_h, x, \tilde{\Xi}_h + \eta) - \nabla_\xi \tilde{J}_{1,h}(x, \tilde{\Xi}_h))\eta, \end{aligned}$$

and from this we can write

$$(4.21)\quad \tilde{\varphi}_h = -y\cdot\eta + \tilde{\nabla}_x^2\tilde{J}_{2,h}(\tilde{X}_h, \xi, \tilde{X}_h+y)y\cdot y + \tilde{\nabla}_\xi^2\tilde{J}_{1,h}(\tilde{\Xi}_h, x, \tilde{\Xi}_h+\eta)\eta\cdot\eta\,,$$

where

$$(4.22)\quad \begin{cases} \tilde{\nabla}_x^2\tilde{J}_{2,h}(x,\xi,x') = \int_0^1 (1-\theta)\vec{\nabla}_x\nabla_x J_{2,h}(x+\theta(x'-x),\xi)d\theta\,, \\ \tilde{\nabla}_\xi^2\tilde{J}_{1,h}(\xi,x',\xi') = \int_0^1 (1-\theta)\vec{\nabla}_\xi\nabla_\xi J_{1,h}(x,\xi+\theta(\xi'-\xi))d\theta\,. \end{cases}$$

Then, from (4.20), (4.12) we have

$$(4.23)\quad \begin{cases} \text{i)}\quad |\partial_\xi^\alpha\partial_x^\beta\tilde{\varphi}_h| \leqq C_{\alpha,\beta}(1+\sum_{k=1}^2 |J_{k,h}|_{2,|\alpha+\beta|-1})^{|\alpha+\beta|+1}(|y|+|\eta|) \\ \qquad\qquad (|\alpha+\beta|\geqq 1)\,, \\ \text{ii)}\quad |\partial_\xi^\alpha\partial_x^\beta\partial_\eta^{\alpha'}\partial_y^{\beta'}\tilde{\varphi}_h| \\ \qquad \leqq C_{\alpha,\beta\,\alpha',\beta'}(1+\sum_{k=1}^2 |J_{k,h}|_{2,|\alpha+\beta+\alpha'+\beta'|-1})^{|\alpha+\beta|+1}\langle y;\eta\rangle \\ \qquad\qquad (|\alpha+\beta|\geqq 1,\ |\alpha'+\beta'|\geqq 1)\,. \end{cases}$$

From (4.21), (4.12) we have

$$(4.24)\quad \begin{cases} \text{i)}\quad |\partial_\xi^\alpha\partial_x^\beta\tilde{\varphi}_h| \leqq C_{\alpha,\beta}(1+\sum_{k=1}^2 |J_{k,h}|_{2,|\alpha+\beta|})^{|\alpha+\beta|+1}(|y|+|\eta|)^2 \\ \qquad\qquad (|\alpha+\beta|\geqq 1)\,, \\ \text{ii)}\quad |\partial_\xi^\alpha\partial_x^\beta(\nabla_y\tilde{\varphi}_h, \nabla_\eta\tilde{\varphi}_h)| \\ \qquad \leqq C_{\alpha,\beta}(1+\sum_{k=1}^2 |J_{k,h}|_{2,1+|\alpha+\beta|})^{|\alpha+\beta|+1}(|y|+|\eta|)\langle y;\eta\rangle \\ \qquad\qquad (|\alpha+\beta|\geqq 1)\,, \\ \text{iii)}\quad |\partial_\xi^\alpha\partial_x^\beta\partial_\eta^{\alpha'}\partial_y^{\beta'}\tilde{\varphi}_h| \\ \qquad \leqq C_{\alpha,\beta,\alpha',\beta'}(1+\sum_{k=1}^2 |J_{k,h}|_{2,|\alpha+\beta+\alpha'+\beta'|})^{|\alpha+\beta|+1}\langle y;\eta\rangle^2 \\ \qquad\qquad (|\alpha+\beta|\geqq 1,\ |\alpha'+\beta'|\geqq 2)\,. \end{cases}$$

III) Let $\chi_0(y,\eta)$ be a C_0^∞-function in R^{2n} such that

$$(4.25)\quad \begin{cases} 0\leqq \chi_0(y,\eta)\leqq 1 \quad \text{on } R^{2n}\,, \\ \chi_0(y,\eta) = 1\ (|y|+|\eta|\leqq 1/2),\ = 0\ (|y|+|\eta|\geqq 1)\,. \end{cases}$$

Set

$$(4.26)\quad \tilde{q}_{0,h}(x,\xi) = h^{-2n\sigma}\iint e^{ih^{-2\sigma}\tilde{\varphi}_h}\chi_0 d\eta dy\,,$$

and, letting $\chi_\infty(y,\eta) = 1-\chi_0(y,\eta)$, set

$$(4.27)\quad \tilde{q}_{\infty,h}(x,\xi) = h^{-2n\sigma}\mathrm{O}_s\text{-}\iint e^{ih^{-2\sigma}\tilde{\varphi}_h}\chi_\infty d\eta dy\,.$$

Now, setting

$$(4.28)\quad \begin{cases} \text{i)}\quad \Gamma = 1+h^{-2\sigma}(|\nabla_y\tilde{\varphi}_h|^2+|\nabla_\eta\tilde{\varphi}_h|^2)\,, \\ \text{ii)}\quad L_h = \Gamma^{-1}\{1-i(\nabla_y\tilde{\varphi}_h\cdot\nabla_y+\nabla_\eta\tilde{\varphi}_h\cdot\nabla_\eta)\}\,, \end{cases}$$

we write for $l \geqq 2n+1$

$$\tilde{q}_{0,h}(x,\xi) = h^{-2n\sigma}\iint e^{ih^{-2\sigma}\tilde{\varphi}_h}({}^tL_h)^l \chi_0 d\eta dy\,. \tag{4.29}$$

Then, if we use (4.16), (4.18) and (4.19), by induction we see that $({}^tL_h)^l\chi_0$ has the form

$$\begin{aligned}&({}^tL_h)^l\chi_0\\ &= \frac{1}{\Gamma^l}\sum_{\substack{|\mu|/2\leqq j\leqq l\\ |\nu|\leqq l}} a_{j,\mu,\nu,h}\frac{(h^{-\sigma}(y,\eta))^\mu}{\Gamma^j}\partial^\nu_{(y,\eta)}\chi_0\,,\end{aligned} \tag{4.30}$$

where $a_{j,\mu,\nu,h}$ are functions of $(y,\eta;x,\xi)$ such that

$$|\partial^\alpha_\xi\partial^\beta_x a_{j,\mu,\nu,h}| \leqq C_{\alpha,\beta}(1+\sum_{k=1}^2 |J_{k,h}|_{2,l+|\alpha+\beta|})^{(|\alpha+\beta|+1)l} \tag{4.31}$$

Here we used the fact that $|y|+|\eta|\leqq 1$ on supp χ_0.

From (4.30) and (4.24)-ii) we see that we can write

$$\begin{aligned}&\partial^\alpha_\xi\partial^\beta_x({}^tL_h)^l\chi_0\\ &= \frac{1}{\Gamma^l}\sum_{\substack{|\mu|/2\leqq j\leqq l+|\alpha+\beta|\\ |\nu|\leqq l}} a^{\alpha,\beta}_{j,\mu,\nu,h}\frac{(h^{-\sigma}(y,\eta))^\mu}{\Gamma^j}\partial^\nu_{(y,\eta)}\chi_0\,,\end{aligned} \tag{4.32}$$

where $a^{\alpha,\beta}_{j,\mu,\nu,h}$ are functions of $(y,\eta;x,\xi)$ satisfying the estimates of the form (4.31).

Then, for any α,β if we set $l=|\alpha+\beta|+2n+1$, we have from (4.18), (4.24)-i), (4.29) and (4.32) that

$$\begin{aligned}&|\partial^\alpha_\xi\partial^\beta_x\tilde{q}_{0,h}(x,\xi)|\\ &\leqq C_{\alpha,\beta}h^{-2n\sigma}\iint\frac{d\eta dy}{\{1+h^{-2\sigma}(|y|+|\eta|)^2\}^{l-|\alpha+\beta|}}\\ &\quad\times(1+\sum_{k=1}^2|J_{k,h}|_{2,l+|\alpha+\beta|})^{(l+|\alpha+\beta|)(|\alpha+\beta|+1)}\\ &\leqq C'_{\alpha,\beta}(1+\sum_{k=1}^2|J_{k,h}|_{2,2|\alpha+\beta|+2n+1})^{(2|\alpha+\beta|+2n+1)(|\alpha+\beta|+1)}\,.\end{aligned}$$

Hence, we have

$$|\tilde{q}_{0,h}|^{(0)}_l \leqq C_l(1+\sum_{k=1}^2|J_{k,h}|_{2,2l+2n+1})^{(2l+2n+1)(l+1)}\,. \tag{4.33}$$

IV) For $\tilde{q}_{\infty,h}(x,\xi)$, setting

$$\begin{cases}\text{i)} & \Gamma_1 = |\nabla_y\tilde{\varphi}_h|^2+|\nabla_\eta\tilde{\varphi}_h|^2\,,\\ \text{ii)} & L_{1,h} = -ih^{2\sigma}\Gamma_1^{-1}(\nabla_y\tilde{\varphi}_h\cdot\nabla_y+\nabla_\eta\tilde{\varphi}_h\cdot\nabla_\eta)\,,\end{cases} \tag{4.34}$$

we write with $l\geqq 2n+1$

$$\tilde{q}_{\infty\ h}(x,\xi) = h^{-2n\sigma}\iint e^{ih^{-2\sigma}\tilde{\varphi}_h}({}^tL_{1,h})^l\chi_\infty d\eta dy\,. \tag{4.35}$$

Then, by induction we see that $({}^tL_{1,h})^l\chi_\infty$ has the form

$$({}^tL_{1,h})^l\chi_\infty = \frac{h^{2\sigma l}}{\Gamma_1^{2l}}\sum_{\substack{|\mu|\leqq 3l\\ |\nu|\leqq l}} a_{\mu,\nu,h}(y,\eta)^\mu\partial^\nu_{(y,\eta)}\chi_\infty\,, \tag{4.36}$$

where $a_{\mu,\nu,h}$ are functions such that

$$|\partial_\xi^\alpha\partial_x^\beta a_{\mu,\nu,h}|\leqq C_{\alpha,\beta}(1+\sum_{k=1}^2|J_{k,h}|_{2,l+|\alpha+\beta|})^{l(|\alpha+\beta|+1)}\,. \tag{4.37}$$

Then, for any α,β if we set $l=|\alpha+\beta|+2n+1$, we have from (4.18), (4.19) and (4.35) that

$$\begin{aligned}
&|\partial_\xi^\alpha\partial_x^\beta\tilde{q}_{\infty,h}(x,\xi)|\\
&\leqq C_{\alpha,\beta}(1+\sum_{k=1}^2|J_{k,h}|_{2,2|\alpha+\beta|+2n+1})^{(|\alpha+\beta|+2n+1)(|\alpha+\beta|+1)}\\
&\quad\times h^{2\sigma(l-|\alpha+\beta|-n)}\iint_{|y|+|\eta|\geqq 1/2}\frac{d\eta dy}{(|y|+|\eta|)^{l-|\alpha+\beta|}}\\
&\leqq C'_{\alpha,\beta}(1+\sum_{k=1}^2|J_{k,h}|_{2,2|\alpha+\beta|+2n+1})^{(|\alpha+\beta|+2n+1)(|\alpha+\beta|+1)}\,.
\end{aligned} \tag{4.38}$$

Hence we get

$$|\tilde{q}_{\infty,h}|_l^{(0)}\leqq C'_l(1+\sum_{k=1}^2|J_{k,h}|_{2,2l+2n+1})^{(l+1)(l+2n+1)}\,. \tag{4.39}$$

From (4.33) and (4.39) we get (4.4).

V) In order to get (4.4)′ we write

$$\tilde{\varphi}_h = -y\cdot\eta+\tilde{\gamma}_h(y,\eta;x,\xi)\,. \tag{4.40}$$

Then, we can write

$$e^{ih^{-2\sigma}\tilde{\varphi}_h}-e^{-ih^{-2\sigma}y\cdot\eta} = ie^{-ih^{-2\sigma}y\cdot\eta}h^{-2\sigma}\tilde{\gamma}_h\int_0^1 e^{i\theta h^{-2\sigma}\tilde{\gamma}_h}d\theta\,. \tag{4.41}$$

Hence, noting

$$h^{-2n\sigma}\mathrm{O_s}-\iint e^{-ih^{-2\sigma}y\cdot\eta}d\eta dy = 1\,,$$

and setting

$$\begin{cases}\tilde{\varphi}_{\theta,h} = -y\cdot\eta+\theta\tilde{\gamma}_h\,,\\ \tilde{q}_{\theta,h} = ih^{-2n\sigma}\mathrm{O_s}-\iint e^{ih^{-2\sigma}\tilde{\varphi}_{\theta,h}}(h^{-2\sigma}\tilde{\gamma}_h)d\eta dy\,,\end{cases} \tag{4.42}$$

we have

$$(4.43)\qquad \tilde{q}_h(x,\xi)-1=\int_0^1 \tilde{q}_{\theta,h}(x,\xi)d\theta.$$

For $\tilde{\gamma}_h$ we have by (4.21), (4.12) the estimates

$$(4.44)\quad \begin{cases} \text{i)} & |\partial_\xi^\alpha \partial_x^\beta \tilde{\gamma}_h| \leqq C_{\alpha,\beta}(\sum_{k=1}^{2} |J_{k,h}|_{2,|\alpha+\beta|}) \\ & \quad \times(1+\sum_{k=1}^{2} |J_{k,h}|_{2,|\alpha+\beta|})^{|\alpha+\beta|}(|y|+|\eta|)^2, \\ \text{ii)} & |\partial_\xi^\alpha \partial_x^\beta (\nabla_y \tilde{\gamma}_h, \nabla_\eta \tilde{\gamma}_h)| \leqq C_{\alpha,\beta}(\sum_{k=1}^{2} |J_{k,h}|_{2,1+|\alpha+\beta|}) \\ & \quad \times(1+\sum_{k=1}^{2} |J_{k,h}|_{2,1+|\alpha+\beta|})^{|\alpha+\beta|}(|y|+|\eta|)\langle y;\eta\rangle, \\ \text{iii)} & |\partial_\xi^\alpha \partial_x^\beta \partial_\eta^{\alpha'} \partial_y^{\beta'} \tilde{\gamma}_h| \leqq C_{\alpha,\beta,\alpha',\beta'}(\sum_{k=1}^{2} |J_{k,h}|_{2,|\alpha+\beta+\alpha'+\beta'|}) \\ & \quad \times(1+\sum_{k=1}^{2} |J_{k,h}|_{2,|\alpha+\beta+\alpha'+\beta'|})^{|\alpha+\beta|}\langle y;\eta\rangle^2 \quad (|\alpha'+\beta'|\geqq 2). \end{cases}$$

Then, replacing $\tilde{\varphi}_h$ of (4.23) by $\tilde{\varphi}_{\theta,h}$ we get (4.4)′ in a way similar to the proof for $\tilde{\varphi}_h$ in II)-IV). Q.E.D.

The following theorem gives a representation formula for the multi-product of Fourier integral operators.

