

## UNIT-REGULAR RINGS AND SIMPLE SELF-INJECTIVE RINGS

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Unit-regular algebras over a field are investigated from the point of view of a directed abelian group with order-unit.

In the section 1 we show that if  $(K_0(R), [R])$  is an ultrasimplicial abelian group for a unit-regular algebra  $R$  over a field  $F$ , then  $R$  has a subalgebra  $T$  such that  $T$  is an ultramatricial  $F$ -algebra and  $R$  is generated as a ring by  $T$  and units of  $R$ .

In the section 2 we discuss a simple left and right self-injective ring  $R$  which is not artinian. Let  $F$  be the center of  $R$  and  $F_\infty$  be the completion of a ring which is a direct limit of  $M_2(F) \rightarrow M_{2^2}(F) \rightarrow \dots$ , where homomorphisms are diagonal maps. We show that there exists a subalgebra  $S$  of  $R$  such that  $S$  is isomorphic to  $F_\infty$  as a  $F$ -algebra and that every idempotent of  $R$  is conjugate to an idempotent of  $S$  and that every element of  $R$  has the form  $uev$ , where  $u, v$  are units in  $R$  and  $e$  is an idempotent of  $S$ .

We take most of our terminologies and notations from Goodearl's recent book [3], and rely as well on this work for statements of known results.

Throughout this paper a ring is an associative ring with identity and modules are unitary

### 1. Unit-regular algebras

DEFINITION [2]. A ring  $R$  is *unit-regular* if for each  $x \in R$  there is some unit (i.e. invertible element)  $u \in R$  such that  $xux = x$ .

DEFINITION [3, p. 200]. For any ring  $R$  the *Grothendieck group*  $K_0(R)$  is an abelian group with generators  $[A]$ , where  $A$  is any finitely generated projective right  $R$ -modules, and with relations  $[A] + [B] = [C]$  whenever  $A \oplus B \cong C$ . Two generators  $[A], [B]$  equal in  $K_0(R)$  if and only if  $A \oplus R^n \cong B \oplus R^n$  for some positive integer  $n$ . Every element of  $K_0(R)$  has the form  $[A] - [B]$  for suitable modules  $A, B$ .

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DEFINITION [3, p. 202]. A *partially ordered abelian group* is an abelian group  $G$  equipped with a partial order  $\leq$  which is translation invariant (i.e.  $x \leq y$  implies  $x+z \leq y+z$ ). The *positive cone* of  $G$  is the set  $G^+ = \{x \in G; x \geq 0\}$ . If the partial order on  $G$  is directed (upward or downward), then  $G$  is called a *directed abelian group*. An *order-unit* in  $G$  is an element  $u > 0$  such that for any  $x \in G$ , there exists a positive integer  $n$  for which  $x \leq nu$ . We denote by a pair  $(G, u)$  a partially ordered abelian group  $G$  with order-unit  $u$ . We always consider a morphism  $f: (G, u) \rightarrow (G', u')$  as a order-preserving group homomorphism such that  $f(u) = u'$ .

DEFINITION. For modules  $A, B$ ,  $A \leq B$  implies that  $A$  is isomorphic to a submodule of  $B$ .

The following Lemma is a fundamental result for a unit-regular ring.

**Lemma 1** [5, Props. 2.1 and 2.2]. *Let  $R$  be a unit-regular ring and let  $A, B, C$  and  $D$  be finitely generated projective right  $R$ -modules. We define  $[A]-[B] \leq [C]-[D]$  if and only if  $A \oplus D \leq B \oplus C$ . Then we have following results:*

- (1)  $(K_0(R), \leq)$  is a directed abelian group.
- (2)  $[A] > 0$  for all non-zero finitely generated projective right  $R$ -modules.
- (3)  $K_0(R)^+$  consists of all elements of  $K_0(R)$  of the form  $[A]$ .
- (4) If  $R \neq 0$ , then  $[R]$  is an order-unit in  $K_0(R)$ .
- (5)  $[A]-[B] = [C]-[D]$  in  $K_0(R)$  if and only if  $A \oplus D \cong B \oplus C$ .

We note that  $(K_0(-), [-])$  is a functor which preserves direct limits from the category of all unit-regular rings to the category of all partially ordered abelian groups with order-unit ([3, Prop. 15.11]).

