Muto, H. and Urakawa, H.
Osaka J. Math.
17 (1980), 471–484

ON THE LEAST POSITIVE EIGENVALUE OF LAPLACIAN FOR COMPACT HOMOGENEOUS SPACES

HIDEO MUTO AND HAJIME URAKAWA

(Received May 21, 1979)

Introduction and statement of results

Let M be an n-dimensional compact smooth manifold. Two Riemannian metrics g_1 and g_2 on M are called to be *homothetically equivalent* if there exists a diffeomorphism Φ of M onto itself such that $\Phi^* g_1$ coincides g_2 with a constant multiple.

Let $M=G/K$ be a compact homogeneous space, where G is a compact Lie group and K is a closed subgroup of G. A Riemannian metric g on M is called be *G-invariant* if all the translations τ_x by elements x in G on M are isometric with respect to the metric g (cf. [3]). Let us consider the elementary, but non-trivial problem: *How many G-invariant mutually homothetically inequivalent Riemannian metrics are there on $M=G/K$?*

If the linear isotropy action of K on the tangent space $T_o(M)$ of M at the origin $o=\{K\}\in M$ (cf. [3]) is irreducible over $\boldsymbol{R}$, then there exists a unique (up to homothetic equivalence) G-invariant Riemannian metric on M (cf. [9]). So the above problem is reduced to the case that the linear isotropy action of K is *reducible* over $\boldsymbol{R}$, that is, the tangent space $T_o(M)$ is decomposed into two proper subspaces invariant by the linear isotropy action of K. In this case, many people would have the following conjecture: *If a compact homogeneous space $M=G/K$ (with some additional assumptions) has the reducible isotropy action of K over $\boldsymbol{R}$, then it would have uncountably many mutually homothetically inequivalent G-invariant metrics.*

One of the purposes of this paper is to show that the above conjecture is affirmative.

Now we assume that a compact homogeneous space G/K has the condition (C): The linear isotropy action of K on the tangent apace $T_o(M)$ of M at the origin o is reducible and includes the identity representation of K on $T_o(M)$. Let $\mathfrak{g}$ be the Lie algebra of all left invariant vector fields on G and let $\mathfrak{k}$ be the subalgebra of $\mathfrak{g}$ corresponding to the subgroup K. Since G is compact, there exists an $\mathrm{Ad}(G)$-invariant inner product B on $\mathfrak{g}$. Let $\mathfrak{m}$ be the ortho-complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to B. Then we have the decomposition

$\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathrm{Ad}(k)\mathfrak{m}=\mathfrak{m}\ (k\in K)$. The isomorphism of $\mathfrak{m}$ onto $T_o(M)$ given by $X\mapsto X_0$ (the tangent vector at o) is K-equivariant, that is $(\mathrm{Ad}(k)X)_o=\tau_{k*}X_o\ (k\in K)$, where τ_{k*} is the differentiation of the translation τ_k at o. So the condition (C) means the following condition (C′):

(C′) *There exists a non-zero element Z in $\mathfrak{m}$ such that* $\mathrm{Ad}(k)Z=Z\ (k\in K)$.

We notice that every G-invariant Riemannian metric g on $M=G/K$ is given by an $\mathrm{Ad}(K)$-invariant inner product (,) on $\mathfrak{m}$ (cf. in 2.2). Thus, to answer the above conjecture, we may choose a suitable homothetically invariant ratio which takes continuously different values among the above G-invariant Riemannian metrics. For this purpose, let us consider the following ratio. For a Riemannian metric g on M, let $-\Delta_g$ be the Laplace-Beltrami operator acting on smooth functions on M and let $\lambda_1(g)$ be the least positive eigenvalue of Δ_g. Then we notice (cf. [1]) that the ratio $\lambda_1(g)\mathrm{vol}(M,g)^{2/n}$ is homothetically invariant, that is, if two Riemannian metrics g_1 and g_2 are homothetically equivalent, then it holds that

$$\lambda_1(g_1)\ \mathrm{vol}\ (M, g_1)^{2/n} = \lambda_1(g_2)\ \mathrm{vol}\ (M, g_2)^{2/n}\ .$$

Now, under the above preparations, we can state the following results.

