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ON THE GROUPS $J_{Z_{m,q}}(*)$

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1. Introduction

Let G be a compact topological group. If V is an orthogonal representation space of G, we denote by $S(V)$ its unit sphere with respect to some G-invariant inner product. Two orthogonal representation spaces V and W of G are called *J-equivalent* if there exists an orthogonal representation space U such that $S(V\oplus U)$ and $S(W\oplus U)$ are G-homotopy equivalent. Let $RO(G)$ denote the real representation ring of G, and let $T_G(*)\subset RO(G)$ denote an additive subgroup consisting of all elements $V-W$ such that V and W are J-equivalent.

In [6] and [7], Kawakubo considered the quotient group $J_G(*)=RO(G)/T_G(*)$ and the natural epimorphism $J_G: RO(G)\rightarrow J_G(*)$, and determined the structure of $J_G(*)$ for compact abelian topological groups G.

The purpose of this paper is to determine $J_G(*)$ in case G is the metacyclic group

$$Z_{m,q}=\{a, b\,|\,a^m=b^q=e,\ bab^{-1}=a^r\}\ ,$$

where m is a positive odd integer, q is an odd prime integer, $(r-1, m)=1$ and r is a primitive q-th root oi 1 mod m. Our main results are Theorem 7.3 and Corollary 7.4.

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2. The metacyclic group $Z_{m,q}$

In this section we recall some well-known results about the metacyclic group $Z_{m,q}$. The metacyclic group $Z_{m,q}$ is a non-abelian group of order mq and every element of $Z_{m,q}$ is written in the form

$$g=a^ib^j\ ,\qquad 0\leqq i\leqq m-1,\quad 0\leqq j\leqq q-1\ .$$

Let $m=p_1^{r(1)}p_2^{r(2)}\cdots p_t^{r(t)}$ be a prime decomposition of m. We can check easily from the definition of $Z_{m,q}$ the following:

(2.1) $(m, r)=1$,
(2.2) $q|(p_i-1)$ for $1 \leqq i \leqq t$ and $q|(m-1)$,
(2.3) $(m, q)=1$.

The metacyclic group $Z_{m,q}$ has the following two subgroups:

(2.4) $\boldsymbol{Z}_m=\langle a\rangle$,
(2.5) $K_q=\langle b\rangle$.

The groups $\boldsymbol{Z}_m$ and K_q are cyclic groups of order m and q respectively and we have

Lemma 2.6. *The group $\boldsymbol{Z}_m$ is a normal subgroup of $Z_{m,q}$ and K_q is a subgroup satisfying $N(K_q)=K_q$ where $N(K_q)$ denotes the normalizer of K_q in $Z_{m,q}$.*

Proof. Obviously $\boldsymbol{Z}_m$ is a normal subgroup of $Z_{m,q}$. Let $g=a^ib^j$ be an arbitrary element of $N(K_q)$. Then we have $g^{-1}bg=b^{-1}a^{i(r-1)}b^{j+1}\in K_q$. Hence $a^{i(r-1)}\in \boldsymbol{Z}_m\cap K_q=\{e\}$. Therefore we obtain $m|i$ and $g=a^ib^j=b^j\in K_q$. Namely $N(K_q)\subset K_q$. q.e.d.

Lemma 2.7. *Let $H\ (\neq\{e\})$ be a subgroup of $Z_{m,q}$. If H satisfies $H\cap \boldsymbol{Z}_m=\{e\}$, then H and K_q are conjugate.*

Proof. By assumption, there exists an element $a^ib^j\in H$ which satisfies $j\not\equiv 0 \mod q$. Hence we obtain $\boldsymbol{Z}_mH=Z_{m,q}$. Thus there exists a canonical isomorphism

$$Z_{m,q}/\boldsymbol{Z}_m\cong H/H\cap \boldsymbol{Z}_m\,.$$

Therefore $q=|Z_{m,q}:\boldsymbol{Z}_m|=|H:H\cap \boldsymbol{Z}_m|=|H|$. Since K_q is a Sylow q-subgroup of $Z_{m,q}$, H and K_q are conjugate. q.e.d.

REMARK 2.8. Let H be an arbitrary subgroup of $Z_{m,q}$. By Lemma 2.7, H satisfies one of the following:

(i) $H=\{e\}$,
(ii) H is conjugate to K_q,
(iii) $H\cap \boldsymbol{Z}_m\neq\{e\}$.

REMARK 2.9. In general the metacyclic group $Z_{m,q}$ depends on not only the integers m, q but also the integer r. But the group $J_{Z_{m,q}}(*)$ depends only on the integers m, q (see Theorem 7.3).