DEFINITION [3, pp. 216 and 219]. A *simplicial directed abelian group*  $G$  is a directed abelian group whose positive cone  $G^+$  has the form  $\mathbf{Z}^+x_1 + \cdots + \mathbf{Z}^+x_n$  for some linearly independent elements (over  $\mathbf{Z}$ )  $x_1, \dots, x_n$ . An *ultrasimplicial directed abelian group*  $G$  is isomorphic to a direct limit of a sequence  $(G_1, u_1) \rightarrow (G_2, u_2) \rightarrow \cdots$  of simplicial directed abelian groups with order-unit.

DEFINITION [3, pp. 217 and 219]. Given a field  $F$ , we define a *matricial  $F$ -algebra* to be any  $F$ -algebra of the form  $M_{p(1)}(F) \times \cdots \times M_{p(n)}(F)$  for any positive integers  $p(1), \dots, p(n)$ . An  $F$ -algebra  $R$  is called *ultramatricial* if  $R$  is isomorphic to a direct limit (in the category of  $F$ -algebras) of a sequence  $R_1 \rightarrow R_2 \rightarrow \cdots$  of matricial  $F$ -algebras.

**Lemma 2** [3, Th. 15.24]. *For an ultramatricial  $F$ -algebra  $R$ ,  $(K_0(R), [R])$  is an ultrasimplicial directed abelian group with order-unit. Conversely for an ultrasimplicial directed abelian group with order-unit  $(G, u)$ , there exists an ultramatricial  $F$ -algebra  $R$  such that  $(K_0(R), [R]) \cong (G, u)$ .*

The following lemma is a generalization of [3, (a) of Lemma 15.23].

**Lemma 3.** *Let  $F$  be a field and  $R$  be an ultramatricial  $F$ -algebra and  $S$  be a unit-regular  $F$ -algebra. If a morphism  $f: (K_0(R), [R]) \rightarrow (K_0(S), [S])$  is given, there exists a  $F$ -algebra homomorphism  $\varphi: R \rightarrow S$  such that  $K_0(\varphi) = f$ .*

*Proof.* Let  $R$  be the direct limit of a sequence  $R_1 \xrightarrow{\pi_1} R_2 \xrightarrow{\pi_2} \dots$  of matricial  $F$ -algebras. Let  $\theta_n: R_n \rightarrow R$  be natural homomorphisms for all  $n$ . Then  $(K_0(R), [R])$  is the direct limit of direct system  $\{(K_0(R_n), [R_n]), K_0(\pi_n)\}$  and  $K_0(\theta_n)$  are natural homomorphisms. Put  $f_n = fK_0(\theta_n)$  for all  $n$ . Then by [3, Lemma 15.23], there exist  $F$ -algebra homomorphisms  $\psi_n: R_n \rightarrow S$  such that  $K_0(\psi_n) = f_n$  for all  $n$ . We shall construct  $F$ -algebra homomorphisms  $\varphi_n: R_n \rightarrow S$  for  $n=1, 2, \dots$  such that  $K_0(\varphi_n) = f_n$  and  $\varphi_{n+1}\pi_n = \varphi_n$  for all  $n$ . Put  $\varphi_1 = \psi_1$ , and assume that we have  $\varphi_k$  for all  $k \leq n$ . Two algebra homomorphisms  $\varphi_n, \psi_{n+1}\pi_n: R_n \rightarrow S$  satisfy  $K_0(\varphi_n) = K_0(\psi_{n+1}\pi_n)$ . Thus we can choose an inner automorphism  $g$  of  $S$  such that  $\varphi_n = g\psi_{n+1}\pi_n$  by [3, Lemma 15.23]. Put  $\varphi_{n+1} = g\psi_{n+1}$ . Noting that  $K_0(g)$  is an identity map on  $K_0(S)$ , we have  $K_0(\varphi_{n+1}) = f_{n+1}$ . For a sequence  $\varphi_1, \varphi_2, \dots$ , there exists a unique  $F$ -algebra homomorphism  $\varphi: R \rightarrow S$  such that  $\varphi\theta_n = \varphi_n$  for all  $n$ . We have  $K_0(\varphi)K_0(\theta_n) = K_0(\varphi_n) = f_n = fK_0(\theta_n)$  for all  $n$ . Then we can conclude  $K_0(\varphi) = f$  by the uniqueness.