Main Theorem. *Let $M=G/K$ be an n-dimensional compact homogeneous space ($n\geqq 2$), where G is a compact connected Lie group and K is a closed connected subgroup of G. Let $\mathfrak{g}$ be the Lie algebra of all left invariant vector fields on G and let $\mathfrak{k}$ be the subalgebra of $\mathfrak{g}$ corresponding to K. Let B be an* $\mathrm{Ad}(G)$*-invariant inner product on $\mathfrak{g}$, and let $\mathfrak{m}$ be the orthocomplement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to B. We assume the condition (C′): There exists a non-zero element Z in $\mathfrak{m}$ such that* $\mathrm{Ad}(k)Z=Z\ (k\in K)$. *Then there exists an one-parameter family of G-invariant Riemannian metrics g_t $(0<t<\infty)$ on M such that*

(1) $\mathrm{vol}\ (M, g_t)$ *is constant in t,*

(2) $\lim_{t\to 0} \lambda_1(g_t)=0$ (*if an one-parameter subgroup* $\{\exp (sZ);\ s\in \boldsymbol{R}\}$ *is closed in* G),

and

(3) $\lim_{t\to\infty} \lambda_1(g_t)=\infty$ (*if G is semi-simple*).

Thus we have immediately the following corollary.

Corollary. *Let $M=G/K$ be as in the above Theorem. Assume that the condition (C′) holds. If either the one-parameter subgroup* $\{\exp (sZ);\ s\in \boldsymbol{R}\}$ *is closed in G or G is semi-simple, then there exist uncountablely many mutually homothetically inequivalent G-invariant Riemannian metrics on M.*

REMARK 1. For examples, the following ones satisfy the conditions of the Main Theorem: real Stiefel manifolds $SO(n+p)/SO(n)$, $p\geqq 2$, $n\geqq 1$; complex Stiefel manifolds $SU(n+p)/SU(n)$, $p\geqq 1$, $n\geqq 1$; quoternion Stiefel manifolds $Sp(n+p)/Sp(n)$, $p\geqq 1$, $n\geqq 1$; $G/(H/T_1)$, where G/H is an irreducible hermitian symmetric space and T_1 is the connected component of the center of H; and compact connected semi-simple group manifolds. On the other hand, a compact flag manifold G/T, where G is a compact semi-simple Lie group and T is a maximal torus in G, does not satisfy the condition (C′) of the above Theorem, but it has the reducible isotropy action.

REMARK 2. Main Theorem is an extension of the results obtained by [7] and [8]. The above Corollary is a generalization of the results of [4] and [5].

Finally, we should express our gratitude to Professor M. Takeuchi and also Professor S. Tanno who suggested us this probrem and encouraged us during the preparation for this paper.

1. The Laplace-Beltrami operator on reductive homogeneous spaces

1.1. Let $M=G/K$ be an n-dimensional homogeneous space, where G is a connected Lie group and K is a closed subgroup of G. In this section, we do not assume necessarily the compactness of M. Let $\mathfrak{g}$ be the Lie algebra of all left invariant vector fields on G and let $\mathfrak{k}$ be the subalgebra of $\mathfrak{g}$ corresponding to the subgroup K.

DEFINITION (cf. [3] p. 200 or [2] p. 389). The coset space $M=G/K$ is called to be *reductive* if there exists a subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$ (direct sum) and $\mathrm{Ad}(k)\mathfrak{m}=\mathfrak{m}$ for all $k\in K$.

In this section, we consider a reductive homogeneous space $M=G/K$. First, we prepare some notations. (See [2] and [6]).

Let $C^\infty(G)$ be the space of all complex valued C^∞ functions on G, $C^\infty(G, K)$ the space of all elements f in $C^\infty(G)$ such that $f(gk)=f(g)$ for each $g\in G$ and $k\in K$, and $C^\infty(M)$ the space of all complex valued C^∞ functions on M. Let π be the natural projection of G onto G/K. Put $o=\{K\}\in M=G/K$. Then the mapping $f\mapsto\tilde{f}$, where $\tilde{f}=f\circ\pi$, gives an isomorphism of $C^\infty(M)$ onto $C^\infty(G, K)$.