Theorem 4.2. *Let* $\phi_{j,h}(x,\xi)\in P_{\rho,\delta}(\tau_j,\tilde{l};h)$, $j=1,2,\cdots$, *and let* $\tau_\infty \leqq \bar{\tau}$ *with* $\tilde{l}$ *and* $\bar{\tau}$ *of* Theorem 3.8. *Define* $\Phi_{j,h}$ *and* $\Phi_{\nu,j,h}$ *by*

$$(4.45)\quad \begin{cases} \text{i)} & \Phi_{j,h}=\Phi_{1,h}\#\cdots\#\Phi_{j,h} \quad (j=1,\cdots,\nu+1), \\ & \Phi_{0,h}=x\cdot\xi, \\ \text{ii)} & \Phi_{\nu,j,h}=\phi_{j,h}\#\cdots\#\phi_{\nu+1,h} \quad (j=1,\cdots,\nu+1), \\ & \Phi_{\nu,\nu+2,h}=x\cdot\xi, \end{cases}$$

for $\nu\geqq 1$. *Let* $r_{j,h}(x,\xi)$, $r_{\nu,j,h}(\xi,x')\in B^0_{\rho,\delta}(h)$ *be the symbols* (*found in* Theorem 3.8) *for* $\Phi_{j,h}$, $\Phi_{\nu,j,h}$, *respectively, such that*

$$(4.46)\qquad R_{j,h}(\Phi^*_{j,h})I(\Phi_{j,h})=I(\Phi_{\nu,j,h})R_{\nu,j,h}(\Phi^*_{\nu,j,h})=I.$$

Set for $p_{j,h}(x,\xi)\in B^{m_j}_{\rho,\delta}(h)$ $(j=1,\cdots,\nu+1)$

$$(4.47)\quad \begin{cases} \text{i)} & Q_{j,h}=I(\Phi_{j-1,h})P_{j,h}(\phi_{j,h})R_{j,h}(\Phi^*_{j,h})\ (\in \boldsymbol{B}^{m_j}_{\rho,\delta}(h)), \\ \text{ii)} & Q_{\nu,j,h}=R_{\nu,j,h}(\Phi^*_{\nu,j,h})P_{j,h}(\phi_{j,h})I(\Phi_{\nu,j+1,h})\ (\in \boldsymbol{B}^{m_j}_{\rho,\delta}(h)). \end{cases}$$

Then, we have the representation formula

$$(4.48)\qquad \begin{aligned} & P_{1,h}(\phi_{1,h})\cdots P_{\nu+1,h}(\phi_{\nu+1,h}) \\ & = Q_{1,h}\cdots Q_{\nu+1,h}I(\Phi_{\nu+1,h}) \\ & = I(\Phi_{\nu+1,h})Q_{\nu,1,h}\cdots Q_{\nu,\nu+1,h}. \end{aligned}$$

Moreover, for the symbols

$$(4.49)\qquad \begin{cases} q_{j,h}(x,\xi)=\sigma(Q_{j,h})(x,\xi)\in B^{m_j}_{\rho,\delta}(h)\,,\\ q_{\nu,j,h}(x,\xi)=\sigma(Q_{\nu,j,h})(x,\xi)\in B^{m_j}_{\rho,\delta}(h)\,,\end{cases}$$

we have the estimates

$$(4.50)\qquad \begin{cases} \text{i)}\quad |q_{j,h}|_l^{(m_j)}\leqq C_l\exp\big(c_l(1+\sum_{s=1}^{j}|J_{s,h}|_{2,k})^{k+2}\big)|p_{j,h}|_{l+6n_0}^{(m_j)}\,,\\ \text{ii)}\quad |q_{\nu,j,h}|_l^{(m_j)}\leqq C_l\exp\big(c_l(1+\sum_{s=j}^{\nu+1}|J_{s,h}|_{2,k})^{k+2}\big)|p_{j,h}|_{l+6n_0}^{(m_j)}\,,\end{cases}$$

where $n_0>n$ is even; $k=2l+15n_0+1$; *and C_l, c_l are positive constants.*

Proof. We give the proof only for $Q_{j,h}$. Then, the proof for $Q_{\nu,j,h}$ is done in the similar way. The formula (4.48) is clear.

We write

$$Q_{j,h}=(I(\Phi_{j-1,h})P_{j,h}(\phi_{j,h}))R_{j,h}(\phi^*_{j,h})\,.$$

Then, by Theorem 4.1 and Theorem 3.5 we see that $Q_{j,h}\in \boldsymbol{B}^{m_j}_{\rho,\delta}(h)$.

By Theorem 3.8 there exist symbols $t_{j,h}(\xi,x')\in B^0_{\rho,\delta}(h)$ such that

$$(4.51)\qquad I=I(\phi_{j,h})T_{j,h}(\phi^*_{j,h})$$

and

$$(4.52)\qquad |t_{j,h}|_l^{(0)}\leqq C_l\exp(2|J_{j,h}|_{2,l+7n_0})\,.$$

Then we can write

$$(4.53)\qquad Q_{j,h}=(I(\Phi_{j-1,h})I(\phi_{j,h}))(T_{j,h}(\phi^*_{j,h})P_{j,h}(\phi_{j,h}))R_{j,h}(\Phi^*_{j,h})\,.$$

By Theorem 3.5-ii) there exist symbols $s_{j,h}(x,\xi)\in B^{m_j}_{\rho,\delta}(h)$ such that

$$(4.54)\qquad T_{j,h}(\phi^*_{h,j})P_{j,h}(\phi_{j,h})=S_{j,h}$$

and by (4.52)

$$(4.55)\qquad \begin{aligned} |s_{j,h}|_l^{(m_j)}&\leqq C_l\exp(|J_{j,h}|_{2,l+2n_0})|t_{j,h}|_{l+2n_0}^{(0)}|p_{j,h}|_{l+2n_0}^{(m_j)}\\ &\leqq C_l\exp(3|J_{j,h}|_{2,l+9n_0})|p_{j,h}|_{l+2n_0}^{(m_j)}\,.\end{aligned}$$

Hence we can write

$$(4.56)\qquad Q_{j,h}=I(\Phi_{j-1,h})I(\phi_{j,h})S_{j,h}R_{j,h}(\Phi^*_{j,h})\,.$$

By Theorem 4.1 there exist symbols $u_{j,h}(x,\xi)\in B^0_{\rho,\delta}(h)$ such that

$$(4.57)\qquad I(\Phi_{j-1,h})I(\phi_{j,h})=U_{j,h}(\Phi_{j,h})$$

and

$$(4.58)\qquad |u_{j,h}|_l^{(0)}\leqq C_l\exp(|\boldsymbol{J}_{j-1,h}|_{2,2l+2n+1}+|J_{j,h}|_{2,2l+2n+1})$$

where $\boldsymbol{J}_{j-1,h}=\Phi_{j-1,h}-x\cdot\xi$. Then, we have

$$Q_{j,h} = U_{j,h}(\Phi_{j,h})S_{j,h}R_{j,h}(\Phi^*_{j,h}). \tag{4.59}$$

By Theorem 3.7-ii) there exist symbols $k_{j,h}(x,\xi)\in B^{m_j}_{\rho,\delta}(h)$ such that

$$U_{j,h}(\Phi_{j,h})S_{j,h} = K_{j,h}(\Phi_{j,h}) \tag{4.60}$$

and by (4.55) and (4.58)

$$\begin{aligned} |k_{j,h}|^{(m_j)}_l &\leqq C_l \exp(|\boldsymbol{J}_{j,h}|_{2,l+2n_0-1})|u_{j,h}|^{(0)}_{l+2n_0}|s_{j,h}|^{(m_j)}_{l+2n_0} \\ &\leqq C'_l \exp\Big(\sum_{s=j-1}^{j}|\boldsymbol{J}_{s,h}|_{2,k'}+4|J_{j,h}|_{2,k'}\Big)|p_{j,h}|^{(m_j)}_{l+4n_0}, \end{aligned} \tag{4.61}$$

where $k'=2l+11n_0+1$. Then we have

$$Q_{j,h} = K_{j,h}(\Phi_{j,h})R_{j,h}(\Phi^*_{j,h}). \tag{4.62}$$

Finally by Theorem 3.5–i) there exist symbols $q_{j,h}(x,\xi)\in B^{m_j}_{\rho,\delta}(h)$ such that

$$K_{j,h}(\Phi_{j,h})R_{j,h}(\Phi^*_{j,h}) = q_{j,h}(X, D_x) \tag{4.63}$$

and by (4.61)

$$\begin{aligned} |q_{j,h}|^{(m_j)}_l &\leqq C_l \exp(|\boldsymbol{J}_{j,h}|_{2,l+2n_0})|k_{j,h}|^{(m_j)}_{l+2n_0}|r_{j,h}|^{(0)}_{l+2n_0} \\ &\leqq C'_l \exp\Big(2\sum_{s=j-1}^{j}|\boldsymbol{J}_{s,h}|_{2,k}+4|J_{j,h}|_{2,k}\Big)|p_{j,h}|^{(m_j)}_{l+6n_0}|r_{j,h}|^{(0)}_{l+2n_0}. \end{aligned} \tag{4.64}$$

By the definition of $r_{j,h}$ and Theorem 3.8 we have

$$|r_{j,h}|^{(0)}_{l+2n_0}\leqq C_l \exp(2|\boldsymbol{J}_{j,h}|_{2,l+9n_0}). \tag{4.65}$$

Thus, noting by (2.23) and (2.20) that

$$|\boldsymbol{J}_{j,h}|_{2,l}\leqq C''_l\Big(1+\sum_{s=1}^{j}|J_{s,h}|_{2,l}\Big)^{l+2}$$

for any l, we get (4.50)-i) from (4.64) and (4.65). Q.E.D.

We conclude this section with the following theorem which summarizes the calculus of Fourier integral operators we have studied.

Thecrem 4.3. *Let $n_0>n$ be an even integer and put $\tilde{l}=21n_0+1$. Let $\tilde{\tau}>0$ be sufficiently small as in* Theorem 3.8. *Let $\phi_{j,h}(x,\xi)\in P_{\rho,\delta}(\tau_j,\tilde{l}:h)$ for $j=1,2,\cdots$, and let $\tau_\infty\equiv\sum_{j=1}^{\infty}\tau_j\leqq\tilde{\tau}$. Let $\nu\geqq1$ be an integer and put $\Phi_{\nu+1,h}=\phi_{1,h}\#\cdots\#\phi_{\nu+1,h}$. Let $p_{j,h}(x,\xi)\in B^{m_j}_{\rho,\delta}(h)$ for $j=1,\cdots,\nu+1$.*

Then there exists a symbol $r_{\nu+1,h}(x,\xi)\in B^{\overline{m}_{\nu+1}}_{\rho,\delta}(h)$ $(\overline{m}_{\nu+1}=m_1+\cdots+m_{\nu+1})$ such that

$$P_{1,h}(\phi_{1,h})\cdots P_{\nu+1,h}(\phi_{\nu+1,h}) = R_{\nu+1,h}(\Phi_{\nu+1,h}) \tag{4.66}$$

and

$$
\begin{aligned}
&|r_{\nu+1,h}|_{l}^{(\overline{m}_{\nu+1})}\\
(4.67)\qquad &\leq \tilde{C}_l^{\nu+2}\exp(\tilde{c}_l(1+\sum_{s=1}^{\nu+1}|J_{s,h}|_{2,k_1})^{k_1+2})\\
&\times\sum_{l_1+\cdots+l_{\nu+1}\leq l+2n_0}\prod_{j=1}^{\nu+1}|p_{j,h}|_{9n_0+l_j}^{(m_j)},
\end{aligned}
$$

where $J_{j,h}=\phi_{j,h}-x\cdot\xi$; $k_1=2l+25n_0+1$; *and* $\tilde{C}_l$, $\tilde{c}_l$ *are positive constants.*

Proof. By Theorem 4.2 we can write

$$
\begin{aligned}
(4.68)\qquad &P_{1,h}(\phi_{1,h})\cdots P_{\nu+1,h}(\phi_{\nu+1,h})\\
&=I(\Phi_{\nu+1,h})Q_{1,h}\cdots Q_{\nu+1,h},
\end{aligned}
$$

where $Q_{j,h}$ is defined by (4.47) of Theorem 4.2. By Theorem 1.8 there exists a symbol $s_{\nu+1,h}(x,\xi)\in B_{\rho,\delta}^{\overline{m}_{\nu+1}}(h)$ such that

$$
(4.69)\qquad Q_{1,h}\cdots Q_{\nu+1,h}=S_{\nu+1,h}
$$

and

$$
(4.70)\qquad |s_{\nu+1,h}|_{l}^{(\overline{m}_{\nu+1})}\leq C_0^{\nu+1}\sum_{l_1+\cdots+l_{\nu+1}\leq l}\prod_{j=1}^{\nu+1}|q_{j,h}|_{3n_0+l_j}^{(m_j)}.
$$

In (4.70) we note that $|q_{j,h}|_{3n_0+l_j}^{(m_j)}=|q_{j,h}|_{3n_0}^{(m_j)}$ except l numbers of $\{q_{j,h}\}_{j=1}^{\nu+1}$. Then, setting $l=3n_0$ in (4.50), we have by Theorem 4.2

$$
(4.71)\qquad |q_{j,h}|_{3n_0}^{(m_j)}\leq C_l'|p_{j,h}|_{9n_0}^{(m_j)},
$$

and for $|q_{j,h}|_{3n_0+l_j}^{(m_j)}$ we have

$$
\begin{aligned}
(4.72)\qquad &|q_{j,h}|_{3n_0+l_j}^{(m_j)}\leq C_l\exp(c_l(1+\sum_{s=1}^{j}|J_{s,h}|_{2,n_j})^{n_j+2})|p_{j,h}|_{l_j+9n_0}^{(m_j)}\\
&\qquad (n_j=2l_j+21n_0+1).
\end{aligned}
$$

Hence, from (4.70)-(4.72) we have for $k_1'=2l+21n_0+1$

$$
\begin{aligned}
(4.73)\qquad &|s_{\nu+1,h}|_{l}^{(\overline{m}_{\nu+1})}\leq C_1^{\nu+1}\exp(lc_l(1+\sum_{s=1}^{\nu+1}|J_{s,h}|_{2,k_1'})^{k_1'+2})\\
&\times\sum_{l_1+\cdots+l_{\nu+1}\leq l}\prod_{j=1}^{\nu+1}|p_{j,h}|_{l_j+9n_0}^{(m_j)}.
\end{aligned}
$$

On the other hand, by Theorem 3.7-ii) there exists a symbol $r_{\nu+1,h}(x,\xi)\in B_{\rho,\delta}^{\overline{m}_{\nu+1}}(h)$ such that

$$
(4.74)\qquad I(\Phi_{\nu+1,h})S_{\nu+1,h}=R_{\nu+1,h}(\Phi_{\nu+1,h})
$$

and

$$
(4.75)\qquad |r_{\nu+1,h}|_{l}^{(\overline{m}_{\nu+1})}\leq C_l\exp(|J_{\nu+1,h}|_{2,l+2n_0-1})|s_{\nu+1,h}|_{l+2n_0}^{(\overline{m}_{\nu+1})}.
$$

Hence, from (4.73) and (4.75) we have (4.67) for positive constants $\tilde{C}_l$, $\tilde{c}_l$. Q.E.D.

5. Approximate fundameutal solution

In this section, using the theory developed in sections 1–4, we shall construct the approximate fundamental solution for the Cauchy problem of a Schrödinger equation.

For a Fréchet space V we denote by $\mathscr{B}^m([0,T];V)$ $(0<T\leqq 1)$ the set of V-valued C^m-functions $u(t)$: $[0,T]\ni t\mapsto u(t)\in V$. Let $\mathscr{B}^{k,\infty}(R^{2n})$ $(k\geqq 1)$ denote the Fréchet space of C^∞-functions $F(x,\xi)$ in R^{2n}, such that $\partial_\xi^\alpha\partial_x^\beta F(x,\xi)$ $(|\alpha+\beta|\geqq k)$ are all bounded, and provided with semi-norms $|F|_l=|F|_{k,l}$ $(l=0,1,\cdots)$ defined by

$$(5.1)\qquad \begin{aligned}|F|_l=&\sum_{|\alpha+\beta|\leqq k-1}\sup_{x,\xi}\{|\partial_\xi^\alpha D_x^\beta F(x,\xi)|/\langle x;\xi\rangle^{k-|\alpha+\beta|}\}\\&+\sum_{k\leqq|\alpha+\beta|\leqq k+l}\sup_{x,\xi}\{|\partial_\xi^\alpha D_x^\beta F(x,\xi)|\}\,.\end{aligned}$$

Now considar a real-valued symbol $H(t,x,\xi)$ with a parameter $t\in[0,T]$, which belongs to $\mathscr{B}^0(I_T;\mathscr{B}^{2,\infty}(R^{2n}))$ with $I_T=[0,T]$, and set

$$(5.2)\qquad H_h(t,x,\xi)=h^{\delta-\rho}H(t,h^{-\delta}x,h^\rho\xi)\quad(0\leqq\delta\leqq\rho\leqq 1)\,.$$

Let $K_h(t,x,\xi)$ be a symbol which has the form

$$(5.3)\qquad \begin{cases}K_h(t,x,\xi)=H_h(t,x,\xi)+\tilde{H}_h(t,x,\xi)\,,\\ \tilde{H}_h(t,x,\xi)\in\mathscr{B}^0(I_T;B^0_{\rho,\delta}(h))\ \ (I_T=[0,T],\ 0\leqq\delta\leqq\rho\leqq 1)\,.\end{cases}$$

REMARK. By the careful check of the discussions in what follows we can replace the conditions $H(t,x,\xi)\in\mathscr{B}^0(I_T;\mathscr{B}^{2,\infty}(R^{2n}))$ and $\tilde{H}_h(t,x,\xi)\in\mathscr{B}^0(I_T;B^0_{\rho,\delta}(h))$ by the weaker conditions:

"$H(t,x,\xi)$ and $\tilde{H}_h(t,x,\xi)$, $0\leqq t\leqq T$, are bounded in $\mathscr{B}^{2,\infty}(R^{2n})$ and in $B^0_{\rho,\delta}(h)$, respectively, and $\partial_\xi^\alpha\partial_x^\beta H(t,x,\xi)$ and $\partial_\xi^\alpha\partial_x^\beta\tilde{H}_h(t,x,\xi)$ are continuous on $[0,T]\times R^{2n}$ for any α, β".