3. The real representation ring $RO(\boldsymbol{Z}_{m,q})$

In this section we determine the additive generators of the real representation ring $RO(Z_{m,q})$. First we recall the results, due to Curtis and Reiner [2;

§47], about the additive generators of the complex representation ring $R(Z_{m,q})$.

The metacyclic group $Z_{m,q}$ has the following unitary representations:

(3.1) the trivial one-dimensional representation 1_{C^1},

(3.2) the complex q-dimensional representations T_h ($h\in\boldsymbol{Z}$) defined by

$$T_h(a)=\begin{pmatrix} L_0 & & & & \\ & L_1 & & 0 & \\ & & L_2 & & \\ & 0 & & \ddots & \\ & & & & L_{q-1}\end{pmatrix}\in U(q)$$

and

$$T_h(b)=\begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 & 1 \\ 1 & 0 & & & & 0 \\ & 1 & 0 & 0 & & \vdots \\ & & 1 & & & \vdots \\ & 0 & & \ddots & & \vdots \\ & & & & 1 & 0\end{pmatrix}\in U(q)\,,$$

where $L_j=\exp(2\pi hr^j\sqrt{-1}/m)$ for $0\leqq j\leqq q-1$,

(3.3) the complex one-dimensional representations ρ_d ($d\in\boldsymbol{Z}$) defined by

$$\rho_d(a)=1\in U(1)$$

and

$$\rho_d(b)=\exp(2\pi d\sqrt{-1}/q)\in U(1)\,.$$

The representations T_h ($h\in\boldsymbol{Z}$) satisfy the following (see [2; §47]):

(3.4) *If* $(h,m)=1$, *then* T_h *is irreducible.*

(3.5) *When* T_h *and* T_k *are irreducible,* T_h *and* T_k *are inequivalent if and only if* $r^jh\not\equiv k \bmod m$ *for* $0\leqq j\leqq q-1$.

Denote by $FR(Z_{m,q})$ the subgroup of $R(Z_{m,q})$ generated by $\{T_h\,|\,(h,m)=1,\ h\in\boldsymbol{Z}\}$. When n is an integer such that $n\,|\,m$ and $n>1$, we obtain the metacyclic group $Z_{n,q}=\{c,d\,|\,c^n=d^q=e,\ dcd^{-1}=c^r\}$ and define the natural epimorphism $\pi_n\colon Z_{m,q}\to Z_{n,q}$ by $\pi_n(a^ib^j)=c^id^j$.

Theorem 3.6. *There is an isomorphism (additively)*

$$R(Z_{m,q})\cong A'\oplus B'\oplus\bigoplus_{n|m,n>1}FR(Z_{n,q})\,,$$

where A' *is the subgroup of* $R(Z_{m,q})$ *generated by* 1_{C^1} *and* B' *is the subgroup of* $R(Z_{m,q})$ *generated by* $\{\rho_d\,|\,(d,q)=1,\ d\in\boldsymbol{Z}\}$.

Proof. It follows that

$$R(Z_{m,q})=A'\oplus B'\oplus\bigoplus_{n|m,n>1}\pi_n^*(FR(Z_{n,q}))$$

(see [2; §47]). Since $\pi_n^* | FR(Z_{n,q})$: $FR(Z_{n,q}) \to R(Z_{m,q})$ is injective, we obtain the result. q.e.d.

If χ is a complex representation, then the real representation $r(\chi)$ is defined to be the underlying real representation of χ, and $\bar{\chi}$ denotes the complex conjugate representation of χ.

Lemma 3.7. *If $(h, m)=1$, then T_h and $\bar{T}_h$ are inequivalent.*

Proof. Suppose that T_h is equivalent to $\bar{T}_h \simeq T_{-h}$. It follows from (3.5) that there exists an integer j $(0 \leqq j \leqq q-1)$ such that $r^j h \equiv -h \bmod m$. Since $(h, m)=1$, we have $r^j \equiv -1 \bmod m$. Thus we obtain $1 \equiv (r^j)^q \equiv (-1)^q \equiv -1 \bmod m$. This is a contradiction. Therefore T_h is inequivalent to $\bar{T}_h$. q.e.d.

Denote by $FRO(Z_{m,q})$ the subgroup of $RO(Z_{m,q})$ generated by $\{r(T_h) | (h, m)=1, h \in \boldsymbol{Z}\}$. Now we have

Theorem 3.8. *There is an isomorphism (additively)*

$$RO(Z_{m,q}) \simeq A \oplus B \oplus \bigoplus_{n|m, n>1} FRO(Z_{n,q}),$$

where A is the subgroup of $RO(Z_{m,q})$ generated by the trivial one-dimensional representation $1_{\boldsymbol{R}^1}$ and B is the subgroup of $RO(Z_{m,q})$ generated by $\{r(\rho_d) | (d, q)=1, d \in \boldsymbol{Z}\}$.