Let $\boldsymbol{D}(G)$ be the space of all differential operators on G which are invariant by left translations L_x, $(x\in G)$, $\boldsymbol{D}_0(G)$ the space of all elements in $\boldsymbol{D}(G)$ which are invariant by right translations R_k, $(k\in K)$, and $\boldsymbol{D}(M)$ the space of all differential operators on M which are invariant by the translations τ_x $(x\in G)$ on M. Then, for every $D\in\boldsymbol{D}_0(G)$, we can define $\varpi(D)\in\boldsymbol{D}(M)$ by

$$(\varpi(D)f)^{\sim}=D\tilde{f}\,,\quad f\in C^\infty(M)\,.$$

Let $S(\mathfrak{m})$ be the symmetric algebra over $\mathfrak{m}$. Then $S(\mathfrak{m})$ can be regarded as a K-module by the adjoint action of K on $\mathfrak{m}$. Let $S(\mathfrak{m})_K$ be the set of all elements in $S(\mathfrak{m})$ which are invariant by the action $\mathrm{Ad}(k)$, $k\in K$. Let $S(\mathfrak{m})_K{}^C$ be the complexification of $S(\mathfrak{m})_K$. Then the following theorem holds.

Theorem (cf. [2] or [6]).

(1) *The mapping* ϖ; $\boldsymbol{D}_0(G)\to\boldsymbol{D}(M)$ *is homomorphism of* $\boldsymbol{D}_0(G)$ *onto* $\boldsymbol{D}(M)$.

(2) *There exists an isomorphism* $\hat{\lambda}$ *of* $S(\mathfrak{m})_K{}^C$ *onto* $\boldsymbol{D}(M)$ *which is given as follows: Let* $\{Y_1,\cdots,Y_n\}$ *be a basis of* $\mathfrak{m}$. *Then, for every polynomial* $P(Y_1,\cdots,Y_n)$ *in* $S(\mathfrak{m})_K{}^C$,

(1.1)
$$[\hat{\lambda}(P(Y_1,\cdots,Y_n))f](x\cdot o)=\left[P\left(\frac{\partial}{\partial y_1},\cdots,\frac{\partial}{\partial y_n}\right)f(x\exp(\textstyle\sum_{i=1}^n y_iY_i)\cdot o)\right](0)\,.$$

Here, in the right hand side, we regard $f(x\exp(\sum_{i=1}^n y_iY_i)\cdot o)$ *as a function in* $(y_1,\cdots,y_n)$ *and* $P\left(\frac{\partial}{\partial y_1},\cdots,\frac{\partial}{\partial y_n}\right)$ *expresses the differential operator which is given by substituting* $\frac{\partial}{\partial y_1},\cdots,\frac{\partial}{\partial y_n}$ *into the polynomial* $P(Y_1,\cdots,Y_n)$.

1.2. Every G-invariant Riemannian metric on a reductive homogeneous space $M=G/K$ is given as follows (cf. [3]): Let $(\,,)$ be an $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{m}$. Then there exists a unique G-invariant Riemannian metric on M such that

$$(g)_o(X_o,Y_o)=(X,Y)\,,\quad X,Y\in\mathfrak{m}\,.$$

Here the tangent vectors X_o, $Y_o\in T_o(M)$ of M at the origin $o=\{K\}$ correspond to elements X, Y in $\mathfrak{m}$.