When $0\leqq\delta<\rho\leqq 1$, we assume further that $\tilde{H}_h(t,v,\xi)$ has the asymptotic expansion

$$(5.4)\qquad \tilde{H}_h(t,x,\xi)\sim\sum_{j=0}^\infty h^{(\rho-\delta)j}H_j(t,h^{-\delta}x,h^\rho\xi)\quad(\mathrm{mod}\ \mathscr{B}^0(I_T;B^\infty_{\rho,\delta}(h)))\,,$$

where

$$(5.5)\qquad H_j(t,x,\xi)\in\mathscr{B}^0(I_T;\mathscr{B}(R^{2n})),\ j=0,1,\cdots\,.$$

For $K_h(t)=K_h(t,X,D_x)$ we consider the Cauchy problem of Schrödinger type

$$(5.6)\qquad \begin{cases}L_hu\equiv(D_t+K_h(t,X,D_x))u=0\ \text{ on }\ [0,T_0]\,,\\ u|_{t=s}=\varphi(x)\in\mathscr{S}(0\leqq s\leqq T_0)\end{cases}$$

for some small $0<T_0\leqq T$.

Let $H_h^w(t)=H_h^w(t,X,D_x)$ be the Weyl operator for the symbol $H_h(t,x,\xi)$ defined by

$$
\begin{aligned}
&H_h^w(t,x,\xi) \\
&= h^{\rho-\delta}\mathrm{O}_s-\iint e^{-iy\cdot\eta}H(t,h^{-\delta}(x+\frac{y}{2}),h^{\rho}(\xi+\eta))đ\eta dy\,.
\end{aligned}
\tag{5.7}
$$

Then, it is easy to see that $H_h^w(t,x,\xi)$ has the form (5.3). Furthermore, when $0\leqq\delta<\rho\leqq 1$, we have the asymptotic expansion

$$
\begin{aligned}
&H_h^w(t,x,\xi) = H_h(t,x,\xi)+\tilde{H}_h(t,x,\xi)\,, \\
&\tilde{H}_h(t,x,\xi)\sim\sum_{\alpha\neq 0}\frac{h^{(\rho-\delta)(|\alpha|-1)}}{2^{|\alpha|}\alpha!}H_{(\alpha)}^{(\alpha)}(t,h^{-\delta}x,h^{\rho}\xi)\,.
\end{aligned}
\tag{5.8}
$$

Since $H_h^w(t,X,D_x)=H_h(t,\dfrac{X+X'}{2},D_x)$, we see that $H_h^w(t)$ is symmetric in the sense

$$
(H_h^w(t)v,w) = (v,H_h^w(t)w) \quad \text{for} \quad v,w\in\mathcal{S}\,. \tag{5.9}
$$

For $H_h(t,X,D_x)$ we have

$$
(H_h(t,X,D_x)v,w) = (v,H_h(t,X',D_x)w) \quad (v,w\in\mathcal{S})\,.
$$

So we see that $H_h(t,X,D_x)$ is symmetric, it and only if

$$
H_h(t,x,\xi) = \mathrm{O}_s-\iint e^{-iy\cdot\eta}H_h(t,x+y,\xi+\eta)đ\eta dy \tag{5.10}
$$

Consider the Hamiltonian operator

$$
\begin{aligned}
&H_h(t) = H_h(t,X,D_x) \\
&= h^{-1}\{h^2\sum_{j,k=1}^{n}a_{jk}(t)D_{x_j}D_{x_k}+h\sum_{\substack{j,k=1\\ j\neq k}}^{n}b_{jk}(t)x_jD_{x_k}+V(t,x)\}
\end{aligned}
\tag{5.11}
$$

where $a_{jk}(t)$, $b_{jk}(t)$ are real valued continuous functions on $[0,T]$, and $V(t,x)$ is a real-valued function of class $\mathcal{B}^0(I_T:\mathcal{B}^{2,\infty}(R^{2n}))$. Then, it is easy to see that $H_h(t,X,D_x)$ is symmetric, since (5.10) holds for $H_h(t,x,\xi)$.

In what follows we shall construct the fundamental solution $U_h(t,s)$ for the Cauchy problem (5.6), that is,

$$
\begin{cases}
L_hU_h(t,s) = 0 \quad (0\leqq s,t\leqq T_0)\,, \\
U_h(s,s) = I\,.
\end{cases}
\tag{5.12}
$$

Let $(q(t,s;x,\xi),p(t,s;x,\xi))$ be the solution of the Hamilton equation

$$
\begin{cases}
\dfrac{d}{dt}q(t,s) = \nabla_{\xi}H(t,q(t,s),p(t,s))\,, \\[2ex]
\dfrac{d}{dt}p(t,s) = -\nabla_{x}H(t,q(t,s),p(t,s))
\end{cases}
\tag{5.13}
$$

on $[0, T]$ with the initial condition

(5.14) $$q(s, s) = x, p(s, s) = \xi \quad (0 \leqq s \leqq T).$$

Then, we summarize from [6] the fundamental results as follows.

Proposition 5.1. i) *The solution* $(q(t,s;x,\xi), p(t,s;x,\xi))$ *belongs to* $\mathcal{B}^1(I_T{}^2; \mathcal{B}^{1,\infty}(R^{2n}))$ *with* $I_T{}^2 = [0, T] \times [0, T]$ *and satisfies*

(5.15) $$\text{“}\{(q(t, s; x, \xi)-x)/(t-s), (p(t, s; x, \xi)-\xi)/(t-s)\}_{0 \leqq s,t \leqq T} \text{ is bounded in } \mathcal{B}^{1,\infty}(R^{2n})\text{”},$$

where $\mathcal{B}^1(I_T{}^2; \mathcal{B}^{1,\infty}(R^{2n}))$ *is understood to be the space of* C^1*-mappings from* $I_T{}^2 = [0, T]^2$ to $\mathcal{B}^{1,\infty}(R^{2n})$.

ii) *Take a small* $T_0 (0 < T_0 \leqq T)$. *Then, for* $(t, s) \in I_{T_0}{}^2$ *there exist the inverse* C^∞ *diffeomorphisms* $x \mapsto y(t,s;x,\xi)$ *and* $\xi \mapsto \eta(t,s;x,\xi)$ *of the mappings* $y \mapsto x = q(t,s;y,\xi)$ *and* $\eta \mapsto \xi = p(t,s;x,\eta)$, *respectively, and they satisfy*

(5.16) $$y(t, s; x, \xi), \eta(t, s; x, \xi) \in \mathcal{B}^1(I_{T_0}{}^2; \mathcal{B}^{1,\infty}(R^{2n})) \quad (I_{T_0} = [0, T_0])$$

and

(5.17) $$\text{“}\{(y(t, s; x, \xi)-x)/(t-s), (\eta(t, s; x, \xi)-\xi)/(t-s)\}_{0 \leqq s,t \leqq T_0} \text{ is bounded in } \mathcal{B}^{1,\infty}(R^{2n})\text{”}.$$

Now we construct the solutlon of the Hamilton-Jacobi equation

(5.18) $$\begin{cases} \partial_t \phi(t, s; x, \xi) + H(t, x, \nabla_x \phi(t, s; x, \xi)) = 0 \quad \text{on} \quad [0, T_0]^2 \times R^{2n}, \\ \phi(s, s; x, \xi) = x \cdot \xi \quad (0 \leqq s \leqq T_0) \end{cases}$$

as follows (cf. [8]). Define $\phi(t,s;x,\xi)$ by

(5.19) $$\phi(t, s; x, \xi) = u(t, s; y(t, s; x, \xi), \xi),$$

where $u(t,s;y,\eta)$ is defined by

(5.20) $$u(t, s; y, \eta) = y \cdot \eta + \int_s^t (\xi \cdot \nabla_\xi H - H)(\tau, q(\tau, s; y, \eta), p(\tau, s; y, \eta)) d\tau.$$

Then we have

Proposition 5.2. *For the solution* $\phi(t,s;x,\xi)$ *of* (5.18) *we have*

(5.21) $$\begin{cases} \nabla_x \phi(t, s; x, \xi) = \eta(s, t; x, \xi), \\ \nabla_\xi \phi(t, s; x, \xi) = y(t, s; x, \xi), \end{cases}$$

(5.22) $$\partial_s \phi(t, s; x, \xi) - H(s, \nabla_\xi \phi(t, s; x, \xi), \xi) = 0 \quad \text{on} \quad [0, T_0]^2 \times R^{2n},$$

and for $J(t,s;x,\xi)=\phi(t,s;x,\xi)-x\cdot\xi$

$$\text{“}\{J(t,s;x,\xi)/(t-s)\}_{0\leqq s,t\leqq T_0} \text{ is bounded in } \mathscr{B}^{2,\infty}(R^{2n})\text{”}. \tag{5.23}$$

Furthermore, for any fixed l *there exist* $\tilde{T}_l(0<\tilde{T}_l\leqq T_0)$ *and* $c_l(\geqq 1)$ *such that* $c_l\tilde{T}_l<1$ *and*

$$\phi(t,s;x,\xi)\in P(c_l|t-s|,l) \quad \text{on} \quad [0,\tilde{T}_l]^2. \tag{5.24}$$

Proof. As to (5.21) see Proposition 3.5 of [6]. Then using Proposition 5.1, we obtain (5.23). For (5.22) see Theorem 2.1 of [9]. (5.24) is an immediate consequence of (5.23). Q.E.D.

Now define $\phi_h(t,s)=\phi_h(t,s;x,\xi)$ for $0<h<1$ by

$$\phi_h(t,s;x,\xi)=h^{\delta-\rho}\phi(t,s;h^{-\delta}x,h^{\rho}\xi). \tag{5.25}$$

Then we have

Proposition 5.3. *The phase function* $\phi_h(t,s;x,\xi)$ *satisfies*

$$\begin{cases}\partial_t\phi_h(t,s;x,\xi)+H_h(t,x,\nabla_x\phi_h(t,s;x,\xi))=0 \quad \text{on} \quad [0,T_0]^2\times R^{2n}, \\ \phi_h(s,s;x,\xi)=x\cdot\xi,\end{cases} \tag{5.26}$$

and

$$\partial_s\phi_h(t,s;x,\xi)-H_h(s,\nabla_\xi\phi_h(t,s;x,\xi),\xi)=0 \quad \text{on} \quad [0,T_0]^2\times R^{2n}. \tag{5.27}$$

Furthermore, for any fixed l *we have with* $0<\tilde{T}_l\leqq T_0$ *and* c_l *of* Proposition 5.2

$$\phi_h(t,s;x,\xi)\in P_{\rho,\delta}(c_l|t-s|,l;h) \quad \text{on} \quad [0,\tilde{T}_l]^2. \tag{5.28}$$

Proof. We obtain (5.26) and (5.27) easily from (5.18) and (5.22), and we get (5.28) from (5.24). Q.E.D.

In the following we switch to another small $T_0>0$ such that $T_0\leqq\tilde{T}_0$, if necessary.

Now we first define two kinds of approximate fundamental solution as follows. Let $E_h(\phi_h(t,s))=e_h(\phi_h(t,s);t,s;X,D_x)$ be the Fourier integral operator with the phase function $\phi_h(t,s)$ and the symbol $e_h(t,s,x;\xi)$ of class $\mathscr{B}^1(I_{T_0}{}^2;B^0_{\rho,\delta}(h))$ $(I_{T_0}{}^2=[0,T_0]^2)$.

DEFINITION 5.4. We say that $E_h(\phi_h(t,s))$ is the *approximate fundamental solution of order zero* and *order infinity* for the problem (5.6), when $E_h(\phi_h(t,s))$ satisfies, respectively,

$$\begin{cases}\text{i)} \quad \sigma(L_hE_h(\phi_h(t,s)))\in\mathscr{B}^0(I_{T_0}{}^2;B^0_{\rho,\delta}(h)), \\ \text{ii)} \quad E_h(\phi_h(s,s))=I \quad (0\leqq s\leqq T_0)\end{cases} \tag{5.29}$$

and

$$(5.30)\qquad \begin{cases} \text{i)} & \sigma(L_h E_h(\phi_h(t,s)))\in\mathscr{B}^0(I_{T_0}{}^2; B^{\infty}_{\rho,\delta}(h))\,, \\ \text{ii)} & E_h(\phi_h(s,s)) = I \quad (0\leqq s\leqq T_0)\,. \end{cases}$$

In order to make the discussion clear in what follows we introduce the following

DEFINITION 5.5. We say that a C^∞-function $P_h(x,\xi)$ on R^{2n} with a parameter $h\in(0,1)$ belongs to the class $B^{m;l}_{\rho,\delta}(h)$ for real m,l and $0\leqq\delta\leqq\rho\leqq 1$, when $p_h(x,\xi)\langle h^{-\delta}x;h^{\rho}\xi\rangle^{-l}$ belongs to $B^m_{\rho,\delta}(h)$.

REMARK 1°. By the definition we have $B^{m;0}_{\rho,\delta}(h)=B^m_{\rho,\delta}(h)$.

2°. Set $t_h(x,\xi)=\langle h^{-\delta}x;h^{\rho}\xi\rangle$ and $t_{l,h}(x,\xi)=t_h(x,\xi)^l$ for real l. Then we have

$$(5.31)\qquad |t_{l,h}{}^{(\alpha)}_{(\beta)}(x,\xi)|\leqq C_{\alpha,\beta,l}h^{\rho|\alpha|-\delta|\beta|}t_{l-|\alpha+\beta|,h}(x,\xi)\,.$$

3°. If $m\geqq m'$ and $l\leqq l'$, we have $B^{m;l}_{\rho,\delta}(h)\subset B^{m';l'}_{\rho,\delta}(h)$.

4°. If a C^∞-function $s_{l,h}(x,\xi)$ satisfies

$$(5.32)\qquad |s_{l,h}{}^{(\alpha)}_{(\beta)}(x,\xi)|\leqq C_{\alpha,\beta,l}h^{\rho|\alpha|-\delta|\beta|}t_{l-|\alpha+\beta|,h}(x,\xi)\,,$$

then we have

$$(5.33)\qquad s_{l,h}{}^{(\alpha)}_{(\beta)}(x,\xi)\in B^{\rho|\alpha|-\delta|\beta|,l-|\alpha+\beta|}_{\rho,\delta}(h)\,.$$

In particular, by (5.31) we have

$$(5.34)\qquad t_{l,h}{}^{(\alpha)}_{(\beta)}(x,\xi)\in B^{\rho|\alpha|-\delta|\beta|,l-|\alpha+\beta|}_{\rho,\delta}(h)\,.$$

5°. For $p(x,\xi)\in\mathscr{B}^{k,\infty}(R^{2n})$, set $p_h(x,\xi)=p(h^{-\delta}x,h^{\rho}\xi)$. Then p_h satisfies (5.32) with $s_{l,h}=p_h$ and $l=k$. Thus we have

$$(5.35)\qquad p_h{}^{(\alpha)}_{(\beta)}(x,\xi)\in B^{\rho|\alpha|-\delta|\beta|,k-|\alpha+\beta|}_{\rho,\delta}(h)\,.$$

Proposition 5.6. *Let $l\geqq 0$ and let $\bar{l}$ denote the minimum integer not less than l. Let $p_h(x,\xi)\in B^{m;l}_{\rho,\delta}(h)$ and $\phi_h(x,\xi)\in P_{\rho,\delta}(\tau,0;h)$, and consider the pseudo-differential operator $P_h=p_h(X,D_x)$ defined by* (1.10) *and the Fourier integral operator $P_h(\phi_h)=p_h(\phi_h;X,D_x)$ defined by* (3.1) *or* (3.1)′. *Then, we have the following*

1) *$P_h, P_h(\phi_h)$: $\mathscr{S}\to\mathscr{S}$ are continuous.*

2) *Assume further that*

$$(5.36)\qquad p_h{}^{(\alpha)}_{(\beta)}(x,\xi)\in B^{m+\rho|\alpha|-\delta|\beta|,l-|\alpha+\beta|}_{\rho,\delta}(h)\quad(|\alpha+\beta|\leqq\bar{l})$$

and let $Q_h(\phi_h)=q_h(\phi_h;X,D_x)\in \boldsymbol{B}^{m'}_{\rho,\delta}(\phi_h)$. Then in Theorem 3.7 *we have for $r_h(x,\xi)$ defined by* (3.36) *and* (3.37) (*resp.* (3.40) *and* (3.41)) *that*

$$(5.37)\qquad \begin{cases} \text{i)} & r_h(x,\xi)\in B^{m+m',l}_{\rho,\delta}(h)\,, \\ \text{ii)} & r_h(\phi_h;X,D_x) = P_hQ_h(\phi_h)\ (\textit{resp. } Q_h(\phi_h)P_h)\,. \end{cases}$$

Furthermore we have the expansion formulae

$$r_h(x,\xi')-\sum_{|\alpha|<N}\frac{1}{\alpha!}D_{x'}^{\alpha}\{p_h^{(\alpha)}(x,\tilde{\nabla}_x\phi_h(x,\xi',x'))q_h(x',\xi')\}_{|x'=x}$$

(5.38) (*resp.*

$$r_h(x,\xi')-\sum_{|\alpha|<N}\frac{1}{\alpha!}\partial_{\xi}^{\alpha}\{q_h(x,\xi)p_{h(\alpha)}(\tilde{\nabla}_{\xi}\phi_h(\xi,x,\xi'),\xi')\}_{|\xi=\xi'})$$
$$\in B_{\rho,\delta}^{m+m'+(\rho-\delta)N}(h) \text{ for any } N\geqq \bar{l}.$$

REMARK 1°. We should note that, in general, the symbol $p_h(x,\xi)$ of the Fourier integral operator $P_h(\phi_h)$ in Proposition 5.6 is not bounded on $R_x^n\times R_\xi^n$ for any fixed $h\in(0,1)$. The statement 1) means that $P_h(\phi_h)\colon \mathcal{S}\to\mathcal{S}$ is well defined and the statement 2) means that Theorem 3.7 holds for the present Fourier integral operator $Q_h(\phi_h)$ in a slightly modified form.