Proof. The result follows easily from Theorem 3.6 and Adams [1; Theorem 3.57].

In the following we write T_h and ρ_d instead of $r(T_h)$ and $r(\rho_d)$ respectively. We use the same symbol as a representation for its representation space.

Remark 3.9. The representation T_h is identified with the following unitary representation space:

$$\begin{cases} T_h(a) \circ (z_0, z_1, \cdots, z_{q-1}) = (\exp(2\pi h\sqrt{-1}/m) z_0, \exp(2\pi h r\sqrt{-1}/m) z_1, \cdots, \\ \qquad\qquad\qquad\qquad\qquad\qquad \exp(2\pi h r^{q-1}\sqrt{-1}/m) z_{q-1}), \\ T_h(b) \circ (z_0, z_1, \cdots, z_{q-1}) = (z_{q-1}, z_0, z_1, \cdots, z_{q-2}), \end{cases}$$

where $(z_0, z_1, \cdots, z_{q-1}) \in \boldsymbol{C}^q$. Moreover we regard $\boldsymbol{R}^1$ as $1_{\boldsymbol{R}^1}$.

4. *G*-homotopy equivalences of spheres of *G*-representation spaces

We begin by fixing some notations. Let G be a finite group and X be a

G-space. We denote the isotropy group at $x\in X$ by G_x. For a subgroup H of G, (H) denotes the conjugacy class of H in G and we set

$$X^H = \{x\in X \mid G_x\supset H\} .$$

For a G-map $f: X_1\to X_2$, we denote by f^H the restriction $f\mid X_1^H: X_1^H\to X_2^H$. If V is a unitary G-representation space, then for a subgroup H of G, $S(V)^H$ has a canonical orientation defined by the complex structure of V^H. Let V, W be unitary G-representation spaces and $f: S(V)\to S(W)$ be a G-map. Then for a subgroup H of G satisfying $\dim S(V)^H=\dim S(W)^H$, we have the degree of the map $f^H: S(V)^H\to S(W)^H$. When $S(V)^H=S(W)^H=\phi$, we define $\deg f^H=1$. Since G is a finite group, there are only finite conjugacy classes of subgroups of G, say

$$\{(H_1), (H_2), \cdots, (H_n)\} .$$

By Theorem 1.1 of James-Segal [5], we have

Theorem 4.1. *Let V, W be unitary G-representation spaces which satisfy the condition* $\dim S(V)^{H_i}=\dim S(W)^{H_i}$ *for* $1\leqq i\leqq n$. *If there exists a G-map $f: S(V)\to S(W)$ such that*

$$|\deg f^{H_i}|=1 \qquad \text{for} \quad 1\leqq i\leqq n ,$$

then $S(V)$ and $S(W)$ are G-homotopy equivalent.

5. The group $J_{Z_{m,q}}(B)$

Let ρ_{a_i} $(1\leqq i\leqq n)$ and ρ_{b_j} $(1\leqq j\leqq n)$ be non-trivial $Z_{m,q}$-representation spaces defined by (3.3). We set

$$M = \rho_{a_1}\oplus\rho_{a_2}\oplus\cdots\oplus\rho_{a_n} , \quad M'=\rho_{b_1}\oplus\rho_{b_2}\oplus\cdots\oplus\rho_{b_n} .$$

Theorem 5.1. *The following three conditions are equivalent:*
(i) *$S(M)$ and $S(M')$ are $Z_{m,q}$-homotopy equivalent,*
(ii) *M and M' are J-equivalent,*
(iii) $\prod_{i=1}^n a_i\equiv\pm\prod_{j=1}^n b_j \bmod q$.

Proof. From the definition of ρ_d $(d\in \boldsymbol{Z})$, it suffices to consider the K_q-actions instead of the $Z_{m,q}$-actions. The K_q-representation $\rho_d|K_q$ is defined by $(\rho_d|K_q)(b)=\exp(2\pi d\sqrt{-1}/q)$. Since $(a_i, q)=(b_j, q)=1$ for $1\leqq i$, $j\leqq n$, it follows from Kawakubo [7; Theorem 2.6] that (i), (ii) and (iii) are equivalent. q.e.d.

Corollary 5.2. *There is an isomorphism*

$$J_{Z_{m,q}}(B)\cong\boldsymbol{Z}\oplus\boldsymbol{Z}_{(q-1)/2} .$$

Proof. See Kawakubo [7; §2 and §3].