For this metric g on M, let $-\Delta_g$ be the Laplace-Beltrami operator on M, that is

$$\Delta_g f=-\textstyle\sum_{i,j=1}^n g^{ij}\left(\frac{\partial^2 f}{\partial y_i\partial y_j}-\sum_{k=1}^n\Gamma_{ij}^k\frac{\partial f}{\partial y_k}\right),$$

for every $f\in C^\infty(M)$. Here (g^{ij}) is the inverse matrix of (g_{ij}), (g_{ij}) is the components of g with respect to the local coordinate $(y_1,\cdots,y_n)$ of M and Γ_{ij}^k is the Christoffel symbol of the Riemannian connection for g. Since the translations τ_x, $x\in G$ on M are isometries with respect to g, then the operator Δ_g belongs to $\boldsymbol{D}(M)$ (cf. [2] p. 387). So we investigate to express Δ_g explicitely in terms of $S(\mathfrak{m})_K{}^C$, using the above theorem.

Lemma 1.1. *Let* $\{Y_i\}_{i=1}^n$ *be an orthonormal basis of* $\mathfrak{m}$ *with respect to the above* $\mathrm{Ad}(K)$*-invariant inner product* $(\,,)$. *Then the following polynomials belong to* $S(\mathfrak{m})_K{}^C$:

(1) $\sum_{i=1}^n Y_i^2$,

(2) $\sum_{i=1}^{n} \mathrm{Trace}_{\mathfrak{g}}(\mathrm{ad}(Y_i))Y_i$,
where $\mathrm{Trace}_{\mathfrak{g}}(\mathrm{ad}(Y))$ *is the trace of an endomorphism* $\mathrm{ad}(Y)$ *of* $\mathfrak{g}$ *for every* $Y\in\mathfrak{m}$.

Proof. It is clear that $\sum_{i=1}^{n} Y_i^2$ belongs to $S(\mathfrak{m})_K{}^C$, due to the $\mathrm{Ad}(K)$-invariance of $(,)$. For another orthonormal basis $\{Y_i'\}_{i=1}^{n}$ of $\mathfrak{m}$, we have

$$\sum_{i=1}^{n} \mathrm{Trace}_{\mathfrak{g}}(\mathrm{ad}(Y_i'))Y_i' = \sum_{i=1}^{n} \mathrm{Trace}_{\mathfrak{g}}(\mathrm{ad}(Y_i))Y_i .$$

For $k\in K$, put $Y_i'=\mathrm{Ad}(k)Y_i$. Then $\{Y_i'\}_{i=1}^{n}$ is also orthonormal with respect to $(,)$. So we have

$$\mathrm{Ad}(k)(\sum_{i=1}^{n} \mathrm{Trace}_{\mathfrak{g}}(\mathrm{ad}(Y_i))Y_i) = \sum_{i=1}^{n} \mathrm{Trace}_{\mathfrak{g}}(\mathrm{ad}(\mathrm{Ad}(k^{-1})Y_i'))Y_i'$$
$$= \sum_{i=1}^{n} \mathrm{Trace}_{\mathfrak{g}}(\mathrm{ad}(Y_i'))Y_i') = \sum_{i=1}^{n} \mathrm{Trace}_{\mathfrak{g}}(\mathrm{ad}(Y_i))Y_i .$$

Q.E.D.

Theorem 1. *Let $M=G/K$ be a reductive homogeneous space. For every G-invariant Riemannian metric g on M, we have*

$$\Delta_g = -\hat{\lambda}(\sum_{i=1}^{n} Y_i^2)+\hat{\lambda}(\sum_{i=1}^{n} \mathrm{Trace}_{\mathfrak{g}}(\mathrm{ad}(Y_i))Y_i) .$$

Here $\{Y_i\}_{i=1}^{n}$ is an orthonormal basis of $\mathfrak{m}$ with respect to the $\mathrm{Ad}(K)$*-invariant inner product $(,)$ corresponding to g,* $\mathrm{Trace}_{\mathfrak{g}}(\mathrm{ad}(Y))$ *is the trace of an endomorphism* $\mathrm{ad}(Y)$ *of* $\mathfrak{g}$, *for every* $Y\in\mathfrak{m}$ *and* $\hat{\lambda}$ *is given by* (1.1).