2°. When $0\leqq\delta<\rho\leqq 1$, the expansions (5.38) coincide with (3.39) (resp. (3.42)).

Proof. 1) The continuity of $P_h;\mathcal{S}\to\mathcal{S}$ is clear, and that of $P_h(\phi_h)\colon\mathcal{S}\to\mathcal{S}$ can be proved in completely the same way as that of the proof of Proposition 3.2.

2) We get (5.37)-ii) in the same way as in the proof of Theorem 3.7. To get (5.37)-i) we make in (3.37) (resp. (3.41)) Taylor's expansions of order $N\geqq 0$ for $s_h(x,\xi'+\eta,x+y,\xi')$ in η (resp. y). Then, using (5.36), we see that

$$s_h^{(\alpha,0)}(x,\xi,x',\xi')=p_h^{(\alpha)}(x,\xi+\tilde{\nabla}_xJ_h(x,\xi',x'))q_h(x',\xi')$$

(resp. $s_{h(0,\alpha)}(x,\xi,x',\xi')=q_h(x,\xi)p_{h(\alpha)}(x'+\tilde{\nabla}_{\xi}J_h(\xi,x,\xi'),\xi')$) belongs to $B_{\rho,\delta}^{m+m'+\rho N}(h)$ (resp. $B_{\rho,\delta}^{m+m'-\delta N}(h)$) for $|\alpha|=N(\geqq\bar{l})$. Hence, we obtain (5.38), and setting $N=\bar{l}$ we get (5.37)-i). Q.E.D.

Now, for a fixed $a_h'(t,s;x,\xi)\in\mathcal{B}^1(I_{T_0}{}^2;B_{0,0}^0(h))$ set

(5.39) $$a_h(t,s;x,\xi)=a_h'(t,s;h^{-\delta}x,h^{\rho}\xi)\ (\in\mathcal{B}^1(I_{T_0}{}^2;B_{\rho,\delta}^0(h)))\,,$$

and consider

(5.40) $$\Gamma_h(t,s)\equiv H_h(t,X,D_x)a_h(\phi_h(t,s);t,s;X,D_x)\,.$$

We note that $H_h(t,x,\xi)$ satisfies the condition (5.36) with $m=\delta-\rho$ and $l=2$. Hence, by Proposition 5.6–2) we see that there exists $\gamma_h(t,s;x,\xi)\in\mathcal{B}^0(I_{T_0}{}^2;B_{\rho,\delta}^{\delta-\rho,2}(h))$ such that

(5.41) $$\Gamma_h(t,s)=\gamma_h(\phi_h(t,s);t,s;X,D_x)\,.$$

Furthermore, by (5.38) for $N=2$ there exists $r_h(t,s;x,\xi)\in\mathcal{B}^0(I_{T_0}{}^2;B_{\rho,\delta}^0(h))$ such that

$$\begin{aligned}
&\gamma_h(t, s; x, \xi) \\
&= H_h(t, x, \nabla_x\phi_h)a_h(t, s; x, \xi) \\
&\quad +\sum_{j=1}^{n} H_h^{(j)}(t, x, \nabla_x\phi_h)a_{h(j)}(t, s; x, \xi) \\
&\quad -\frac{i}{2}\{\sum_{j,k=1}^{n} H_h^{(j,k)}(t, x, \nabla_x\phi_h)\frac{\partial^2}{\partial x_j\partial x_k}\phi_h\}a_h(t, s; x, \xi) \\
&\quad +h^{\rho-\delta}r_h(t, s; x, \xi),
\end{aligned} \tag{5.42}$$

where $H_h^{(j)}\equiv\partial_{\xi_j}H_h$, $a_{h(j)}=D_{x_j}a_h$ and $H_h^{(j,k)}=\partial_{\xi_j}\partial_{\xi_k}H_h$. For the operator

$$\tilde{\Gamma}_h(t, s)\equiv\tilde{H}_h(t, X, D_x)a_h(\phi_h(t, s); t, s; X, D_x), \tag{5.43}$$

by Theorem 3.7 we can find $\tilde{\gamma}_h(t,s;x,\xi)\in\mathscr{B}^0(I_{T_0}{}^2; B^0_{\rho,\delta}(h))$ such that

$$\tilde{\Gamma}_h(t, s) = \tilde{\gamma}_h(\phi_h(t, s); t, s; X, D_x), \tag{5.44}$$

and we can write for some $\tilde{r}_h(t,s;x,\xi)\in\mathscr{B}^0(I_{T_0}{}^2; B^0_{\rho,\delta}(h))$

$$\begin{aligned}
&\tilde{\gamma}_h(t, s; x, \xi) \\
&= \tilde{H}_h(t, x, \nabla_x\phi_h)a_h(t, s; x, \xi)+h^{\rho-\delta}\tilde{r}_h(t, s; x, \xi).
\end{aligned} \tag{5.45}$$

On the other hand we have

$$\begin{aligned}
&D_t a_h(\phi_h(t, s); t, s; X, D_x) \\
&= (\partial_t\phi_h\cdot a_h)(\phi_h(t, s); t, s; X, D_x) \\
&\quad +(D_t a_h)(\phi_h(t, s); t, s; X, D_x).
\end{aligned} \tag{5.46}$$

Hence, summarizing (5.40)–(5.46), we see by (5.26) that there exists a symbol $b_h(t,s;x,\xi)\in\mathscr{B}^0(I_{T_0}{}^2; B^0_{\rho,\delta}(h))$ such that

$$\begin{aligned}
&L_h a_h(\phi_h(t, s); t, s; X, D_x) \\
&= (\mathcal{L}_h a_h)(\phi_h(t, s); t, s; X, D_x)+h^{\rho-\delta}b_h(\phi_h(t, s); t, s; X, D_x),
\end{aligned} \tag{5.47}$$

where $\mathcal{L}_h$ is the transport operator defined by

$$\begin{aligned}
&(\mathcal{L}_h a_h)(t, s; x, \xi) \\
&= D_t a_h + \sum_{j=1}^{n} H_h^{(j)}(t, x, \nabla_x\phi_h)D_{x_j}a_h \\
&\quad +\{-\frac{i}{2}(\sum_{j,k=1}^{n} H_h^{(j,k)}(t, x, \nabla_x\phi_h)\frac{\partial^2}{\partial_{x_j}\partial_{x_k}}\phi_h)+\tilde{H}_h(t, x, \nabla_x\phi_h)\}a_h.
\end{aligned} \tag{5.48}$$

Now we set

$$\tilde{H}'_h(t, x, \xi) = \tilde{H}_h(t, h^\delta x, h^{-\rho}\xi)\ (\in\mathscr{B}^0(I_T; B^0_{0,0}(h))). \tag{5.49}$$

Then, from (5.2), (5.25) and (5.39) we can write

$$(5.50)\quad \begin{aligned}&(\mathcal{L}'_h a'_h)(t,s;x,\xi)\equiv(\mathcal{L}_h a_h)(t,s;h^{\delta}x,h^{-\rho}\xi)\\ &=D_t a'_h+\sum_{j=1}^{n}H^{(j)}(t,x,\nabla_x\phi)D_{x_j}a'_h\\ &\quad+\{-\frac{i}{2}(\sum_{j,k=1}^{n}H^{(j,k)}(t,x,\nabla_x\phi)\frac{\partial^2}{\partial_{x_j}\partial_{x_k}}\phi)+\tilde{H}'_h(t,x,\nabla_x\phi)\}a'_h\,.\end{aligned}$$

Theorem 5.7. *Let $I(\phi_h(t,s))$ be the Fourier integral operator with phase function $\phi_h(t,s)$ and symbol 1. Then, $I(\phi_h(t,s))$ is the approximate fundamental solution of order zero for L_h.*

Proof. It is easy to see $I(\phi_h(s,s))=I$. Consider $L_h I(\phi_h(t,s))$. Noting $H_h^{(j,h)}(t,x,\nabla_x\phi_h)\frac{\partial^2}{\partial_{x_j}\partial_{x_k}}\phi_h\in\mathcal{B}^0(I_{T_0}{}^2;B^0_{\rho,\delta}(h))$, by (5.48) we see that $\mathcal{L}_h a_h$ belongs to $\mathcal{B}^0(I_{T_0}{}^2;B^0_{\rho,\delta}(h))$. Hence, from (5.47) we obtain (5.29)-i). Q.E.D.

Theorem 5.8. *When $0\leqq\delta<\rho\leqq 1$, there exists a symbol $e_h(t,s;x,\xi)\in\mathcal{B}^1(I_{T_0}{}^2;B^0_{\rho,\delta}(h))$ such that $E_h(\phi_h(t,s))=e_h(\phi_h(t,s);t,s;X,D_x)$ is the approximate fundamental solution of order infinity for L_h.*

Furthermore, there exists a series of symbols $a_\nu(t,s;x,\xi)\in\mathcal{B}^1(I_{T_0}{}^2;\mathcal{B}(R^{2n}))$ such that

$$(5.51)\quad \begin{cases}a_0(s,s;x,\xi)=1\,,\\ a_\nu(s,s;x,\xi)=0\quad(\nu\geqq 1)\end{cases}$$

and

$$(5.52)\quad e_h(t,s;x,\xi)\sim\sum_{\nu=0}^{\infty}h^{(\rho-\delta)\nu}a_\nu(t,s;h^{-\delta}x,h^{\rho}\xi)\,.$$

Proof. I) Noting (5.4) we define transport operators $\mathcal{M}_h$ and $\mathcal{M}$ corresponding to $\mathcal{L}_h$ and $\mathcal{L}'_h$, respectively, by

$$(5.53)\quad \begin{aligned}\mathcal{M}_h a_h&=D_t a_h+\sum_{j=1}^{n}H_h^{(j)}(t,x,\nabla_x\phi_h)D_{x_j}a_h\\ &\quad+\{-\frac{i}{2}(\sum_{j,k=1}^{n}H_h^{(j,k)}(t,x,\nabla_x\phi_h)\frac{\partial^2}{\partial_{x_j}\partial_{x_k}}\phi_h)+H_{0,h}(t,x,\nabla_x\phi_h)\}a_h\end{aligned}$$

and

$$(5.54)\quad \begin{aligned}\mathcal{M}a&=D_t a+\sum_{j=1}^{n}H^{(j)}(t,x,\nabla_x\phi)D_{x_j}a\\ &\quad+\{-\frac{i}{2}(\sum_{j,k=1}^{n}H^{(j,k)}(t,x,\nabla_x\phi)\frac{\partial^2}{\partial_{x_j}\partial_{x_k}}\phi)+H_0(t,x,\nabla_x\phi)\}a\,,\end{aligned}$$

where $H_0(t,x,\xi)$ is a symbol of (5.5) and

$$(5.55)\quad H_{0,h}(t,x,\xi)=H_0(t,h^{-\delta}x,h^{\rho}\xi)\,.$$

We set for $a_\nu(t,s;x,\xi)\in\mathcal{B}^1(I_{T_0}{}^2;\mathcal{B}(R^{2n}))$

$$(5.56)\quad a_{\nu,h}(t,s;x,\xi)=a_\nu(t,s;h^{-\delta}x,h^{\rho}\xi)\;(\in\mathcal{B}^1(I_{T_0}{}^2;B^0_{\rho,\delta}(h)))\,.$$

Then, in the similar way to the discussion from (5.40) to (5.46) in order to get (5.47) we see by (5.4) that we can write

$$\gamma_{\nu,h}(t, s; x, \xi) \equiv \sigma(L_h a_{\nu,h}(\phi_h(t, s); t, s; X, D_x))$$

$$\sim \mathcal{M}_h a_{\nu,h}(t, s; x, \xi) + \sum_{k=1}^{\infty} h^{(\rho-\delta)k} b_{\nu,k}(t, s; h^{-\delta}x, h^{\rho}\xi) \tag{5.57}$$

$$(\mathrm{mod}\ \mathcal{B}^0(I_{T_0}{}^2; B^{\infty}_{\rho,\delta}(h)))$$

for some $b_{\nu,k}(t,s;x,\xi) \in \mathcal{B}^0(I_{T_0}{}^2; \mathcal{B}(R^{2n}))$ determined by H, H_j $(j=0,1\cdots)$ of (5.4) and a_ν.