6. The group Ker $(J_{Z_{m,q}}|FRO(Z_{m,q}))$

In this section we determine the group Ker $(J_{Z_{m,q}}|FRO(Z_{m,q}))$. Let T_h and T_k be $Z_{m,q}$-representation spaces defined by (3.9). If T_h is contained in $FRO(Z_{m,q})$, then the integer h satisfies $(h, m)=1$. Thus there exists some integer $\bar{h}$ such that $\bar{h}h\equiv 1 \bmod m$. We define a $Z_{m,q}$-map $f_{\bar{h}k}\colon S(T_h)\to S(T_k)$ by

$$f_{\bar{h}k}(z_0, z_1, \cdots, z_{q-1}) = \frac{(z_0^{\bar{h}k}, z_1^{\bar{h}k}, \cdots, z_{q-1}^{\bar{h}k})}{||(z_0^{\bar{h}k}, z_1^{\bar{h}k}, \cdots, z_{q-1}^{\bar{h}k})||}.$$

It is obvious that $f_{\bar{h}k}$ is a well-defined $Z_{m,q}$-map.

Let T_{h_i} $(1\leqq i\leqq n)$ and T_{k_j} $(1\leqq j\leqq n)$ be $Z_{m,q}$-representation spaces contained in $FRO(Z_{m,q})$. We set

$$N = T_{h_1}\oplus T_{h_2}\oplus\cdots\oplus T_{h_n}, \quad N' = T_{k_1}\oplus T_{k_2}\oplus\cdots\oplus T_{k_n}.$$

Let x_0 (resp. y_0) be the point (0, 1) of $S(\boldsymbol{R}^2)\subset S(N\oplus\boldsymbol{R}^2)$ (resp. $S(\boldsymbol{R}^2)\subset S(N'\oplus\boldsymbol{R}^2)$). Since $\boldsymbol{C}^1$ is a complex vector space, the underlying real vector space $\boldsymbol{R}^2$ has a canonical orientation.

Lemma 6.1. *There exists a $Z_{m,q}$-map $F\colon S(N\oplus\boldsymbol{R}^2)\to S(N'\oplus\boldsymbol{R}^2)$ such that*
(i) $F(x_0)=y_0$,
(ii) $\deg F = \prod_{i=1}^{n} (\bar{h}_i k_i)^q$, $\deg F^{K_q} = \prod_{i=1}^{n} \bar{h}_i k_i$ *and* $\deg F^H=1$,
where H is an arbitrary subgroup of $Z_{m,q}$ satisfying $H\cap \boldsymbol{Z}_m\neq\{e\}$.

Proof. First we study the $Z_{m,q}$-map $f_{\bar{h}k}\colon S(T_h)\to S(T_k)$, where T_h and T_k are contained in $FRO(Z_{m,q})$. It follows from the definition of $f_{\bar{h}k}$ that $\deg f_{\bar{h}k}=(\bar{h}k)^q$. For the subgroup K_q, we have

$$S(T_h)^{K_q} = \{(z_0, z_1, \cdots, z_{q-1})\in S(T_h)\,|\, z_0 = z_1 = \cdots = z_{q-1}\},$$
$$S(T_k)^{K_q} = \{(w_0, w_1, \cdots, w_{q-1})\in S(T_k)\,|\, w_0 = w_1 = \cdots = w_{q-1}\}.$$

Hence $\deg (f_{\bar{h}k})^{K_q}=\bar{h}k$. Since $S(T_h)^H=S(T_k)^H=\phi$, we obtain $\deg (f_{\bar{h}k})^H=1$. Then we put

$$F = f_{\bar{h}_1k_1} * f_{\bar{h}_2k_2} *\cdots* f_{\bar{h}_nk_n} * id_{S(\boldsymbol{R}^2)},$$

where $*$ denotes the join. Now F is a $Z_{m,q}$-map from $S(N\oplus\boldsymbol{R}^2)$ to $S(N'\oplus\boldsymbol{R}^2)$ which satisfies the conditions (i) and (ii). q.e.d.

The following lemma is due to Petrie [10].

Lemma 6.2. *Let G be a finite group and V, W be unitary G-representation spaces. Let H be a subgroup of G whose conjugacy class is contained in* Iso$(V)=$

$\{(G_v)\,|\,v\in V\}$. *Suppose that* $f: S(V)\to S(W)$ *is an H-map, then there exists a G-map* $\tau(G, H; f)$: $S(V\oplus \boldsymbol{R}^2)\to S(W\oplus \boldsymbol{R}^2)$ *which satisfies the following conditions*:

(i) $\tau(G, H; f)(x_0)=y_0$ *where* x_0, y_0 *are those in Lemma* 6.1.