Proof. Since both hand sides of the above equality belong to $\boldsymbol{D}(M)$, we may prove, at the origin o of M,

$$\Delta_g f(o) = -\hat{\lambda}(\sum_{i=1}^{n} Y_i^2)f(o)+\hat{\lambda}(\sum_{i=1}^{n} \mathrm{Trace}_{\mathfrak{g}}(\mathrm{ad}\, Y_i))Y_i)f(o) ,$$

for all $f\in C^\infty(M)$. Take a local coordinate $(y_1, \cdots, y_n)$ around the origin o defined by the mapping $\exp(\sum_{i=1}^{n} y_i Y_i)\cdot o\mapsto(y_1, \cdots, y_n)$. Put $\mathrm{Exp}=\pi\circ\exp$, a mapping $\mathfrak{m}$ into M. For $x=\exp(X)$, $X\in\mathfrak{m}$, such that $x\cdot o$ belongs to the above local coordinate neighborhood of the origin o, we have (cf. [2])

$$\left(\frac{\partial}{\partial y_i}\right)_{x\cdot o} = \mathrm{Exp}_{*_X}(Y_i) = \pi_{*_x}\circ\exp_{*_X}(Y_i)$$
$$= \pi_{*_x}\circ L_{x*_e}\circ\sum_{m=0}^{\infty}\frac{(-\mathrm{ad}(X))^m}{(m+1)!}(Y_i) = \tau_{x*_o}\circ\pi_{*_e}\circ\sum_{m=0}^{\infty}\frac{(-\mathrm{ad}(X))^m}{(m+1)!}(Y_i) .$$

Here for a smooth mapping Φ, Φ_{*_p} denotes its differential at a point p of M. Then

$$g_{ij}(x\cdot o) = g_{x\cdot o}\left(\left(\frac{\partial}{\partial y_i}\right)_{x\cdot o}, \left(\frac{\partial}{\partial y_j}\right)_{x\cdot o}\right)$$
$$= (g)_0\left(\pi_{*_e}\left(\sum_{m=0}^{\infty}\frac{(-\mathrm{ad}(X))^m}{(m+1)!}(Y_i)\right)\right), \pi_{*_e}\left(\sum_{m=0}^{\infty}\frac{((-\mathrm{ad}(X))^m}{(m+1)!}(Y_j)\right)\right)$$
$$= \left(\left(\sum_{m=0}^{\infty}\frac{(-\mathrm{ad}(X))^m}{(m+1)!}(Y_i)\right)_{\mathfrak{m}}, \sum_{m=0}^{\infty}\frac{(-\mathrm{ad}(X))^m}{(m+1)!}(Y_j)\right)_{\mathfrak{m}}\right),$$

where, $W_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-component of an element W in $\mathfrak{g}$ corresponding to the decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$. Hence we have

$$
(1.2)\qquad \begin{cases} g_{ij}(o)=\delta_{ij}\,, & \text{and} \\ \left(\dfrac{\partial}{\partial y_k}\right)_0 g_{ij}=-\dfrac{1}{2}(c_{ki}^j+c_{kj}^i)\,, \end{cases}
$$

where we put $[Y_i, Y_j]_{\mathfrak{m}}=\sum_{k=1}^n c_{ij}^k Y_k\,(1\leqq i,j\leqq n)$. In fact,

$$
\begin{aligned}
\left(\frac{\partial}{\partial y_k}\right)_0 g_{ij} &= \left[\frac{d}{ds}g_{ij}(\exp(sY_k)\cdot o)\right]_{s=0} \\
&= \left[\frac{d}{ds}\left(\left(\sum_{m=0}^{\infty}\frac{(-s\,\mathrm{ad}(Y_k))^m}{(m+1)!}(Y_i)\right)_{\mathfrak{m}}\,,\right.\right. \\
&\qquad \left.\left.\left(\sum_{m=0}^{\infty}\frac{(-s\,\mathrm{ad}(Y_k))^m}{(m+1)!}(Y_j)\right)_{\mathfrak{m}}\right)\right]_{s=0} \\
&= -\frac{1}{2}([Y_k, Y_i]_{\mathfrak{m}}, Y_j)+(Y_i, [Y_k, Y_j]_{\mathfrak{m}})) \\
&= -\frac{1}{2}(c_{ki}^j+c_{kj}^i)\,.
\end{aligned}
$$