II) Now, we first determine $a_{0,h}(t,s;x,\xi)$ by

$$\begin{cases} \mathcal{M}_h a_{0,h}(t, s) = 0 & \text{on } [0, T_0]^2 \times R^{2n}, \\ a_{0,h}(s, s) = 1 & \text{on } [0, T_0] \times R^{2n}, \end{cases} \tag{5.58}$$

which is equivalent to

$$\begin{cases} \mathcal{M}_0 a(t, s) = 0 & \text{on } [0, T_0]^2 \times R^{2n}, \\ a_0(s, s) = 1 & \text{on } [0, T_0] \times R^{2n}. \end{cases} \tag{5.59}$$

Then, $a_0(t,s)$ can be solved as

$$\begin{aligned} &a_0(t, s; x, \xi) \\ &= \exp\Big[-\int_s^t \Big\{\frac{1}{2}\Big(\sum_{j,k=1}^{n} H^{(j,k)}(\tau, X(\tau), \eta(s, t; X(\tau), \xi))\Big) \\ &\quad \times \frac{\partial^2}{\partial_{x_j}\partial_{x_k}}\phi(\tau, s; X(\tau), \xi)\Big) + iH_0(\tau, X(\tau), \eta(s, t; X(\tau), \xi))\Big\} d\tau\Big], \end{aligned} \tag{5.60}$$

where $X(\tau) = q(\tau, s; y(t,s;x,\xi), \xi)$, and $y(t,s)$ and $\eta(t,s)$ are the functions in Proposition 5.1-ii). Then, it is easy to see that

$$a_0(t, s; x, \xi) \in \mathcal{B}^1(I_{T_0}{}^2; \mathcal{B}(R^{2n})). \tag{5.61}$$

Now, by induction we determine $a_{\nu,h}(t,s;x,\xi)$ $(\nu=1,2,\cdots)$ by the equations

$$\begin{cases} \mathcal{M}_h a_{\nu,h} + \sum_{l=0}^{\nu-1} b_{l,\nu-l}(t, s; h^{-\delta}x, h^{\rho}\xi) = 0 & \text{on } [0, T_0]^2 \times R^{2n}, \\ a_{\nu,h}(s, s) = 0 \quad \text{on } [0, T_0] \times R^{2n}, \end{cases} \tag{5.62}$$

which are equivalent to

$$\begin{cases} \mathcal{M} a_\nu + \sum_{l=0}^{\nu-1} b_{l,\nu-l}(t, s; x, \xi) = 0 & \text{on } [0, T_0]^2 \times R^{2n}, \\ a_\nu(s, s) = 0 \quad \text{on } [0, T_0] \times R^{2n}. \end{cases} \tag{5.63}$$

Then, the solutions $a_\nu(t,s)$ are given by

$$\begin{aligned} &a_\nu(t, s; x, \xi) \\ &= -a_0(t, s; x, \xi) \int_s^t \frac{\sum_{l=0}^{\nu-1} b_{l,\nu-l}(\tau, s; X(\tau), \xi)}{a_0(\tau, s; X(\tau), \xi)} d\tau. \end{aligned} \tag{5.64}$$

Then, step by step we can check that

$$a_\nu(t, s; x, \xi) \in \mathscr{B}^1(I_{T_0}{}^2; \mathscr{B}(R^{2n})) \qquad (\nu \geqq 1). \tag{5.65}$$

Now, for any fixed $N \geqq 1$ set

$$e_{N,h}(t, s; x, \xi) = \sum_{\nu=0}^{N-1} h^{(\rho-\delta)\nu} a_{\nu,h}(t, s; x, \xi). \tag{5.66}$$

Then, from (5.57), (5.58) and (5.62) we see that for $E_{N,h}(\phi_h(t,s)) = e_{N,h}(\phi_h(t,s); t, s; X, D_x)$

$$\begin{cases} \sigma(L_h E_{N,h}(\phi_h(t, s))) \in \mathscr{B}^0(I_{T_0}{}^2; B_{\rho,\delta}^{(\rho-\delta)N}(h)), \\ E_{N,h}(\phi_h(s, s)) = I. \end{cases} \tag{5.67}$$

III) Finally, by Theorem 1.3 we can find $e_h(t,s;x,\xi)$ satisfying (5.52) in the form

$$e_h(t, s; x, \xi) = \sum_{\nu=0}^{\infty} h^{(\rho-\delta)\nu} \chi(\varepsilon_\nu^{-1} h) a_{\nu,h}(t, s; x, \xi). \tag{5.68}$$

Noting Proposition 5.6, we can write for $v \in \mathscr{S}$

$$\begin{aligned} & L_h E_h(\phi_h(t, s))v \\ &= \sum_{\nu=0}^{\infty} h^{(\rho-\delta)\nu} \chi(\varepsilon_\nu^{-1} h) L_h a_{\nu,h}(\phi_h(t, s); t, s; X, D_x)v \\ &= L_h E_{N,h}(\phi_h(t, s))v \\ &\quad + \sum_{\nu=N}^{\infty} h^{(\rho-\delta)\nu} \chi(\varepsilon_\nu^{-1} h) L_h a_{\nu,h}(\phi_h(t, s); t, s; X, D_x)v \\ &\quad + b_{N,h}(\phi_h(t, s); t, s; X, D_x)v \end{aligned} \tag{5.69}$$

for some $b_{N,h}(t,s;x,\xi) \in \mathscr{B}^0(I_{T_0}{}^2; B_{\rho,\delta}^{\infty}(h))$. We note from (5.57) and (5.62) that

$$\sigma(L_h a_{\nu,h}(\phi_h(t, s); t, s; X, D_x)) \in \mathscr{B}^0(I_{T_0}{}^2; B_{\rho,\delta}^{0}(h)). \tag{5.70}$$

Hence, taking an appropriately decreasing sequence $\{\varepsilon_j\}_{j=1}^{\infty}$ again if necessary, we see from (5.67), (5.69) and (5.70) that for any N

$$\sigma(L_h E_h(\phi_h(t, s))) \in \mathscr{B}^0(I_{T_0}{}^2; B_{\rho,\delta}^{(\rho-\delta)N}(h)), \tag{5.71}$$

which proves (5.30). Q.E.D.

As a special case of L_h we consider an operator $\tilde{L}_h$ defined by

$$\tilde{L}_h = D_t + H_h(t, X, D_x). \tag{5.72}$$

Then, we can show that the approximate fundamental solutions have stronger properties which are effective to guarantee the convergence of the iterated integral of Feynman's type.

Theorem 5.9. 1) *Set*

$$g_{0,h}(t,s;x,\xi)=\sigma(\tilde{L}_h I(\phi_h(t,s)))\,. \tag{5.73}$$

Then we have

$$\text{“}\{g_{0,h}(t,s;x,\xi)/(t-s)\}_{0\leq s,t\leq T_0} \text{ is bounded in } B^0_{\rho,\delta}(h)\text{”}. \tag{5.74}$$

2) *Let* $\tilde{E}_h(\phi_h(t,s))=\tilde{e}_h(\phi_h(t,s);t,s;X,D_x)$ *be the approximate fundamental solution (of order infinity) for* L_h *which is constructed in* Theorem 5.8 *with* $\tilde{H}_h(t,x,\xi)=0$. *Set*

$$g_{\infty,h}(t,s;x,\xi)=\sigma(\tilde{L}_h\tilde{E}_h(\phi_h(t,s)))\,. \tag{5.75}$$

Then we have

$$\begin{aligned}&\text{“}\{(\tilde{e}_h(t,s;x,\xi)-1)/(t-s)^2\}_{0\leq s,t\leq T_0} \text{ and}\\ &\{\partial_t\tilde{e}_h(r,s;x,\xi)/(t-s),\ \partial_s\tilde{e}_h(t,s;x,\xi)/(t-s)\}_{0\leq s,t\leq T_0}\\ &\text{are bounded in } B^0_{\rho,\delta}(h)\text{”},\end{aligned} \tag{5.76}$$

and

$$\text{“}\{g_{\infty,h}(t,s;x,\xi)/(t-s)\}_{0\leq s,t\leq T_0} \text{ is bounded in } B^\infty_{\rho,\delta}(h)\text{”}. \tag{5.77}$$

Proof. 1) By Propositlon 5.6–2) and Taylor's expansion of order 1 we can write

$$\begin{aligned}&\sigma(H_x(t,X,D_x)I(\phi_h(t,s)))\,(x,\xi')\\ &=\mathrm{O}_s-\iint e^{-iy\cdot\eta}H_h(t,x,\xi'+\eta+\tilde{\nabla}_x J_h(t,s;x,\xi',x+y))\,d\!\!{}^-\eta dy\\ &=H_h(t,x,\nabla_x\phi_h(t,s;x,\xi'))\\ &\quad+\int_0^1[\mathrm{O}_s-\iint e^{-iy\cdot\eta}\{\sum_{j,k=1}^n H_h^{(j,k)}(t,x,\xi'+\theta\eta+\tilde{\nabla}_x J_h(t,s;x,\xi',x+y))\\ &\quad\cdot(\int_0^1\theta_1\frac{\partial^2}{\partial_{x_j}\partial_{x_k}}J_h(t,s;x+\theta_1 y,\xi')d\theta_1)\}\,d\!\!{}^-\eta dy]d\theta\,.\end{aligned} \tag{5.78}$$

Then, noting (5.26) and (5.28) we get (5.74).

2) In (5.60) set $H_0=0$. Then, noting (5.28) we see that

$$\begin{aligned}&\text{“}\{(a_0(t,s)-1)/(t-s)^2,\ \partial_t a_0(t,s)/(t-s),\ \partial_s a_0(t,s)/(t-s)\}_{0\leq s,t\leq T_0}\\ &\text{is bounded in } \mathscr{B}(R^{2n})\text{”},\end{aligned} \tag{5.79}$$

which means that for $a_{0,h}$ defined by (5.56)

$$\begin{aligned}&\text{“}\{(a_{0,h}(t,s)-1)/(t-s)^2,\ \partial_t a_{0,h}(t,s)/(t-s),\ \partial_s a_{0,h}(t,s)/(t-s)\}_{0\leq s,t\leq T_0}\\ &\text{is bounded in } B^0_{\rho,\delta}(R^{2n})\text{”}\end{aligned} \tag{5.80}$$

Now we assume that for $a_{\nu,h}$ defined by (5.56)

$$\begin{aligned}&\text{“}\{a_{\nu,h}(t,s)/(t-s)^2,\ \partial_t a_{\nu,h}(t,s)/(t-s),\ \partial_s a_{\nu,h}(t,s)/(t-s)\}_{0\leq s,t\leq T_0}\\ &\text{is bounded in } B^0_{\rho,\delta}(R^{2n})\text{”}.\end{aligned} \tag{5.81}$$

Consider

$$\begin{aligned}(5.82)\qquad &\gamma_{\nu,h}(t,s;x,\xi)\\ &\equiv \sigma(H_h(t,X,D_x)a_{\nu,h}(\phi_h(t,s);t,s;X,D_x))\\ &= H_h(t,x,\nabla_x\phi_h)a_{\nu,h}+\mathcal{L}_h a_{\nu,h}-D_t a_{\nu,h}\\ &\quad+\sum_{k=1}^{N-1}h^{(\rho-\delta)k}b_{\nu,k,h}(t,s;x,\xi)\\ &\quad+h^{(\rho-\delta)N}c_{\nu,N,h}(t,s;x,\xi)\qquad(\nu\geqq 1)\,,\end{aligned}$$

where

$$(5.83)\qquad b_{\nu,k,h}(t,s;x,\xi)=b_{\nu,k}(t,s;h^{-\delta}x,h^{\rho}\xi)$$

for $b_{\nu,k}$ of (5.57) with $H_0=0$, and $c_{\nu,N,h}$ are the remainder terms.
Set

$$\begin{aligned}(5.84)\qquad &s_h(t,s;x,\xi,x',\xi')\\ &=H_h(t,x,\xi+\tilde{\nabla}_x J_h(t,s;x,\xi',x'))a_{\nu,h}(t,s;x',\xi')\,.\end{aligned}$$

Then by Proposition 5.6–2) we have

$$\begin{aligned}(5.85)\qquad &\gamma_{\nu,h}(t,s;x,\xi')\\ &=\mathrm{O_s}-\iint e^{-iy\cdot\eta}s_h(t,s;x,\xi'+\eta,x+y,\xi')\,đ\eta dy\,.\end{aligned}$$

In (5.85) we make Taylor's expansion. Then using (5.81) we see by (5.84) that

$$(5.86)\qquad \text{“}\{b_{\nu,k,h}(t,s)/(t-s)\}_{0\leqq s,t\leqq T_0}\text{ is bounded in } B^0_{\rho,\delta}(h)\quad (k=1,2,\cdots)\text{”}$$

and

$$(5.87)\qquad \text{“}\{c_{\nu,N,h}(t,s)/(t-s)\}_{0\leqq s,t\leqq T_0}\text{ is bounded in } B^0_{\rho,\delta}(h)\quad (N=1,2,\cdots)\text{”}.$$

Hence, by (5.80) we see that we obtain (5.86) and (5.87) for $\nu=0$. Then, by means of (5.64) we see that for $a_{\nu,h}$ defined by (5.56) the statement (5.81) holds with $\nu=1$. Consequently we obtain (5.81), (5.86), (5.87) for any $\nu=0,1,\cdots$.

Now we remind by Theorem 1.3 that $\tilde{e}_h(t,s;x,\xi)$ has the form

$$\begin{aligned}(5.88)\qquad &\tilde{e}_h(t,s;x,\xi)\\ &=\sum_{\nu=0}^{\infty}h^{(\rho-\delta)\nu}\chi(\varepsilon_\nu^{-1}h)a_{\nu,h}(t,s;x,\xi)\end{aligned}$$

for an appropriately decreasing sequence $\{\varepsilon_j\}_{j=0}^{\infty}$. Then, by (5.81) we get (5.76).

Now from (5.62), (5.82) and (5.86), (5.87) we see that

$$(5.89)\qquad \text{“}\{\sigma(\tilde{L}_h a_{\nu,h}(\phi_h(t,s);t,s;X,D_x))/(t-s)\}_{0\leqq s,t\leqq T_0}\text{ is bounded in } B^{\infty}_{\rho,\delta}(h)\text{”}.$$

Hence, in the similar discussion as in III) of the proof of Theorem 5.8 we can obtain (5.77). Q.E.D.

6. Fundamental solution

In this section, using the approximate fundamental solution, we construct the fundamental solution $U_h(t,s)$ for L_h, and derive the main properties of $U_h(t,s)$.

Theorem 6.1. *For a sufficiently small* $0<T_0\leqq T$ *there exists uniquely the fundamentl solution* $U_h(t,s)$ *in the class of Fourier integral operators with phase function* $\phi_h(t,s)$ *and symbols of class* $\mathscr{B}^1(I_{T_0}{}^2; B^0_{\rho,\delta}(h))$, *and there exist symbols* $d_{0,h}(t,s;x,\xi)\in\mathscr{B}^1(I_{T_0}{}^2; B^0_{\rho,\delta}(h))$ *in case* $0\leqq\delta\leqq\rho\leqq 1$ *and* $d_{\infty,h}(t,s;x,\xi)\in\mathscr{B}^1(I_{T_0}{}^2; B^{\infty}_{\rho,\delta}(h))$ *in case* $0\leqq\delta<\rho\leqq 1$ *such that for*

$$\begin{cases} D_{0,h}(\phi_h(t,s)) = d_{0,h}(\phi_h(t,s);t,s;X,D_x)\,, \\ D_{\infty,h}(\phi_h(t,s)) = d_{\infty,h}(\phi_h(t,s);t,s;X,D_x) \end{cases} \tag{6.1}$$

we can write

$$U_h(t,s) = I(\phi_h(t,s)) + D_{0,h}(\phi_h(t,s)) \qquad (0\leqq\delta\leqq\rho\leqq 1) \tag{6.2}$$

and

$$U_h(t,s) = E_h(\phi_h(t,s)) + D_{\infty,h}(\phi_h(t,s)) \qquad (0\leqq\delta<\rho\leqq 1)\,, \tag{6.3}$$

where $E_h(\phi_h(t,s))=e_h(\phi_h(t,s);t,s;X,D_x)$ *is the approximate fundamental solution of order infinity (given in* Theorem 5.8).

Furthermore, we have

$$\text{“}\{d_{0,h}(t,s)/(t-s)\}_{0\leqq s,t\leqq T_0} \text{ is bounded in } B^0_{\rho,\delta}(h)\text{”}. \tag{6.4}$$

and

$$\text{“}\{d_{\infty,h}(t,s)/(t-s)\}_{0\leqq s,t\leqq T_0} \text{ is bounded in } B^{\infty}_{\rho,\delta}(h)\text{”}. \tag{6.5}$$

Proof. We consider only the case $0\leqq\delta<\rho\leqq 1$ for $E_h(\phi_h(t,s))$. Then the case $0\leqq\delta\leqq\rho\leqq 1$ is proved similarly for $I(\phi_h(t,s))$.

I) Let $\tilde{l}=21n_0+1$ ($n_0>n$, even) and choose a sufficiently small $0<\tilde{\tau}<1$ such that Theorem 3.8 holds. Take a small $0<T_0\leqq T$ such that the constant $c_{\tilde{l}}$ of (5.8) in Proposition 5.3 satisfies

$$c_{\tilde{l}}T_0\leqq\tilde{\tau}\,. \tag{6.6}$$

Then, we see that for any subdivision Δ: $t\gtrless t_1\gtrless\cdots\gtrless t_\nu\gtrless s$ $(t,s\in[0,T_0])$

$$\phi_h(t,t_1)\#\phi_h(t_1,t_2)\#\cdots\#\phi_h(t_\nu,s)$$

is well defined. On the other hand we can easily see that

$$\phi_h(t,\theta)\#\phi_h(\theta,s)=\phi_h(t,s)\quad (t,s\in[0,T_0],\ t\gtrless\theta\gtrless s) \tag{6.7}$$

holds (cf. for example, the proof of Theorem 2.3 in [10]). Hence, we have

$$\begin{gathered}\phi_h(t,t_1)\#\phi_h(t_1,t_2)\#\cdots\#\phi_h(t_\nu,s)=\phi_h(t,s)\\ (t,s\in[0,T_0],\ t\gtrless t_1\gtrless\cdots t_\nu\gtrless s)\,.\end{gathered} \tag{6.8}$$

Now we define

$$\begin{aligned}W_{\nu,h}(\phi_h(t,s))=w_{\nu,h}(\phi_h(t,s);t,s;X,D_x)\\ (\nu=1,2,\cdots)\end{aligned} \tag{6.9}$$

by

$$W_{1,h}(\phi_h(t,s))=-iL_hE_h(\phi_h(t,s)) \tag{6.10}$$

and

$$\begin{aligned}W_{\nu+1,h}(\phi_h(t,s))&=\int_s^t W_{1,h}(\phi_h(t,\theta))W_{\nu,h}(\phi_h(\theta,s))d\theta\\ &=\int_s^t\int_s^{t_1}\cdots\int_s^{t_{\nu-1}}W_{1,h}(\phi_h(t,t_1))W_{1,h}(\phi_h(t_1,t_2))\cdots W_{1,h}(\phi_h(t_{\nu,s}))d\boldsymbol{t}_\nu\,,\end{aligned} \tag{6.11}$$

where $\boldsymbol{t}_\nu=(t_1,\cdots,t_\nu)$ and $d\boldsymbol{t}_\nu=dt_1\cdots dt_\nu$. Then we see by Theorem 5.8 that there exists a symbol

$$w_{1,h}(t,s;x,\xi)\in\mathscr{B}^0(I_{T_0}{}^2;B^\infty_{\rho,\delta}(h))$$

such that we have (6.9) with $\nu=1$.