(ii) *Let K be a subgroup of G such that* $\dim V^K=\dim W^K$. *If there exists some element* g_0 *of G such that* $g_0^{-1}Kg_0\subset H$, *we have*

$$\deg \tau(G, H; f)^K = |(G/H)^K| \deg f^{g_0^{-1}Kg_0}.$$

On the other hand, if $g^{-1}Kg\not\subset H$ *for any element g of G, we have*

$$\deg \tau(G, H; f)^K = 0.$$

Proof. By Meyerhoff-Petrie [9; Theorem 2.2] and Petrie [10; Lemma 2.3], there exists a G-map $\tilde{f}$: $S(V\oplus \boldsymbol{R}^1)\to S(W\oplus \boldsymbol{R}^1)$ which satisfies the condition (ii). Then we obtain a G-map $\tau(G, H; f)=\tilde{f}*id_{S(\boldsymbol{R}^1)}$: $S(V\oplus \boldsymbol{R}^2)\to S(W\oplus \boldsymbol{R}^2)$. It is obvious that the G-map $\tau(G, H; f)$ satisfies the conditions (i) and (ii).

q.e.d.

Lemma 6.3. *There exist two* $Z_{m,q}$*-maps* θ, ψ: $S(N\oplus \boldsymbol{R}^2)\to S(N'\oplus \boldsymbol{R}^2)$ *which satisfy the following two conditions*:

(i) $\theta(x_0)=\psi(x_0)=y_0$,

(ii) $\deg\theta=mq$, $\deg\theta^{K_q}=\deg\theta^H=0$, $\deg\psi=m$, $\deg\psi^{K_q}=1$ *and* $\deg\psi^H=0$, *where H is an arbitrary subgroup of* $Z_{m,q}$ *satisfying* $H\cap \boldsymbol{Z}_m\neq\{e\}$.

Proof. We recall that N, N' are unitary $Z_{m,q}$-representation spaces and remark that $\mathrm{Iso}(N)=\{(e), (K_q), (Z_{m,q})\}$. Apply Lemma 6.2 to the identity map id: $S(N)\to S(N')$ which is an $\{e\}$-map, then we have a $Z_{m,q}$-map $\theta=\tau(Z_{m,q}, \{e\}; id)$: $S(N\oplus \boldsymbol{R}^2)\to S(N'\oplus \boldsymbol{R}^2)$ such that $\theta(x_0)=y_0$, $\deg\theta=|z_{m,q}|=mq$ and $\deg\theta^{K_q}=\deg\theta^H=0$. Moreover the identity map is not only an $\{e\}$-map but also a K_q-map. We also have a $Z_{m,q}$-map $\psi=\tau(Z_{m,q}, K_q; id)$: $S(N\oplus \boldsymbol{R}^2)\to S(N'\oplus \boldsymbol{R}^2)$ such that $\psi(x_0)=y_0$, $\deg\psi=|Z_{m,q}/K_q|=m$, $\deg\psi^{K_q}=|(Z_{m,q}/K_q)^{K_q}|=|N(K_q)/K_q|=1$ and $\deg\psi^H=0$. q.e.d.

Now we have

Theorem 6.4. *The following three conditions are equivalent*:

(i) $S(N\oplus \boldsymbol{R}^2)$ *and* $S(N'\oplus \boldsymbol{R}^2)$ *are* $Z_{m,q}$*-homotopy equivalent,*

(ii) *N and N' are J-equivalent,*

(iii) $\prod_{i=1}^{n} h_i^q \equiv \pm \prod_{j=1}^{n} k_j^q \mod m$.

Proof. Obviously (i) implies (ii).

First we show that (ii) implies (iii). By assumption, there exists an orthogonal $Z_{m,q}$-representation space U such that $S(N\oplus U)$ and $S(N'\oplus U)$

are $Z_{m,q}$-homotopy equivalent. Obviously $S(N\oplus U)$ and $S(N'\oplus U)$ are also $\boldsymbol{Z}_m$-homotopy equivalent. Let $\mu_d\ (d\in\boldsymbol{Z})$ be the complex one-dimensional $\boldsymbol{Z}_m$-representations defined by $\mu_d(a)=\exp(2\pi d\sqrt{-1}/m)$. Then we have $T_h|\boldsymbol{Z}_m\cong\mu_h\oplus\mu_{hr}\oplus\mu_{hr^2}\oplus\cdots\oplus\mu_{hr^{q-1}}$ as $\boldsymbol{Z}_m$-representations. The integers h_ir^s, k_jr^s satisfy $(h_ir^s, m)=(k_jr^s, m)=1$ for $1\leqq i,\ j\leqq n$ and $0\leqq s\leqq q-1$. It follows from Kawakubo [7; Theorem 2.6] that $r^{q(q-1)n/2}\prod_{i=1}^n h_i^q\equiv\pm r^{q(q-1)n/2}\prod_{j=1}^n k_j^q \bmod m$. Since $r^q\equiv 1 \bmod m$, we obtain the condition (iii).