**Lemma 4.** *Let  $F$  be a field and  $R$  be a unit-regular  $F$ -algebra. For any ultrasimplicial directed subgroup  $G$  with order-unit  $[R]$  of  $(K_0(R), [R])$ , there exists a subalgebra  $T$  of  $R$  such that  $T$  is an ultramatricial  $F$ -algebra and that  $K_0(i): (K_0(T), [T]) \cong (G, [R])$ , where  $i$  is the inclusion map.*

*Proof.* By Lemma 2, there exists an ultramatricial  $F$ -algebra  $T'$  such that  $(K_0(T'), [T']) \cong (G, [R])$ . Let  $f: (K_0(T'), [T']) \rightarrow (G, [R])$  be an order-preserving group isomorphism such that  $f([T']) = [R]$ . Then we can choose an  $F$ -algebra homomorphism  $\varphi: T' \rightarrow R$  such that  $K_0(\varphi) = f$  by Lemma 3. For any  $x \in \text{Ker } \varphi$ ,  $0 = [\varphi(x)R] = K_0(\varphi)[xT'] = f([xT'])$ . Therefore  $[xT'] = 0$ , and hence we have  $xT' = 0$  by Lemma 1. Since  $\varphi$  is monomorphism, we have a desired algebra  $T = \varphi(T')$ .

**Theorem 1.** *Let  $R$  be a unit-regular algebra over a field  $F$ , and assume that  $(K_0(R), [R])$  is ultrasimplicial. Then there exists a subalgebra  $T$  of  $R$  such that*

- (a)  $T$  is an ultramatricial  $F$ -algebra
- (b) every idempotent of  $R$  is conjugate to an idempotent of  $T$ .
- (c) every element of  $R$  is a product of a unit and a conjugate of an idempotent of  $T$ , i.e. every element of  $R$  has the form  $uev$ , where  $u, v$ , are units of  $R$  and  $e$  is an idempotent of  $T$ .

Proof. By Lemma 4, there exists an ultramatricial  $F$ -algebra  $T$  of  $R$  such that  $K_0(i): (K_0(T), [T]) \cong (K_0(R), [R])$ , where  $i: T \rightarrow R$  is the inclusion. Let  $e$  be any idempotent of  $R$ . There exist finitely generated projective right  $T$ -modules  $A, B$  such that  $[eR] = K_0(i)[A]$ ,  $[(1-e)R] = K_0(i)[B]$ . Since  $K_0(i)([T]) = K_0(i)([A \oplus B])$ , then  $[T] = [A \oplus B]$ . We have  $T \cong A \oplus B$  by Lemma 1. We choose an idempotent  $f$  of  $T$  such that  $fT \cong A$ . Since  $[eR] = K_0(i)([fT]) = [fR]$ , then  $eR \cong fR$  by Lemma 1. Therefore we have  $e = u^{-1}fu$  for some unit  $u$  of  $R$  by [7, Th. 2]. For every  $x \in R$ , there exists a unit  $u$  of  $R$  such that  $xu$  is an idempotent. Then (c) is an immediate consequence of (b).

REMARK. The example  $R$  given in [3, Example 15.28] is a unit-regular algebra and  $(K_0(R), [R])$  is an ultrasimplicial directed abelian group but is not ultramatricial.

## 2. Simple self-injective rings

DEFINITION [3, p. 80]. A regular ring  $R$  satisfies the *comparability axiom* if we have either  $J \leq K$  or  $K \leq J$  for any two principal right ideals  $J, K$ . A ring  $R$  is *directly finite* if  $xy = 1$  implies  $yx = 1$  for  $x, y \in R$ .

DEFINITION [3, p. 226]. A *rank function* of a regular ring  $R$  is a map  $N: R \rightarrow [0, 1]$  such that

- (a)  $N(1) = 1$
- (b)  $N(xy) \leq \min \{N(x), N(y)\}$  for all  $x, y \in R$
- (c)  $N(e+f) = N(e) + N(f)$  for all orthogonal idempotents  $e, f \in R$
- (d)  $N(x) > 0$  for all non-zero  $x \in R$ .

Let  $R$  be any simple unit-regular ring which satisfies the comparability axiom. By [3, Cor. 18.12 and Th. 18.17],  $R$  has a unique rank function  $N$  which is determined by the rule:  $N(x) = \sup \{kn^{-1}; k, n \in \mathbf{Z}, k \geq 0, n > 0, (R_R)^k \leq (xR)^n\}$ . If a regular ring  $R$  has a rank function  $N$ ,  $N$  induces a metric  $\delta$  on  $R$  by the rule  $\delta(x, y) = N(x-y)$  for all  $x, y \in R$  ([3, p. 282]). We call this metric a *rank-metric induced by  $N$*  or  *$N$ -metric*. Moreover the completion  $\bar{R}$  of  $R$  with respect to  $N$ -metric is a unit-regular left and right self-injective ring ([3, Th. 19.7]). We call  $\bar{R}$  the  $N$ -completion of  $R$ .