Furthermore, we see from Theorem 4.3 and (6.8) that there exist symbols

$$\tilde{w}_{\nu+1,h}(t,\boldsymbol{t}_\nu,s;x,\xi)\in\mathscr{B}^0(\Omega_\nu;B^\infty_{\rho,\delta}(h)) \tag{6.12}$$

such that

$$\begin{aligned}&W_{1,h}(\phi_h(t,t_1))\cdots W_{1,h}(\phi_h(t_\nu,s))\\ &=\tilde{w}_{\nu+1,h}(\phi_h(t,s);t,\boldsymbol{t}_\nu,s;X,D_x)\,,\end{aligned} \tag{6.13}$$

where Ω_ν denotes the domain defined by

$$\Omega_\nu=\{(t,\boldsymbol{t}_\nu,s)\,|\,t,s\in[0,T_0],\ t\gtrless t_1\gtrless\cdots\gtrless t_\nu\gtrless s\}\,. \tag{6.14}$$

Hence, we see that there exists a symbol

$$w_{\nu,h}(t,s;x,\xi)\in\mathscr{B}^0(I_{T_0}{}^2;B^\infty_{\rho,\delta}(h))\quad(\nu=1,2,\cdots) \tag{6.15}$$

such that (6.9) holds for any ν.

II) Next we investigate the convergence of

$$d'_{\infty,h}(t,s;x,\xi)=\sum_{\nu=1}^{\infty}w_{\nu,h}(t,s;x,\xi)\,. \tag{6.16}$$

We note that for any $N\geqq 0$ we see that

$$\begin{aligned} w_{1,h}(t,s;x,\xi)&\in\mathscr{B}^0(I_{T_0}{}^2;B^N_{\rho,\delta}(h))\\ &\subset\mathscr{B}^0(I_{T_0}{}^2;B^0_{\rho,\delta}(h)) \end{aligned} \tag{6.17}$$

and that

$$|w_{1,h}(t,s)|_l^{(0)}\leqq|w_{1,h}(t,s)|_l^{(N)}\leqq\|w_{1,h}\|_l^{(N)}\,, \tag{6.18}$$

where $\|w_{1,h}\|_l^{(N)}=\max_{t,s\in[0,T_0]}|w_{1,h}(t,s)|_l^{(N)}$.

In (6.13) we regard $w_{1,h}(t_j,t_{j+1})$ as

$$\begin{cases} w_{1,h}(t,t_1;x,\xi)\in\mathscr{B}^0(I_{T_0}{}^2;B^N_{\rho,\delta}(h))\,,\\ w_{1,h}(t_j,t_{j+1};x,\xi)\in\mathscr{B}^0(I_{T_0}{}^2;B^0_{\rho,\delta}(h))\\ \qquad\qquad (j=1,\cdots,\nu,\ t_{\nu+1}=s)\,. \end{cases} \tag{6.19}$$

Then, noting (6.18) we have by Theorem 4.3

$$|\tilde{w}_{\nu+1,h}(t,t_\nu,s)|_l^{(N)}\leqq(C_l\|w_{1,h}\|_{l'}^{(N)})^{\nu+1} \tag{6.20}$$

for an integer l' and a constant C_l. Hence, noting that

$$w_{\nu+1,h}(t,s;x,\xi)=\int_s^t\int_s^{t_1}\cdots\int_s^{t_{\nu-1}}\tilde{w}_{\nu+1,h}(t,t_\nu,s;x,\xi)dt_\nu\,, \tag{6.21}$$

we obtain

$$\begin{aligned} |w_{\nu+1,h}(t,s)|_l^{(N)}&\leqq\frac{|t-s|^\nu}{\nu!}(C_l\|w_{1,h}\|_{l'}^{(N)})^{\nu+1}\\ &\leqq\frac{1}{\nu!}(T_0C_l\|w_{1,h}\|_{l'}^{(N)})^{\nu+1}\,, \end{aligned} \tag{6.22}$$

from which we get

$$\|w_{\nu+1,h}\|_l^{(N)}\leqq\frac{1}{\nu!}(T_0C_l\|w_{1,h}\|_{l'}^{(N)})^{\nu+1}\,. \tag{6.23}$$

Hence, we see that the series (6.16) converges in $\mathscr{B}^0(I_{T_0}{}^2;B^N_{\rho,\delta}(h))$ for any N, which means that the series (6.16) convergas in $\mathscr{B}^0(I_{T_0}{}^2;B^\infty_{\rho,\delta}(h))$.

III) Setting $D'_{\infty,h}(\phi_h(t,s))=d'_{\infty,h}(\phi_h(t,s);t,s;X,D_x)$, we define $D_{\infty,h}(\phi_h(t,s))$ by

$$D_{\infty,h}(\phi_h(t,s))=\int_s^t E_h(\phi_h(t,\theta))D'_{\infty,h}(\phi_h(\theta,s))d\theta \tag{6.24}$$

and consider (6.3). Then, noting Proposition 5.6, we have for $v\in\mathscr{S}$

$$\begin{aligned} L_hU_h(t,s)v&=L_hE_h(\phi_h(t,s))v\\ &-iD'_{\infty,h}(\phi_h(t,s))v+\int_s^t L_hE_h(\phi_h(t,\theta))D'_{\infty,h}(\phi_h(\theta,s))vd\theta\,. \end{aligned}$$

Then, by the definition (6.10), (6.16), (6.11) we have

$$L_hU_h(t, s)v = iW_{1,h}(\phi_h(t, s))v - iD'_{\infty,h}(\phi_h(t, s))v$$
$$+i\int_s^t W_{1,h}(\phi_h(t, \theta))D'_{\infty,h}(\phi_h(\theta, s))vd\theta = 0 ,$$

which means, together with $U_h(s,s)=I$, that $U_h(t,s)$ is the desired fundamental solution for L_h.

Replacing $E_h(\phi_h(t,s))$ by $I(\phi_h(t,s))$, we define $W_{\nu,h}(\phi_h(t,s))$ by (6.10) and (6.11). Then, fixing $N=0$ in II), we get the convergence of $d'_{\infty,h}(t,s;x,\xi)=\sum_{\nu=1}^{\infty} w_{\nu,h}(t,s;x,\xi)$ in $\mathcal{B}^0(I_{T_0}{}^2;B^0_{\rho,\delta}(h))$, and see that $U_h(t,s)$ defined by (6.2) is also the fundamental solution for L_h.

IV) Finally we prove the uniqueness of $U_h(t,s)$. Consider L_h^* defined by

$$L_h^* = D_t + K'_h(t, D_x, X') , \tag{6.25}$$

where $K'_h(t,\xi,x')$ is defined by

$$K'_h(t, \xi, x') = H_h(t, x', \xi) + \overline{\tilde{H}_h(t, x', \xi)} . \tag{6.26}$$

Then, we have

$$\int_{t_0}^{t_1}(L_hu, \tilde{u})dt = \int_{t_0}^{t_1}(u, L_h^*\tilde{u})dt \tag{6.27}$$

for $u, \tilde{u}\in\mathcal{B}^1([t_0,t_1];\mathcal{S})$ $(0\leqq t_0<t_1\leqq T_0)$ such that $u(t_0)\tilde{u}(t_0)=u(t_1)\tilde{u}(t_1)=0$. On the other hand, by (5.2) we see that there exists $\tilde{H}'_h(t,x,\xi)\in\mathcal{B}^0(I_T;B^0_{\rho,\delta}(h))$ such that

$$K'_h(t, D_x, X') = H_h(t, X, D_x) + \tilde{H}'_h(t, X, D_x) . \tag{6.28}$$

Then, by the existence part of the present theorem we can construct the fundamental solutlon $U_h^*(t,s)$ for L_h^* of the form (6.2).

Now assume that there exists another fundamentl solution $U'_h(t,s)$ in the class of Fourier integral operators. Set for $v\in\mathcal{S}$ and a fixed $s\in[0,T_0]$

$$u(t, s) = (U'_h(t, s) - U_h(t, s))v . \tag{6.29}$$

Then, we see that

$$L_hu(t, s) = 0 \quad \text{on} \quad [0, T_0], \quad u(s, s) = 0 . \tag{6.30}$$

Set

$$\tilde{u}(t, s) = i\int_{T_0}^t U_h^*(t, \theta)u(\theta, s)d\theta .$$

Then, we have

$$L_h^*\tilde{u}(t, s) = u(t, s) \quad \text{on} \quad [0, T_0], \quad \tilde{u}(T_0, s) = 0 . \tag{6.31}$$

Then, noting (6.27), we have by (6.30), (6.31)

$$(6.32)\qquad \begin{aligned} 0 &= \int_s^{T_0}(L_h u, \tilde{u})dt = \int_s^{T_0}(u, L_h^* \tilde{u})dt \\ &= \int_s^{T_0}(u(t, s), u(t, s))dt . \end{aligned}$$

Hence, we get $u(t,s)=0$ on $[s, T_0]$. Then, by (6.29) we have

$$(U_h'(t, s) - U_h(t, s))v = 0$$

for any $v \in \mathcal{S}$. From this we see that the symbols of $U_h'(t,s)$ and $U_h(t,s)$ coincide. Q.E.D.

The fundamental solution $U_h(t,s)$ of Theorem 6.1 has the following properties.

Theorem 6.2. *Let $U_h(t,s)$ be the fundamental solution constructed in* Theorem 6.1. *Then, we have the following*:

1) *The Cauchy problem*

$$(6.33)\qquad \begin{cases} L_h u = f(t) \in \mathcal{B}^0(I_{T_0}; \mathcal{S}) , \\ u|_{t=s} = v \in \mathcal{S} \quad (0 \leqq s \leqq T_0) \end{cases}$$

can be solved uniquely by

$$(6.34)\qquad u_h(t, s) = U_h(t, s)v + i\int_s^t U_h(t, \theta)f(\theta)d\theta \quad (\in \mathcal{B}^1(I_{T_0}; \mathcal{S})) .$$

2) *The following relations hold*:

$$(6.35)\qquad U_h(t, \theta)U_h(\theta, s) = U_h(t, s) ,$$

$$(6.36)\qquad D_s U_h(t, s) - U_h(t, s)K_h(s, X, D_x) = 0 .$$

Proof. 1) It is easy to see that $u_h(t,s)$ given by (6.34) satisfies (6.33). Now let $u_1(t,s) \in \mathcal{B}^1(I_{T_0}; \mathcal{S})$ satisfy

$$(6.37)\qquad \begin{cases} L_h u_1(t, s) = 0 \quad \text{on} \quad [0, T_0] , \\ u_1|_{t=s} = 0 . \end{cases}$$

Set

$$(6.38)\qquad \tilde{u}_1(t, s) = i\int_{T_0}^t U_h^*(t, \theta)u_1(\theta, s)d\theta .$$

Then, we have

$$L_h^* \tilde{u}_1(t, s) = u_1(t, s) \quad \text{on} \quad [0, T_0],\ \tilde{u}_1(T_0, s) = 0 .$$

Hence, we have

$$\begin{aligned}&\int_s^{T_0}(u_1(t, s), u_1(t, s))dt\\&=\int_s^{T_0}(u_1(t, s), L_h^*\tilde{u}_1(t, s))dt\\&=\int_s^{T_0}(L_h u_1(t, s), \tilde{u}_1(t, s))dt = 0 .\end{aligned}$$

So we have $u_1(t,s)=0$ on $[s, T_0]$. Replacing T_0 by 0 in (6.38) we get $u_1(t,s)=0$ on $[0,s]$. Hence, the uniqueness of the solution of (6.33) is proved.

2) For $v\in\mathcal{S}$ set

$$u(t, \theta, s) = U_h(t, \theta)U_h(\theta, s)v .$$

Then, we have

$$\text{(6.39)}\qquad \begin{cases}L_h u(t, \theta, s) = 0 \quad \text{on} \quad [0, T_0] ,\\ u(\theta, \theta, s) = U_h(\theta, s)v .\end{cases}$$

On the other hand consider $u(t,s)=U_h(t,s)v$. Then we have

$$\text{(6.40)}\qquad \begin{cases}L_h u(t, s) = 0 \quad \text{on} \quad [0, T_0] ,\\ u(\theta, s) = U_h(\theta, s)v .\end{cases}$$

Hence, by the uniqueness of the solutlon of the problem (6.33) we get $U_h(t,\theta)U_h(\theta,s)v=U_h(t,s)v$ for any $v\in\mathcal{S}$. So we get (6.35).

From (6.35) we have for $v\in\mathcal{S}$

$$\begin{aligned}0 &= D_\theta(U_h(t, \theta)U_h(\theta, s))v\\&= D_\theta U_h(t, \theta)\cdot U_h(\theta, s)v+U_h(t, \theta)\cdot D_\theta U_h(\theta, s)v\\&= D_\theta U_h(t, \theta)\cdot U_h(\theta, s)v-U_h(t, \theta)\cdot K_h(\theta, X, D_x)U_h(\theta, s)v .\end{aligned}$$

Hence, setting $\theta=s$, we get (6.36). Q.E.D.

For $\tilde{L}_h=D_t+H_h(t,X,D_x)$ we have

Theorem 6.3. *Let $\tilde{U}_h(t,s)$ be the fundamental solution for $\tilde{L}_h$, and let*

$$\text{(6.41)}\qquad \begin{cases}\tilde{D}_{0,h}(\phi_h(t, s)) = \tilde{d}_{0,h}(\phi_h(t, s); t, s; X, D_x) ,\\ \tilde{D}_{\infty,h}(\phi_h(t, s)) = \tilde{d}_{\infty,h}(\phi_h(t, s); t, s; X, D_x)\end{cases}$$

corresdond to $D_{0,h}(\phi_h(t,s))$, $D_{\infty,h}(\phi_h(t,s))$ in Theorem 6.1, *respectively. Then, we have*

$$\text{(6.42)}\qquad \begin{aligned}&\text{“}\{\tilde{d}_{0,h}(t, s)/(t-s)^2, \partial_t\tilde{d}_{0,t}(t, s)/(t-s), \partial_s\tilde{d}_{0,h}(t, s)/(t-s)\}_{0\leq s, t\leq T_0}\\&\textit{is bounded in } B^0_{\rho,\delta}(h)\text{”},\end{aligned}$$

and

(6.43) $$\text{“}\{\tilde{d}_{\infty,h}(t,s)/(t-s)^2,\ \partial_t\tilde{d}_{\infty,h}(t,s)/(t-s),\ \partial_s\tilde{d}_{\infty,h}(t,s)/(t-s)\}_{0\leqq s,t\leqq T_0}$$ *is bounded in* $B^{\infty}_{\rho,\delta}(h)$”.

Proof. In II) of the proof of Theorem 6.1 we have from Theorem 5.9 that

$$|w_{1,h}(t,t_1)|_l^{(N)}\leqq C_l'|t-t_1|\ (t,t_1\in[0,T_0])$$

for the case $\tilde{d}_{0,h}(t,s)$ with $N=0$ and for the case $\tilde{d}_{\infty,h}(t,s)$ with any $N\geqq 0$. Then we can get (6.42) and (6.43). Q.E.D.

In what follows we investigate the L^2-properties of $U_h(t,s)$. Let $H_0=L^2(R^n)$ and let H_2 denote the Hilbert space obtained as the completion of $\mathcal{S}$ with respect to the norm

$$\|v\|_2=\{\sum_{|\alpha+\beta|\leqq 2}\|x^\alpha D_x^\beta v(x)\|^2_{L^2(R^n)}\}^{1/2}. \tag{6.44}$$

We denote H_2 by $H_{2,h}$, when $\|v\|_2$ is replaced by an equivalent norm

$$\|v\|_{2,h}=\{\sum_{|\alpha+\beta|\leqq 2}\|(h^{-\delta}x)^\alpha(h^\rho D_x)^\beta v(x)\|^2_{L^2(R^n)}\}^{1/2}, \tag{6.45}$$

and similarly H_0 by $H_{0,h}$ when $\|v\|_{L^2(R^n)}$ is replaced by $h^{\rho-\delta}\|v\|_{L^2(R^n)}$. We often write $\|v\|_0=\|v\|_{L^2(R^n)}$.