Next we show that (iii) implies (i). By Lemma 6.1, there exists a $Z_{m,q}$-map $F\colon S(N\oplus\boldsymbol{R}^2)\to S(N'\oplus\boldsymbol{R}^2)$ such that

$$(6.4.1)\quad \begin{cases} F(x_0)=y_0\,, \\ \deg F=\prod_{i=1}^n(\bar{h}_ik_i)^q,\ \deg F^{K_q}=\prod_{i=1}^n\bar{h}_ik_i \\ \text{and}\quad \deg F^H=1 \quad\text{where}\quad H\cap\boldsymbol{Z}_m\neq\{e\}. \end{cases}$$

On the other hand, by Lemma 6.3, there exists a $Z_{m,q}$-map $\psi\colon S(N\oplus\boldsymbol{R}^2)\to S(N'\oplus\boldsymbol{R}^2)$ such that

$$(6.4.2)\quad \begin{cases} \psi(x_0)=y_0\,, \\ \deg\psi=m,\ \deg\psi^{K_q}=1 \text{ and } \deg\psi^H=0 \text{ where } H\cap\boldsymbol{Z}_m\neq\{e\}\,. \end{cases}$$

We define $\varepsilon\ (=\pm1)$ by $\prod_{i=1}^n h_i^q\equiv\varepsilon\prod_{j=1}^n k_j^q \bmod m$. The $Z_{m,q}$-homotopy classes of $Z_{m,q}$-maps from $S(N\oplus\boldsymbol{R}^2)$ to $S(N'\oplus\boldsymbol{R}^2)$ sending x_0 to y_0 form a group. Therefore by (6.4.1) and (6.4.2), we obtain a $Z_{m,q}$-map $F_2=F-(\prod_{i=1}^n\bar{h}_ik_i-\varepsilon)\psi$ which satisfies the following condition:

$$(6.4.3)\quad \begin{cases} F_2(x_0)=y_0\,, \\ \deg F_2=\prod_{i=1}^n(\bar{h}_ik_i)^q-(\prod_{i=1}^n\bar{h}_ik_i-\varepsilon)m\,,\quad \deg F_2^{K_q}=\varepsilon \\ \text{and}\quad \deg F_2^H=1 \quad\text{where}\quad H\cap\boldsymbol{Z}_m\neq\{e\}\,. \end{cases}$$

By Lemma 6.3, there exists a $Z_{m,q}$-map $\theta\colon S(N\oplus\boldsymbol{R}^2)\to S(N'\oplus\boldsymbol{R}^2)$ such that

$$(6.4.4)\quad \begin{cases} \theta(x_0)=y_0\,, \\ \deg\theta=mq \quad\text{and}\quad \deg\theta^{K_q}=\deg\theta^H=0 \quad\text{where}\quad H\cap\boldsymbol{Z}_m\neq\{e\}. \end{cases}$$

On the other hand, by the assumption (iii), we have

$$\prod_{i=1}^n(\bar{h}_ik_i)^q\equiv\varepsilon \qquad \bmod m.$$

Then we obtain

(6.4.5) $$\prod_{i=1}^{n}(\bar{h}_i k_i)^q-(\prod_{i=1}^{n}\bar{h}_i k_i-\varepsilon)m-\varepsilon\equiv 0 \qquad \bmod m.$$

Moreover it is well-known that

$$\prod_{i=1}^{n}(\bar{h}_i k_i)^q\equiv \prod_{i=1}^{n}\bar{h}_i k_i \qquad \bmod q.$$

Hence we obtain (see (2.2))

(6.4.6) $$\begin{aligned}\prod_{i=1}^{n}(\bar{h}_i k_i)^q-(\prod_{i=1}^{n}\bar{h}_i k_i-\varepsilon)m-\varepsilon &\equiv (1-m)\prod_{i=1}^{n}\bar{h}_i k_i+\varepsilon(m-1)\\ &\equiv 0 \qquad \bmod q.\end{aligned}$$

Since m and q are relatively prime integers, by (6.4.5) and (6.4.6), we obtain

$$\prod_{i=1}^{n}(\bar{h}_i k_i)^q-(\prod_{i=1}^{n}\bar{h}_i k_i-\varepsilon)m-\varepsilon\equiv 0 \qquad \bmod mq.$$