DEFINITION [1]. A *factor sequence*  $\mu = (p(1), p(2), \dots)$  is an infinite sequence of positive integers such that  $p(n) | p(n+1)$  for all  $n$  and  $p(n) \rightarrow \infty$  when  $n \rightarrow \infty$ .

Let  $F$  be a field. For a factor sequence  $\mu$ ,  $F_\mu$  denotes the ultramatricial  $F$ -algebra determined by a direct system  $\{M_{p(n)}(F), f_n\}$ , where  $f_n: M_{p(n)}(F) \rightarrow M_{p(n+1)}(F)$  is a block diagonal homomorphism.  $F_\mu$  is a simple unit regular ring

which satisfies the comparability axiom. Then  $F_\mu$  has a unique rank function  $N$ . The  $N$ -completion  $\bar{F}_\mu$  of  $F_\mu$  is a simple unit-regular left and right self-injective ring which is not artinian and its center is isomorphic to  $F$  ([4, Th. 2.8]). Since  $\bar{F}_\mu \cong \bar{F}_\nu$  as  $F$ -algebra by [10] for any two factor sequences  $\mu, \nu$ , then we denote the completion of  $F_\mu$  by  $F_\infty$  instead of  $\bar{F}_\mu$ .

Let  $R$  be a simple left and right self-injective ring which is not artinian.  $R$  has a unique rank function  $N$  and it is complete with respect to  $N$ -metric ([3, Cor. 21.14]). The center  $F$  of  $R$  is a field ([3, Cor. 1.15]). Now we investigate the relation between  $R, F_\mu$  and  $F_\infty$ . We note that the additive group  $\mathbf{R}$  of all real numbers is considered as partially ordered abelian group with normal order.

**Theorem 2.** *Let  $R$  be a simple left and right self-injective ring which is not artinian and let  $F$  be the center of  $R$ . Then the following results hold:*

(1) *For any factor sequence  $\mu$ , there exist a subalgebra  $T$  of  $R$  which is isomorphic to  $F_\mu$  as  $F$ -algebra.*

(2) *For every non-zero idempotent  $e$  of  $R$ , there exists a (single or infinite) sequence  $\{e_1, e_2, \dots\}$  of orthogonal non-zero idempotents in a subalgebra conjugate to  $T$  such that  $\bigoplus_n Re_n$  is essential in  $Re$  and also  $\bigoplus_n e_n R$  is essential in  $eR$ .*

(3) *There exists a subalgebra  $S$  of  $R$  such that*

(a)  *$S$  is isomorphic to  $F_\infty$  as  $F$ -algebra*

(b) *every idempotent of  $R$  is conjugate to an idempotent of  $S$ .*

(c) *every element of  $R$  is a product of a unit of  $R$  and a conjugate of an idempotent of  $S$ .*

Proof. (1) Since  $R$  is a simple (unit-) regular right self-injective ring of "Type II<sub>f</sub>" (See [3, pp. 102, 113 and 120]), then  $(K_0(R), [R]) \cong (\mathbf{R}, 1)$  by [3, Th. 15.8]. Put  $\mu = (p(1), p(2), \dots)$  and  $N$  be the unique rank function of  $F_\mu$ . For  $x \in F_\mu$ , we know  $N(x) = p(n)^{-1} \text{rank}(x)$ , where  $x \in M_{p(n)}(F)$ . A morphism  $\theta: K_0(F_\mu) \rightarrow \mathbf{R}$  induced by the rule  $\theta([xF_\mu]) = N(x)$  for all  $x \in F_\mu$  is a order-preserving group isomorphism such that  $\theta([F_\mu]) = 1$  by [3, Cor. 16, 15]. We have  $\text{Image } \theta = \bigcup_n \mathbf{Z} \cdot p(n)^{-1}$  by easy calculation. This is an ultrasimplicial directed abelian group. Then there exists an ultramatricial  $F$ -algebra  $T$  of  $R$  such that  $(K_0(T), [T]) \cong (\text{Image } \theta, 1)$  by Lemma 3. By [3, Th. 15.26], we have  $T \cong F_\mu$  as  $F$ -algebra.