Proposition 6.4. *Let* $S_{l,h}$ *for real* l *be pseudo-differential operators with symbols* $s_{l,h}(x,\xi)$ *such that*

$$|s_{l,h}{}^{(\alpha)}_{(\beta)}(x,\xi)|\leqq C_{\alpha,\beta,l}h^{\rho|\alpha|-\delta|\beta|}\langle h^{-\delta}x;\ h^\rho\xi\rangle^{l-|\alpha+\beta|}, \tag{6.46}$$

and let $\phi_h(x,\xi)\in P_{\rho,\delta}(\tilde{\tau},\tilde{l};h)$ *for* $\tilde{\tau},\tilde{l}$ *of* Theorem 3.8. *Then, for any* $P_h(\phi_h)=p_h(\phi_h;X,D_x)\in \boldsymbol{B}^m_{\rho,\delta}(\phi_h)$ *we have*

$$S_{-l,h}P_h(\phi_h)S_{l,h}\in \boldsymbol{B}^m_{\rho,\delta}(\phi_h). \tag{6.47}$$

Proof. Consider $P_h(\phi_h)S_{l,h}$ for $l\geqq 0$. Then, by Proposition 5.6 we see that there exists $r_{l,h}(x,\xi)\in B^{m;l}_{\rho,\delta}(h)$ such that

$$P_h(\phi_h)S_{l,h}=R_{l,h}(\phi_h)\equiv r_{l,h}(\phi_h;X,D_x). \tag{6.48}$$

Now, by Theorem 3.8 we write $I=R_h(\phi_h^*)I(\phi_h)$, and by (6.48) write

$$P_h(\phi_h)S_{l,h}=(R_{l,h}(\phi_h)R_h(\phi_h^*))I(\phi_h). \tag{6.49}$$

Then, as in Theorem 3.5-i), setting

$$\begin{aligned} q_{l,h}(x,\xi,x') &= r_{l,h}(x,\tilde{\nabla}_x\phi_h^{-1}(x,\xi,x')) \\ &\times r_h(\tilde{\nabla}_x\phi_h^{-1}(x,\xi,x'),x')\left|\frac{D(\tilde{\nabla}_x\phi_h^{-1})}{D(\xi)}(x,\xi,x')\right| \end{aligned} \tag{6.50}$$

and

$$\gamma_{l,h}(x,\xi) = \mathrm{O}_s\text{-}\iint e^{-iy\cdot\eta} q_{l,h}(x,\xi+\eta,x+y)đ\eta dy\,, \tag{6.51}$$

we can write

$$R_{l,h}(\phi_h)R_h(\phi_h^*) = \gamma_{l,h}(X,D_x)\,, \tag{6.52}$$

where $r_h(\xi,x')$ is the symbol of $R_h(\phi_h^*)$.

Then, noting $r_{l,h}(x,\xi)\in B^{m,l}_{\rho,\delta}(h)$ we see that $q_{l,h}(x,\xi,x')$ satisfies

$$|q_{l,h(\beta,\beta')}^{(\alpha)}(x,\xi,x')| \leqq C_{\alpha,\beta,\beta'} h^{m+\rho|\alpha|-\delta|\beta+\beta'|}\langle h^{-\delta}x;\,h^{\rho}\xi;\,h^{-\delta}x'\rangle^{l}\,, \tag{6.53}$$

where $\langle x;\xi;x'\rangle=(1+|x|^2+|\xi|^2+|x'|^2)^{1/2}$.

Then, setting

$$\begin{cases}\tilde q_{l,h}(x,\xi,x') = q_{l,h}(h^{\delta}x,h^{-\rho}\xi,h^{\delta}x')\,,\\ \tilde\gamma_{l,h}(x,\xi) = \gamma_{l,h}(h^{\delta}x,h^{-\rho}\xi)\,,\end{cases}$$

and making a change of variables $y=h^{\delta}\tilde y$, $\eta=h^{-\delta}\tilde\eta$, we can write

$$\tilde\gamma_{l,h}(x,\xi) = \mathrm{O}_s\text{-}\iint e^{-i\tilde y\cdot\tilde\eta}\tilde q_{l,h}(x,\xi+h^{\rho-\delta}\tilde\eta,x+\tilde y)đ\tilde\eta d\tilde y\,. \tag{6.54}$$

Furthermore, we have by (6.53)

$$|\tilde q_{l,h(\beta,\beta')}^{(\alpha)}(x,\xi,x')| \leqq C_{\alpha,\beta,\beta'} h^{m}\langle x;\,\xi;\,x'\rangle^{l}\,.$$

Then, from (6.54) we see that

$$|\tilde\gamma_{l,h(\beta)}^{(\alpha)}(x,\xi)| \leqq C_{\alpha,\beta} h^{m}\langle x;\,\xi\rangle^{l}\,,$$

from which we obtain

$$|\gamma_{l,h(\beta)}^{(\alpha)}(x,\xi)| \leqq C_{\alpha,\beta} h^{m+\rho|\alpha|-\delta|\beta|}\langle h^{-\delta}x;\,h^{\rho}\xi\rangle^{l}\,. \tag{6.55}$$

Finally by (6.49), (6.52) we write

$$S_{-l,h}P_h(\phi_h)S_{l,h} = (S_{-l,h}\gamma_{l,h}(X,D_x))I(\phi_h)\,. \tag{6.56}$$

We define $\mu_{l,h}(x,\xi)$ by

$$\mu_{l,h}(x,\xi) = \mathrm{O}_s\text{-}\iint e^{-iy\cdot\eta}s_{-l,h}(x,\xi+\eta)\gamma_{l,h}(x+y,\xi)đ\eta dy \tag{6.57}$$

Then, we have

$$S_{-l,h}\gamma_{l,h}(X,D_x) = \mu_{l,h}(X,D_x)\,, \tag{6.58}$$

and, noting (6.46) for $-l$ and (6.55), we have, in the same way as the method to get (6.55), that $\mu_{l,h}(x,\xi)\in B^{m}_{\rho,\delta}(h)$. Hence, using Theorem 3.7, we see from

(6.56), (6.58) that (6.47) holds with $l \geqq 0$.

When $l \leqq 0$, we write

$$S_{-l,h}P_h(\phi_h)S_{l,h} \\ = I(\phi_h)((R_h(\phi_h^*)(S_{-l,h}P_h(\phi_h)))S_{l,h}).$$

Then, we get (6.47) with $l \leqq 0$. Q.E.D.

Proposition 6.5. 1) *Let $T_h^{\pm}$ be peudo-differential operators with symbols*

$$t_h^+(x,\xi) = \langle h^{-\delta}x; h^\rho\xi\rangle^2, \quad t_h^-(x,\xi) = \langle h^{-\delta}x; h^\rho\xi\rangle^{-2}. \tag{6.59}$$

Then, there exist pseudo-differential operators $R_{j,h}(j=1,2)$ with symbols $r_{j,h}(x,\xi)$ satisfying

$$|r_{j,h(\beta)}^{(\alpha)}(x,\xi)| \leqq C_{\alpha,\beta}h^{\rho(|\alpha|+1)-\delta(|\beta|+1)}\langle h^{-\delta}x; h^\rho\xi\rangle^{-2-|\alpha+\beta|} \quad (j=1,2) \tag{6.60}$$

such that

$$T_h^+T_h^- = I+R_{1,h},\ T_h^-T_h^+ = I+R_{2,h}. \tag{6.61}$$

Furthermore, we have for a constant $C>0$

$$C^{-1}||T_h^+v||_0 \leqq ||v||_{2,h} \leqq C(||T_h^+v||_0+||v||_0)\ (v\in H_{2,h}). \tag{6.62}$$

2) *Let $\phi_h(x,\xi)\in P_{\rho,\delta}(\tilde{\tau},\tilde{l};h)$ with $\tilde{l},\tilde{\tau}$ of* Theorem 3.8 *and let $P_h(\phi_h)=p_h(\phi_h; X,D_x)\in B_{\rho,\delta}^m(\phi_h)$. Then, we see that $P_h(\phi_h)\colon H_{2,h}\to H_{2,h}$ is continuous and for a constant $C>0$*

$$||P_h(\phi_h)v||_{2,h} \leqq Ch^m||v||_{2,h} \quad \text{for } v\in H_{2,h}. \tag{6.63}$$

Proof. 1) We define $r_{1,h}(x,\xi)$ by

$$r_{1,h}(x,\xi) = \sum_{j=1}^n \int_0^1 \{\mathrm{O}_s - \iint e^{-iy\cdot\eta} \\ \times t_h^{+(j)}(x,\xi+\theta\eta)t_{h(j)}^-(x+y,\xi)d\eta dy\} d\theta. \tag{6.64}$$

Then, by the usual expansion formula of order 1 for $\sigma(T_h^+T_h^-)(x,\xi)$ we see that we can write $T_h^+T_h^- = I+R_{1,h}$. Furthermore, from (6.64) we obtain (6.60) for $j=1$. Similarly we get (6.60), (6.61) for $j=2$.

Now by definition we have for $v\in\mathcal{S}$

$$T_h^+v = \int e^{ix\cdot\xi}(1+|h^{-\delta}x|^2+|h^\rho\xi|^2)\hat{v}(\xi)d\xi.$$

Thus, by Theorem 1.12 we obtain

$$||T_h^+v||_0 \leqq C(||v||_0+|||h^{-\delta}x|^2v||_0+|||h^\rho D_x|^2v||_0) \\ \leqq C||v||_{2,h} \qquad \text{for } v\in\mathcal{S}. \tag{6.65}$$

On the other hand we write for $|\alpha+\beta|\leqq 2$

$$(h^{-\delta}x)^{\alpha}(h^{\rho}D_x)^{\beta}v$$
$$=(h^{-\delta}x)^{\alpha}(h^{\rho}D_x)T_h^{-}\cdot T_h^{+}v-(h^{-\delta}x)^{\alpha}(h^{\rho}D_x)^{\beta}R_{2,h}v\,.$$

Then, noting

$$\sigma((h^{-\delta}x)^{\alpha}(h^{\rho}D_x)^{\beta}T_h^{-}),\ \sigma((h^{-\delta}x)^{\alpha}(h^{\rho}D_x)^{\beta}R_{2,h})\in B^0_{\rho,\delta}(h)\,,$$

we get again by Theorem 1.12

$$\|(h^{-\delta}x)^{\alpha}(h^{\rho}D_x)^{\beta}v\|_0\leqq C'(\|T_h^{+}v\|_0+\|v\|_0)\quad(|\alpha+\beta|\leqq 2)\,,$$

and get

$$\|v\|_{2,h}\leqq C''(\|T_h^{+}v\|_0+\|v\|_0)\quad\text{for}\ \ v\in\mathcal{S}\,. \tag{6.66}$$

Hence, from (6.65) and (6.66) we obtain (6.62).

2) By (6.61) we write for $v\in\mathcal{S}$

$$T_h^{+}P_h(\phi_h)v=T_h^{+}P_h(\phi_h)\,(T_h^{-}\cdot T_h^{+}-R_{2,h})v$$
$$=(T_h^{+}P_h(\phi_h)T_h^{-})\,(T_h^{+}v)-(T_h^{+}P_h(\phi_h)R_{2,h})v\,.$$

Then, from (5.34) with $l=\pm 2$, (6.60) and Proposition 6.4 we have

$$\|T_h^{+}P_h(\phi_h)v\|_0\leqq Ch^{m}(\|T_h^{+}v\|_0+\|v\|_0)\,.$$

Hence, by (6.62) we get (6.63). Q.E.D.

We have finally the following

Theorem 6.6. *Let $U_h(t,s)$ be the fundamental solution for L_h given by* Theorem 6.1. *Then*:

1) *The operators $U_h(t,s)\colon H_0\to H_0$, $H_{2,h}\to H_{2,h}$ are uniformly bounded in $(t,s,h)\in[0,T_0]^2\times(0,1)$.*

2) *$K_h(t,X,D_x)U_h(t,s)$, $U_h(t,s)K_h(t,X,D_x)$, $D_tU_h(t,s)$, $D_sU_h(t,s)\colon H_{2,h}\to H_{0,h}$ are uniformly bounded on $[0,T_0]^2\times(0,1)$.*

3) *As an operator: $H_0\to H_0$ and $H_{2,h}\to H_{2,h}$ we have*

$$U_h(t,\theta)U_h(\theta,s)=U_h(t,s)\quad(t,\theta,s\in[0,T_0])\,. \tag{6.67}$$

4) *As an operator: $H_{2,h}\to H_{0,h}$ we have*

$$\begin{cases}L_hU_h(t,s)=0 & \textit{on}\ \ [0,T_0]^2\,,\\ U_h(s,s)=I & \textit{on}\ \ [0,T_0]\end{cases} \tag{6.68}$$

and

$$D_sU_h(t,s)-U_h(t,s)K_h(s,X,D_x)=0\quad\textit{on}\ \ [0,T_0]\,. \tag{6.69}$$

5) *The Cauchy problem*

$$\text{(6.70)} \qquad \begin{cases} L_h u = f(t) \in \mathscr{B}^0(I_{T_0}; H_0)\,, \\ u|_{t=s} = v \in H_{2,h} \end{cases}$$

has a unique solution $u(t,s)$ *in* $\mathscr{D}^0(I_{T_0}; H_{2,h}) \cap \mathscr{B}^1(I_{T_0}; H_{0,h})$, *represented by*

$$\text{(6.71)} \qquad u(t,s) = U_h(t,s)v + i\int_s^t U_h(\theta,s) f(\theta) d\theta\,.$$

Proof. 1) is easy by Proposition 6.5–2).

2) We note that $K_h(t,x,\xi) = h^{\delta-\rho} H(t, h^{-\delta}x, h^\rho \xi) + \tilde{H}_h(t,x,\xi)$ satisfies

$$\text{(6.72)} \qquad \text{“}\{K_{h(\beta)}^{(\alpha)}(t,x,\xi)\langle h^{-\delta}x; h^\rho\xi\rangle^{|\alpha+\beta|-2}\}_{0\leqq t\leqq T_0} \text{ is bounded in } B_{\rho,\delta}^{\delta-\rho+\rho|\alpha|-\delta|\beta|}(h) \text{ for } |\alpha+\beta|\leqq 2\text{”},$$

and by (6.61)

$$\text{(6.73)} \qquad \begin{aligned} &K_h(t,X,D_x) U_h(t,s) \\ &= (K_h(t) T_h^-)(T_h^+ U_h(t,s)) - (K_h(t) R_{2,h}) U_h(t,s)\,. \end{aligned}$$

Then, noting by (5.31) and (6.60)

$$K_h(t) T_h^-,\ K_h(t) R_{2,h} \in \boldsymbol{B}_{\rho,\delta}^{\delta-\rho}(h)\,,$$

we see by Proposition 6.5 that $K_h(t) U_h(t,s)\colon H_{2,h} \to H_{0,h}$ is uniformly bounded on $[0,T_0]^2 \times (0,1)$. Similarly we get 2) for $U_h(t,s) K_h(t,X,D_x)$.

3) is clear from Proposition 6.5.

4) holds for $\mathscr{S}$ by Theorem 6.2. Then, for $v \in H_{2,h}$, choosing $\{v_j\}_{j=1}^\infty \subset \mathscr{S}$ such that $v_j \to v$ in $H_{2,h}$ we get 4) for $v \in H_{2,h}$. Then 2) for $D_t U_h(t,s)$, $D_s U_h(t,s)$ can be easily obtained.

5) is clear from Theorem 6.2 and 1)–4). Q.E.D.

Corollary. *Let* $K_h(t,X,D_x)$ *be symmetric. Then, we have that* $U_h(t,s)$: $H_0 \to H_0$ *is unitary, and have*

$$\text{(6.74)} \qquad U_h(t,s)^* U_h(t,s) = I \quad \textit{on } H_0\,.$$

Proof. For $v \in \mathscr{S}$ we have by Theorem 6.2

$$\begin{aligned} &\partial_t(U_h(t,s)v,\ U_h(t,s)v) \\ &= (\partial_t U_h(t,s)v,\ U_h(t,s)v) + (U_h(t,s)v,\ \partial_t U_h(t,s)v) \\ &= -(iK_h(t) U_h(t,s)v,\ U_h(t,s)v) - (U_h(t,s)v,\ iK_h(t) U_h(t,s)v) = 0\,. \end{aligned}$$

So we have

$$\begin{aligned} (U_h(t,s)v,\ U_h(t,s)v) &= (U_h(s,s)v,\ U_h(s,s)v) \\ &= (v,v)\,. \end{aligned}$$

Hence, we have (6.74) on $\mathscr{S}$, and using Theorem 6.6–1) we get (6.74) on H_0. Q.E.D.