Let n_0 be an integer such that

(6.4.7) $$\prod_{i=1}^{n}(\bar{h}_i k_i)^q-(\prod_{i=1}^{n}\bar{h}_i k_i-\varepsilon)m-\varepsilon = n_0 mq\,.$$

By (6.4.3), (6.4.4) and (6.4.7), we obtain a $Z_{m,q}$-map $F_3=F_2-n_0\theta$ such that

$$\deg F_3=\deg F_3^{K_q}=\varepsilon \quad\text{and}\quad \deg F_3^H=1 \quad\text{where}\quad H\cap \boldsymbol{Z}_m\neq\{e\}\,.$$

Therefore it follows from Remark 2.8 and Theorem 4.1 that $S(N\oplus\boldsymbol{R}^2)$ and $S(N'\oplus\boldsymbol{R}^2)$ are $Z_{m,q}$-homotopy equivalent. q.e.d.

7. The group $J_{Z_{m,q}}(*)$

In this section we determine the group $J_{Z_{m,q}}(*)$. For this purpose we follow the procedure due to Kawakubo [7; §3 and §4]. To determine the group

$$C_m=J_{Z_{m,q}}(FRO(Z_{m,q}))\,,$$

we define another group C'_m as follows. Let $m=p_1^{r(1)}p_2^{r(2)}\cdots p_t^{r(t)}$ be a prime decomposition of m. We set

$$C'_m=\boldsymbol{Z}\oplus\{\bigoplus_{i=1}^{t}\boldsymbol{Z}_{(p_i^{r(i)}-p_i^{r(i)-1})/q}\}/\boldsymbol{Z}_2\,,$$

where the inclusion of $\boldsymbol{Z}_2$ into $\bigoplus_{i=1}^{t}\boldsymbol{Z}_{(p_i^{r(i)}-p_i^{r(i)-1})/q}$ is given by $1\to\bigoplus_{i=1}^{t}(p_i^{r(i)}-p_i^{r(i)-1})/2q$. Remark that $2q\,|\,(p_i-1)$ for $1\leqq i\leqq t$ (see (2.2)).

We also define a homomorphism

$$J'_m\colon FRO(Z_{m,q}) \to C'_m$$

as follows. As is well-known, there exist integers $\alpha(i)$ for $1 \leqq i \leqq t$ such that $\alpha(i)$ is a primitive root mod $p_i^{r(i)}$ and $\alpha(i) \equiv 1 \bmod p_j^{r(j)}$ for every $j \neq i$. For every integer h with $(h, m)=1$ and for $1 \leqq i \leqq t$, there exists a unique $\mu(h, i) \in \boldsymbol{Z}_{p^{r(i)}p^{r(i)-1}}$ such that

$$h \equiv \prod_{i=1}^{t} \alpha(i)^{\mu(h,i)} \qquad \bmod m\,.$$

Let

$$\omega\colon \bigoplus_{i=1}^{t} \boldsymbol{Z}_{p_i^{r(i)}-p_i^{r(i)-1}} \to \{\bigoplus_{i=1}^{t} \boldsymbol{Z}_{(p_i^{r(i)}-p_i^{r(i)-1})/q}\}/\boldsymbol{Z}_2$$

denote the natural projection. Let $\sum_{j=1}^{u} a(h_j)T_{h_j}$ be an arbitrary element of $FRO(Z_{m,q})$, that is, $a(h_j) \in \boldsymbol{Z}$. We define

$$J'_m(\sum_{j=1}^{u} a(h_j)T_{h_j}) = \sum_{j=1}^{u} a(h_j) \oplus \omega(\bigoplus_{i=1}^{t} \sum_{j=1}^{u} a(h_j)\mu(h_j, i))\,.$$

Denote by J_m the restricted homomorphism $J_{Z_{m,q}} | FRO(Z_{m,q})$. We have

Lemma 7.1. *J'_m is an epimorphism and* $\operatorname{Ker} J_m = \operatorname{Ker} J'_m$. *Hence there is an isomorphism*

$$C_m \cong C'_m$$

Proof. Let a, a_i $(1 \leqq i \leqq t)$ be arbitrary integers. Then we have

$$J'_m((a - \sum_{i=1}^{t} a_i)T_1 + \sum_{i=1}^{t} a_i T_{\alpha(i)}) = a \oplus \omega(\bigoplus_{i=1}^{t} a_i)$$

$$\in C'_m = \boldsymbol{Z} \oplus \{\bigoplus_{i=1}^{t} \boldsymbol{Z}_{(p_i^{r(i)}-p_i^{r(i)-1})/q}\}/\boldsymbol{Z}_2\,.$$

This shows that J'_m is surjective.