Therefore we have

$$
\Gamma_{ij}^k(o)=\frac{1}{2}(c_{ki}^j+c_{kj}^i)\,,
$$

in partiqular, $\Gamma_{ii}^k(o)=c_{ki}^i$. For we have

$$
\begin{aligned}
\Gamma_{ij}^k(o) &= \frac{1}{2}\left(\frac{\partial g_{kj}}{\partial y_i}(0)+\frac{\partial g_{ik}}{\partial y_j}(0)-\frac{\partial g_{ij}}{\partial y_k}(0)\right) \\
&= \frac{1}{2}(c_{ki}^j+c_{kj}^i)\,,
\end{aligned}
$$

by (1.2) and $c_{ki}^j+c_{ik}^j=0$. Thus we have

$$
\Delta_g f(o)=-\sum_{i=1}^n\frac{\partial^2}{\partial y_i^2}f(0)+\sum_{k=1}^n\left(\sum_{i=1}^n c_{ki}^i\right)\frac{\partial f}{\partial y_k}(0)\,.
$$

Notice that $\sum_{i=1}^n c_{ki}^i=\mathrm{Trace}_{\mathfrak{g}}(\mathrm{ad}(Y_k))$, due to the fact $\mathrm{ad}(Y_k)(\mathfrak{k})\subset\mathfrak{m}$ and $[Y_k, Y_i]_{\mathfrak{m}}=\sum_{j=1}^n c_{ki}^j Y_j$. Therefore the right hand side of the above equation coincides with

$$
-\lambda\left(\sum_{i=1}^n Y_i^2\right)f(o)+\left(\sum_{k=1}^n \mathrm{Trace}_{\mathfrak{g}}(\mathrm{ad}(Y_k))Y_k\right)f(o)\,. \qquad \text{Q.E.D.}
$$

Corollary. *Let $M=G/K$ be a reductive homogeneous space, where G is a connected Lie group and K is a closed subgroup of G. Assume that the Lie algebra $\mathfrak{g}$ of G is a unimodular Lie algebra, that is,* $\mathrm{Trace}_{\mathfrak{g}}(\mathrm{ad}(X))=0$ *for every $X\in\mathfrak{g}$.*

Then for every G-invariant Riemannian metric g on M, we have

$$\Delta_{\mathfrak{g}} = -\hat{\lambda}(\textstyle\sum_{i=1}^{n} Y_i^2),$$

where $\{Y_i\}_{i=1}^{n}$ *is an orthonormal basis of* $\mathfrak{m}$ *with respect to the* Ad(K)*-invariant inner product* $(,)$ *corresponding to g.*

REMARK 3. If $K=\{e\}$, the above theorem has been obtained in [8].

REMARK 4. If the Riemannian connection for g is the natural torsion-free connection on M, that is, its inner product $(,)$ on $\mathfrak{m}$ satisfies

$$(X, [Z, Y]_{\mathfrak{m}})+([Z, X]_{\mathfrak{m}}, Y) = 0,$$

for every X, Y and $Z\in\mathfrak{m}$ (cf. [3]), then the above Corollary is well-known (cf. [6]).

2. Proof of Main Theorem. (I)

In this section, the situations of Main Theorem are preserved. Let $M=G/K$ be a compact homogeneous space, where G is a compact connected Lie group and K is a closed connected subgroup of G. Let $\mathfrak{g}$ be the Lie algebra of all left invariant vector fields on G, $\mathfrak{k}$ the subalgebra of $\mathfrak{g}$ corresponding to the subgroup K. Let B be an Ad(G)-invariant inner product on $\mathfrak{g}$. Let $\mathfrak{m}$ be the orthocomplement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to B. Then we have the decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$ such that Ad$(k)\,\mathfrak{m}=\mathfrak{m}$ $(k\in K)$. We assume the condition (C'): There exists a non-zero element Z in $\mathfrak{m}$ such that Ad$(k)Z=Z\,(k\in K)$. Let $\mathfrak{m}_1$ be the subspace of $\mathfrak{m}$ spanned by the element Z. Let $\mathfrak{m}_2$ be the orthocomplement of $\mathfrak{m}_1$ in $\mathfrak{m}$ with respect to B. Then we have a decomposition of $\mathfrak{m}$ such that $\mathfrak{m}=\mathfrak{m}_1+\mathfrak{m}_2$ and Ad$(k)\mathfrak{m}_i=\mathfrak{m}_i$ $(k\in K, i=1, 2)$.