Now, we consider the case $K_h(t,X,D_x)=K_h(X,D_x)$ (independent of t). Then, setting

$$(6.75) \qquad U_h(t)=U_h(t,0)\ (0\leqq t\leqq T_0),=U_h(0,-t)\ (-T_0\leqq t\leqq 0)\,,$$

we get

$$(6.76) \qquad U_h(t,s)=U_h(t-s)\,.$$

For $K_h(X,D_x)$ we define the domain $\mathscr{D}(K_h)$ of $K_h(X,D_x)$ by

$$(6.77) \qquad \mathscr{D}(K_h)=\{v\in H_0\,|\,K_hv\in H_0\}(\subset H_0)\,,$$

where $K_hv\in H_0$ means that, for some $\{v_j\}_{j=1}^{\infty}\subset\mathscr{S}$ satisfying $v_j\to v$ in H_0, K_hv_j converges to some w in H_0 (then we define $K_hv=w$). Let $U_h^*(t)$ be the fundamental solution for L_h^*, and let $\mathscr{D}(K_h^*)$ be defined similarly, where $K_h^*\equiv K_h'(D_x,X')$ (see (6.25)). Then, we have

Theorem 6.7. 1) *Let K_h and K_h^* be considered as the closed operators*

$$(6.78) \qquad \begin{cases} K_h\colon (H_0\supset)\mathscr{D}(K_h)\to H_0\,, \\ K_h^*\colon (H_0\supset)\mathscr{D}(K_h^*)\to H_0\,. \end{cases}$$

Then, we have that K_h^* is the adjoint operator of K_h and have

$$(6.79) \qquad (K_hv,w)=(v,K_h^*w) \quad (v\in\mathscr{D}(K_h),\ w\in\mathscr{D}(K_h^*))\,.$$

2) *If $v\in\mathscr{D}(K_h)$, then we have*

$$(6.80) \qquad U_h(t)v\in\mathscr{D}(K_h)$$

and have

$$(6.81) \qquad K_hU_h(t)v=U_h(t)K_hv,\quad v\in\mathscr{D}(K_h)\,.$$

This holds also for K_h^* and $U_h^*(t)$.

Corollary. *If K_h is symmetric, then $K_h\colon (H_0\supset)\mathscr{D}(K_h)\to H_0$ is self-adjoint.*

Proof of Theorem 6.7. 1) The closedness of K_h and K_h^* is clear. For $v\in H_0$ assume that there exists $\tilde{v}\in H_0$ such that

$$(6.82) \qquad (v,K_h^*w)=(\tilde{v},w) \quad \text{for } w\in\mathscr{D}(K_h^*)\,.$$

Since $\mathscr{S}\subset\mathscr{D}(K_h^*)$, we have (6.82) for $w\in\mathscr{S}$. Hence $K_hv=\tilde{v}$, which means that $\mathscr{D}(K_h)\supset\mathscr{D}((K_h^*)^*)$.

Now assume that $v\in\mathscr{D}(K_h)$. Then, noting Theorem 6.6–2) and choosing $\{w_j\}_{j=1}^{\infty}\subset\mathscr{S}$ so that $w_j\to w\in H_{2,h}$ in $H_{2,h}$, we see that

$$(v, K_h^* w) = (K_h v, w), \quad w \in H_{2,h}\,. \tag{6.83}$$

Now, let $T_{\varepsilon,h}$ be a pseudo-differential operator with symbol $t_{\varepsilon,h}(x,\xi)=\{1+\varepsilon\langle h^{-\delta}x;h^{\rho}\xi\rangle^2\}^{-1}(0<\varepsilon<1)$. Then, it is easy to see that $t_{\varepsilon,h}(x,\xi)$ satisfies

$$|t_{\varepsilon,h(\beta)}^{(\alpha)}(x,\xi)| \leqq C_{\alpha,\beta} h^{\rho|\alpha|-\delta|\beta|}\langle h^{-\delta}x;\, h^{\rho}\xi\rangle^{-|\alpha+\beta|}\,, \tag{6.84}$$

and for any fixed $0<h<1$

$$T_{\varepsilon,h}\tilde{w} \to \tilde{w} \quad (\varepsilon \downarrow 0) \ \text{in} \ H_0 \ \text{for} \ \tilde{w}\in H_0\,. \tag{6.85}$$

Then, noting $T_{\varepsilon,h}w \in H_{2,h}$ for $w\in \mathcal{D}(K_h^*)\subset H_0$, we have by (6.83)

$$(v, K_h^* T_{\varepsilon,h} w) = (K_h v, T_{\varepsilon,h} w) \quad (w\in\mathcal{D}(K_h^*))\,. \tag{6.86}$$

Hence from (6.86) we can write

$$(v, T_{\varepsilon,h} K_h^* w) + (v, [K_h^*, T_{\varepsilon,h}]w) = (K_h v, T_{\varepsilon,h} w)\,, \tag{6.87}$$

where $[K_h^*, T_{\varepsilon,h}] = K_h^* T_{\varepsilon,h} - T_{\varepsilon,h} K_h^*$. Then for $\gamma_{\varepsilon,h}=\sigma([K_h^*,T_{\varepsilon,h}])=\sigma(K_h^* T_{\varepsilon,h})-\sigma(T_{\varepsilon,h}K_h^*)$, by Taylor's expansion of order 1, we see that $\gamma_{\varepsilon,h}(x,\xi)$ satisfies

$$|\gamma_{\varepsilon,h(\beta)}^{(\alpha)}(x,\xi)| \leqq C_{\alpha,\beta} h^{\rho|\alpha|-\delta|\beta|}\langle h^{-\delta}x;\, h^{\rho}\xi\rangle^{-|\alpha+\beta|}\,, \tag{6.88}$$

and get

$$[K_h^*, T_{\varepsilon,h}]w \to 0 \ (\varepsilon\downarrow 0) \ \text{in} \ H_0\,. \tag{6.89}$$

Then, from (6.85), (6.87) and (6.89), letting $\varepsilon \downarrow 0$, we have

$$(v, K_h^* w) = (K_h v, w) \ \text{for} \ w = \mathcal{D}(K_h^*)\,, \tag{6.90}$$

which means that $v\in\mathcal{D}((K_h^*)^*)$. Hence, we get $(K_h^*)^*=K_h$.

2) From Theorem 6.6–(6.68), (6.69), and (6.75) we have

$$K_h U_h(t) v = U_h(t) K_h v \ \text{for} \ v\in H_{2,h}\,. \tag{6.91}$$

Then, using $T_{\varepsilon,h}$, for $v\in\mathcal{D}(K_h)$ we can write

$$\begin{aligned} K_h U_h(t) T_{\varepsilon,h} v &= U_h(t) K_h T_{\varepsilon,h} v \\ &= U_h(t) T_{\varepsilon,h} K_h v + U_h(t)\,[K_h, T_{\varepsilon,h}]v\,. \end{aligned} \tag{6.92}$$

Then, setting $w_{\varepsilon,h}=U_h(t)T_{\varepsilon,h}v$, we have

$$\begin{cases} w_{\varepsilon,h} \to U_h(t)v \\ K_h w_{\varepsilon,h} \to U_h(t) K_h v, \quad (\varepsilon\downarrow 0) \end{cases}$$

which means that $U_h(t)v\in\mathcal{D}(K_h)$ and $K_h U_h(t) v = U_h(t) K_h v$. Q.E.D.

Finally we consider the convergence of the iterated integral of Feynman's type. Let $\tilde{U}_h(t,s)$ be the fundamental solution for $\tilde{L}_h = D_t + H_h(t,X,D_x)$ and let $\tilde{E}_h(\phi_h(t,s))$ be the approximate fundamental solution of order infinity. Let $\tilde{D}_{0,h}$, $\tilde{D}_{\infty,h}$ be as given in Theorem 6.3.

Now for a subdivision Δ_ν for $t,s\in[0,T_0]$ defined by

(6.93) $$\Delta_\nu;\ t \geqq t_1 \geqq t_2 \geqq \cdots \geqq t_\nu \geqq s\,,$$

we set

(6.94) $$I(\Delta_\nu;\ \phi_h(t,s)) = I(\phi_h(t,t_1))I(\phi_h(t_1,t_2)) \cdots \cdots I(\phi_h(t_\nu,s))$$

and

(6.95) $$\tilde{E}_h(\Delta_\nu;\ \phi_h(t,s)) = \tilde{E}_h(\phi_h(t,t_1))\tilde{E}_h(\phi_h(t_1,t_2)) \cdots \cdots \tilde{E}_h(\phi_h(t_\nu,s))\,.$$

Then, by Theorem 4.3 we see that there exist symbols

(6.96) $$\tilde{e}_{0,h}(t,\boldsymbol{t}_\nu,s;x,\xi),\ \tilde{e}_{\infty,h}(t,\boldsymbol{t}_\nu,s;x,\xi) \in \mathscr{B}^1(\Omega_\nu;\ B^0_{\rho,\delta}(h))$$

such that

(6.97) $$\begin{cases} I(\Delta_\nu;\ \phi_h(t,s)) = \tilde{e}_{0,h}(\phi_h(t,s);\,t,\boldsymbol{t}_\nu,s;\,X,D_x)\,, \\ \tilde{E}_h(\Delta_\nu;\ \phi_h(t,s)) = \tilde{e}_{\infty,h}(\phi_h(t,s);\,t,\boldsymbol{t}_\nu,s;\,X,D_x)\,, \end{cases}$$

where Ω_ν is defined by

(6.98) $$\Omega_\nu = \{(t,\boldsymbol{t}_\nu,s)\,|\,t \geqq t_1 \geqq t_2 \geqq \cdots \geqq t_\nu \geqq s\} \quad (t,s \in [0,T_0])\,.$$

Then, we have

Theorem 6.8. *Let $\tilde{u}_h(t,s;x,\xi)$ be the symbol of Fourier integral operator $\tilde{U}_h(t,s)$, that is,*

(6.99) $$\tilde{u}_h(t,s;x,\xi) = 1+\tilde{d}_{0,h}(t,s;x,\xi) = \tilde{e}_h(t,s;x,\xi)+\tilde{d}_{\infty,h}(t,s;x,\xi)\,.$$

Then, we have

(6.100) $$\text{“}\left\{\frac{\tilde{e}_{0,h}(t,\boldsymbol{t}_\nu,s;x,\xi)-\tilde{u}_h(t,s;x,\xi)}{|\Delta_\nu|}\right\}_{\Delta_\nu,0\leqq s,t\leqq T_0} \text{ is bounded in } B^0_{\rho,\delta}(h)\text{”},$$

and

(6.101) $$\text{“}\left\{\frac{\tilde{e}_{\infty,h}(t,\boldsymbol{t}_\nu,s;x,\xi)-\tilde{u}_h(t,s;x,\xi)}{|\Delta_\nu|}\right\}_{\Delta_\nu,0\leqq s,t\leqq T_0} \text{ is bounded in } B^\infty_{\rho,\delta}(h)\text{”},$$

where $|\Delta_\nu| = \max\limits_{1\leqq j\leqq \nu+1} |t_j - t_{j-1}|\,(t_0=t,\ t_{\nu+1}=s)$,

Corollary. *We have for $v \in H_0$*

(6.102) $$\|(I(\Delta_\nu;\ \phi_h(t,s)) - \tilde{U}_h(t,s))v\|_0 \leqq C\,|\Delta_\nu|\,\|v\|_0$$

and

(6.103) $$\|(\tilde{E}_h(\Delta_\nu;\phi_h(t,s))-\tilde{U}_h(t,s))v\|_0\leqq C_N h^N|\Delta_\nu|\,\|v\|_0$$

for any N.

Proof of Theorem 6.8. We only prove (6.101). Then (6.100) will be proved more easily.

By Theorem 6.3 we can write

(6.104) $$\tilde{E}_h(\phi_h(t,s))=\tilde{U}_h(t,s)-\tilde{D}_{\infty,h}(\phi_h(t,s)).$$

Hence, by (6.94) we can write

(6.105) $$\begin{aligned}\tilde{E}_h(\Delta_\nu;\phi_h(t,s))&=(\tilde{U}_h(t,t_1)-\tilde{D}_{\infty,h}(\phi_h(t,t_1)))\\&\times(\tilde{U}_h(t_1,t_2)-\tilde{D}_{\infty,h}(\phi_h(t_1,t_2)))\cdots\\&\cdots(\tilde{U}_h(t_\nu,s)-\tilde{D}_{\infty,h}(\phi_h(t_\nu,s))).\end{aligned}$$

Then, using the group property of $\tilde{U}_h(t,s)$ we can write

(6.106) $$\begin{aligned}&\tilde{E}_h(\Delta_\nu;\phi_h(t,s))\\&=\tilde{U}_h(t,s)+\sum_{j=1}^{\nu+1}(-1)^j\Gamma^{(\nu)}_{j,h}(\Delta_\nu;\phi_h(t,s)),\end{aligned}$$

where

(6.107) $$\begin{aligned}&\Gamma^{(\nu)}_{j,h}(\Delta_\nu;\phi_h(t,s))\\&=\sum_{0\leqq k_1<\cdots<k_j\leqq\nu}\tilde{U}_h(t,t_{k_1})\tilde{D}_{\infty,h}(\phi_h(t_{k_1},t_{k_1+1}))\\&\quad\times\tilde{U}_h(t_{k_1+1},t_{k_2})\tilde{D}_{\infty,h}(\phi_h(t_{k_2},t_{k_2+1}))\cdots\\&\quad\cdots\tilde{D}_{\infty,h}(\phi_h(t_{k_j},t_{k_j+1}))\tilde{U}_h(t_{k_j+1},s).\end{aligned}$$

Now from Theorem 5.9 and Theorem 6.3 we have

(6.108) $$|\tilde{e}_h(t,s)|^{(0)}_l\leqq C_l\quad\text{on}\quad[0,T_0]^2$$

and for any $N\geqq 0$

(6.109) $$\begin{aligned}|\tilde{d}_{\infty,h}(t,s)|^{(0)}_l&\leqq|\tilde{d}_{\infty,h}(t,s)|^{(N)}_l\\&\leqq C_{l,N}|t-s|^2\quad\text{on}\quad[0,T_0]^2.\end{aligned}$$

Then, regarding $\tilde{d}_{\infty,h}(t,s)$ as

$$\begin{cases}\tilde{d}_{\infty,h}(t_{k_1},t_{k_1+1})\in B^N_{\rho,\delta}(h),\\ \tilde{d}_{\infty,h}(t_{k_j},t_{k_j+1})\in B^0_{\rho,\delta}(h)\qquad(j=2,\cdots,\nu),\end{cases}$$

we see from (6.108), (6.109) and Theorem 4.3 that

(6.110) $$\begin{aligned}&|\sigma(\Gamma^{(\nu)}_{j,h})|^{(N)}_l\\&\leqq\sum_{0\leqq k_1<\cdots<k_j\leqq\nu}\tilde{C}^{2j}_{l,N}|t_{k_1}-t_{k_1+1}|^2|t_{k_2}-t_{k_2+1}|^2\cdots|t_{k_j}-t_{k_j+1}|^2\\&\leqq\tilde{C}^{2j}_{l,N}|\Delta_\nu|^j\sum_{0\leqq k_1<\cdots<k_j\leqq\nu}|t_{k_1}-t_{k_1+1}|\,|t_{k_2}-t_{k_2+1}|\cdots|t_{k_j}-t_{k_j+1}|\\&\leqq\tilde{C}^{2j}_{l,N}|\Delta_\nu|^jT_0^j.\end{aligned}$$

Hence, we see for a small $0<T_0\leqq T$ with $\tilde{C}^2_{l,N}T_0<1$ that for any N

$$|\sum_{j=1}^{\nu+1}(-1)^j\sigma(\Gamma^{(\nu)}_{j,h})(x,\xi)|_l^{(N)}\leqq C_N|\Delta_\nu|$$

uniformly in $(t,s)\in[0,T_0]^2$. Q.E.D.

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Hitoshi Kitada

Department of Pure and Applied Sciences
University of Tokyo
Komaba, Meguro-ku Tokyo 153
Japan

Hitoshi Kumano-go

Department of Mathematics
Osaka University
Toyonaka, Osaka 560
Japan