Next we show that $\operatorname{Ker} J_m = \operatorname{Ker} J'_m$. Let $x = \sum_{\lambda=1}^{u} a(h_\lambda)T_{h_\lambda} - \sum_{\nu=1}^{v} b(k_\nu)T_{k_\nu}$ be an arbitrary element of $FRO(Z_{m,q})$, where $a(h_\lambda)$ $(1 \leqq \lambda \leqq u)$ and $b(k_\nu)$ $(1 \leqq \nu \leqq v)$ are non-negative integers. The element x is contained in $\operatorname{Ker} J'_m$ if and only if the following condition (7.1.1) is satisfied.

$$(7.1.1) \qquad \begin{cases} \sum_{\lambda=1}^{u} a(h_\lambda) = \sum_{\nu=1}^{v} b(k_\nu)\,, \\ \omega(\bigoplus_{i=1}^{t} \sum_{\lambda=1}^{u} a(h_\lambda)\mu(h_\lambda, i)) = \omega(\bigoplus_{i=1}^{t} \sum_{\nu=1}^{v} b(k_\nu)\mu(k_\nu, i))\,. \end{cases}$$

It is easy to see that the condition (7.1.1) is equivalent to the following condition (7.1.2):

$$
(7.1.2) \qquad \begin{cases} \sum_{\lambda=1}^{u} a(h_\lambda) = \sum_{\nu=1}^{v} b(k_\nu)\,, \\ \prod_{\lambda=1}^{u} h_\lambda^{a(h_\lambda)q} \equiv \pm \prod_{\nu=1}^{v} k_\nu^{b(k_\nu)q} \qquad \bmod m\,. \end{cases}
$$

By Theorem 6.4, x satisfies the condition (7.1.2) if and only if x is contained in $\mathrm{Ker}\, J_m$. Therefore we have $\mathrm{Ker}\, J_m = \mathrm{Ker}\, J'_m$. q.e.d.

We recall that there is an isomorphism (see Theorem 3.8)

$$RO(Z_{m,q}) \cong A \oplus B \bigoplus_{n|m, n>1} FRO(Z_{n,q})\,.$$

Lemma 7.2. *There is an isomorphism*

$$T_{Z_{m,q}}(*) \cong \{0\} \oplus \mathrm{Ker}\,(J_{Z_{m,q}} | B) \oplus \bigoplus_{n|m, n>1} \mathrm{Ker}\, J_n\,.$$

Proof. The result is easily seen from tom Dieck [3; Proposition 4.1].

It follows from Corollary 5.2 and Lemma 7.2 that

$$J_{Z_{m,q}}(*) \cong \boldsymbol{Z} \oplus \boldsymbol{Z} \oplus \boldsymbol{Z}_{(q-1)/2} \oplus \bigoplus_{n|m, n>1} C_n\,.$$

Therefore we obtain, by Lemma 7.1, the following main theorem.

Theorem 7.3. *There is an isomorphism*

$$J_{Z_{m,q}}(*) \cong \boldsymbol{Z} \oplus \boldsymbol{Z} \oplus \boldsymbol{Z}_{(q-1)/2} \oplus \bigoplus_{m|n, n>1} C'_n\,.$$

Corollary 7.4. *Let V, W be orthogonal $Z_{m,q}$-representation spaces. If V and W are J-equivalent, then $S(V \oplus \boldsymbol{R}^2)$ and $S(W \oplus \boldsymbol{R}^2)$ are $Z_{m,q}$-homotopy equivalent.*

Proof. The result follows easily from Theorems 5.1, 6.5 and Lemma 7.2.

REMARK 7.5. M. Morimoto has succeeded to omit $\boldsymbol{R}^2$ in Corollary 7.4.

8. Appendix

In this section G will be a finite group. Denote by $RO_0(G)$ the additive subgroup of $RO(G)$

$$\{V - W \mid \dim V^H = \dim W^H \quad \text{for every subgroup } H \text{ of } G\}\,.$$

In [3] and [4], tom Dieck and Petrie defined the group $jO(G)$ to be $RO_0(G)/$

$T_G(*)$. Since $T_G(*)\subset RO_0(G)\subset RO(G)$, there exists a short exact sequence

$$0\rightarrow jO(G)\rightarrow J_G(*)\rightarrow RO(G)/RO_0(G)\rightarrow 0\,.$$

Since $RO(G)/RO_0(G)$ is a free abelian group (see Lee-Wasserman [8; §3]), the above short exact sequence is split. Thus we have

Proposition 8.1. *There is an isomorphism*

$$J_G(*)\cong jO(G)\oplus RO(G)/RO_0(G)\,.$$

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