Now let $\mathfrak{t}_{\mathfrak{k}}$ be a maximal abelian subalgebra of $\mathfrak{k}$. Then $\mathfrak{t}_{\mathfrak{k}}+\mathfrak{m}_1$ is an abelain subalgebra of $\mathfrak{g}$. By Zorn's lemma, there exists a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ including $\mathfrak{t}_{\mathfrak{k}}+\mathfrak{m}_1$.

Lemma 2.1. *We have*

$$\mathfrak{t} = \mathfrak{t}_{\mathfrak{k}}+\mathfrak{m}_1+\mathfrak{t}\cap\mathfrak{m}_2.$$

Proof. First, we have $\mathfrak{t}=\mathfrak{t}_{\mathfrak{k}}+\mathfrak{t}\cap\mathfrak{m}$. In fact, every element $Y\in\mathfrak{t}$ is written as $Y=Y_{\mathfrak{k}}+Y_{\mathfrak{m}}$ $(Y_{\mathfrak{k}}\in\mathfrak{k}$, $Y_{\mathfrak{m}}\in\mathfrak{m})$ corresponding to the decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$. But $Y_{\mathfrak{k}}$ belongs to the centralizer of $\mathfrak{t}_{\mathfrak{k}}$ in $\mathfrak{k}$. For, we have $[Y_{\mathfrak{k}}, X]=-[Y_{\mathfrak{m}}, X]$ for every $X\in\mathfrak{t}_{\mathfrak{k}}$, where the right hand side belongs to $\mathfrak{m}$ and the left hand side belongs to $\mathfrak{k}$. Thus $[Y_{\mathfrak{k}}, X]=[Y_{\mathfrak{m}}, X]=0$. Since $\mathfrak{t}_{\mathfrak{k}}$ is a maximal abelian subalgebra of $\mathfrak{k}$, $Y_{\mathfrak{k}}$ belongs to $\mathfrak{t}_{\mathfrak{k}}\subset\mathfrak{t}$.So $Y_{\mathfrak{m}}$ belongs to $\mathfrak{t}$. Next, we have $\mathfrak{t}\cap\mathfrak{m}=\mathfrak{m}_1+\mathfrak{t}\cap\mathfrak{m}_2$. In fact, each element $Y\in\mathfrak{t}\cap\mathfrak{m}$ is decomposed as $Y=Y_{\mathfrak{m}_1}+Y_{\mathfrak{m}_2}$,

$(Y_{\mathfrak{m}_1}\in\mathfrak{m}_1,\ Y_{\mathfrak{m}_2}\in\mathfrak{m}_2)$ corresponding to $\mathfrak{m}=\mathfrak{m}_1+\mathfrak{m}_2$. Then we have $Y_{\mathfrak{m}_2}=Y-Y_{\mathfrak{m}_1}\in\mathfrak{t}$ since $Y_{\mathfrak{m}_1}\in\mathfrak{m}_1\subset\mathfrak{t}$. Q.E.D.

Let $\mathfrak{m}_2'$ be the orthocomplement of $\mathfrak{t}\cap\mathfrak{m}_2$ in $\mathfrak{m}_2$ with respect to B. We choose an orthonormal basis $\{X_i\}_{i=1}^n$ of $\mathfrak{m}$ with respect to B such that $X_1\in\mathfrak{m}_1$ and $\{X_2,\cdots,X_n\}$ is taken corresponding to the decomposition $\mathfrak{m}_2=\mathfrak{t}\cap\mathfrak{m}_2+\mathfrak{m}_2'$.