(2) Again  $N$  denote the rank function of  $T$ . We define  $D(xT) = N(x)$  for all  $x \in T$ . Then  $D$  is a unique dimension function on the set of all finitely generated projective right  $T$ -modules ([3, Prop. 16.8]). Moreover  $D$  is extended to a function on all projective right  $T$ -modules ([9]). Then we may assume that  $D$  is this extended function. A morphism induced by the rule:  $[A] \rightarrow D(A)$  is equal to  $\theta$ . Let  $N'$  be the unique rank function of  $R$ . We note that the restrictive map of  $N'$  to  $T$  is equal to  $N$  by the uniqueness. Now let

$e$  be any non-zero idempotent of  $R$ . First we assume that  $N'(e) \in \text{Image } \theta$ . We choose a finitely generated projective  $T$ -module  $A$  such that  $N'(e) = \theta([A])$ . Since  $\theta([A]) = D(A) \leq 1$ , then  $A \leq T_\tau$  by [9, Lemma 2.2]. There exists an idempotent  $f$  in  $T$  such that  $N'(e) = N'(f)$ . Since  $eR \cong fR$  by [3, Cor. 16.15], then there exists a unit  $u \in R$  such that  $e = u^{-1}fu$  by [7, Th. 2]. Next we assume that  $N'(e) \notin \text{Image } \theta$ . Since  $\text{Image } \theta = \bigcup_n \mathbb{Z}p(n)^{-1}$  is dense in the space of all real numbers with respect to the normal metric, then there exists an infinite sequence  $x_1, x_2, \dots$  of positive real numbers in  $\text{Image } \theta$  such that  $N'(e) = \sum_n x_n$ . We choose finitely generated right  $T$ -modules  $A_n$  such that  $D(A_n) = x_n$  for each  $n$ . Let  $A = \bigoplus_n A_n$  be an outer direct sum. Since  $D(A) = \sum_n D(A_n) = N'(e) \leq 1$  ([9, p. 406]), then  $A \leq T_\tau$  by [9, Lemma 2.3]. Hence we may assume that there exists a countably generated right ideal  $A$  of  $T$  such that  $D(A) = N'(e)$ . By [3, Prop. 2.14], there exists an infinite sequence  $g_1, g_2, \dots$  of orthogonal non-zero idempotents in  $T$  such that  $A = \bigoplus_n g_n T$ . Since  $R$  is left and right self-injective, then there exists an idempotent  $g \in R$  such that  $\bigoplus_n Rg_n$  is essential in  $Rg$  and also  $\bigoplus_n g_n R$  is essential in  $gR$  by [11, Th. 6.4]. Hence we have  $N'(g) = \sum_n N'(g_n)$  by [3, Th. 21.11 and 21.13]. We see that a directed sequence  $\{g_1 + \dots + g_n; n=1, 2, \dots\}$  of idempotents is convergent to  $g$  with respect to  $N'$ -metric. We write  $g = \sum_n g_n$ . Let  $S$  be the closure of  $T$  in  $R$ . Since  $(T, N'$ -metric) is a subspace of  $(R, N'$ -metric), then we have  $S \cong F_\infty$ . We have  $N'(g) = \sum_n N'(g_n) = \sum_n D(g_n T) = D(A) = N'(e)$ . Since  $gR \cong eR$  by [3, Cor. 16.15], then we have  $e = u^{-1}gu$  for some unit  $u \in R$  by [7, Th. 2]. A sequence  $\{u^{-1}g_n u; n=1, 2, \dots\}$  is a family of orthogonal nonzero idempotents in a subalgebra  $u^{-1}Tu$ . Put  $e_n = u^{-1}g_n u$  for all  $n$ . Since  $g = \sum_n g_n$ , then  $e = u^{-1}gu = \sum_n e_n$ . Therefore we conclude  $\bigoplus_n Re_n$  is essential in  $Re$  and also  $\bigoplus_n e_n R$  in  $eR$  by [8, Prop. 3].

(3) We have already shown that the closure  $S$  of  $T$  is a desired algebra which satisfies (a) and (b). For every element  $x \in R$ , there exists a unit  $u \in R$  such that  $xu$  is an idempotent. Applying (2) to  $xu$ , we can conclude (c).

REMARK. The (a) of Theorem 2 was proved in the case  $\mu = (2, 2^2, 2^3, \dots)$  by D. Handelman ([6, Remark (1) of Prof. 9]).

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