Now we define a new inner product $B_t\,(0<t<\infty)$ on $\mathfrak{m}$ by

$$\begin{cases} B_t(X_1, X_1)=t^{n-1}\,, & \\ B_t(X_i, X_j)=\delta_{ij}t^{-1} & (2\leqq i,j\leqq n)\,, \quad \text{and} \\ B_t(X_1, X_i)=0 & (2\leqq i\leqq n)\,, \end{cases}$$

that is, $\{t^{-(n-1)/2}X_1, t^{1/2}X_2,\cdots,t^{1/2}X_n\}$ is an orthonormal basis of $\mathfrak{m}$ with respect to B_t. Then we have

Lemma 2.2. *The above new inner product $B_t\,(0<t<\infty)$ on $\mathfrak{m}$ is* $\mathrm{Ad}(K)$-*invariant.*

Proof. Since K is connected, we may prove

$$B_t([W,X],Y)+B_t(X,[W,Y])=0\,,$$

for $W\in\mathfrak{k}$, $X,Y\in\mathfrak{m}$. It may also be proved that

$$B_t([W,X_i],X_j)+B_t(X_i,[W,X_j])=0\,, \tag{2.1}$$

for each $i,j=1,\cdots,n$. Put $[W,X_j]=\sum_{i=2}^n a_{ij}X_i\,(2\leqq j\leqq n)$. Since $[W,X_1]=0$ and $\mathrm{ad}(W)(\mathfrak{m}_2)\subset\mathfrak{m}_2$, we have

$$a_{ij}+a_{ji}=0 \quad (2\leqq i,j\leqq n)\,, \tag{2.2}$$

due to the $\mathrm{Ad}(K)$-invariance of B. We will prove (2.1) in the following three cases: (1) $i=j=1$, (2) either $i=1$ and $2\leqq j\leqq n$, or $2\leqq i\leqq n$ and $j=1$, (3) $2\leqq i,j\leqq n$. Case (1) is clear. Case (2) follows from the fact that $\mathfrak{m}_1$ is orthogonal to $\mathfrak{m}_2$ by the definition of B_t. Case (3). For $2\leqq i,j\leqq n$, we have

$$B_t([W,X_i],X_j)+B_t(X_i,[W,X_j])=t^{-1}(a_{ji}+a_{ij})=0\,,$$

due to (2.2) and the definition of B_t. Q.E.D.

Due to Lemma 2.2, there exists a unique G-invariant Riemannian metric $g_t\,(0<t<\infty)$ on M suth that

$$(g_t)_o(X_o, Y_o)=B_t(X,Y)\,,$$

for $X,Y\in\mathfrak{m}$ (cf. [3] p. 200, Cor. 3.2). A Riemannian metric g_1 on M corresponds to the inner product B on $\mathfrak{m}$. We will show that the above g_t $(0<t<\infty)$ are desired in Main Theorem.

Lemma 2.3. *We have*

$$\mathrm{vol}\,(M, g_t) = \mathrm{vol}\,(M, g_1)\,, \qquad (0<t<\infty)\,.$$

Proof. Since K is connected and compact, the homogeneous space $M=G/K$ is orientable. Since G is connected, the translations τ_x by $x\in G$ on M preserve the volume element v_{g_t} of (M, g_t). So we may see $(v_{g_t})_o=(v_{g_1})_o\in\bigwedge^n T_o^*M$, where T_o^*M is the cotangent space of M at the origin o. But it is valid due to the definition of $g_t\,(0<t<\infty)$. Q.E.D.

Lemma 2.4. *We have*

$$\Delta_{g_t} = (t^{-(n-1)}-t)(-\hat{\lambda}(X_1^2))+t\Delta_{g_1}\,,$$

where the polynomial X_1^2 belongs to $S(\mathfrak{m})_K{}^C$.

Proof. By the condition (C'), the polynomial X_1^2 belongs to $S(\mathfrak{m})_K{}^C$. Due to Corollary of Theorem 1 and the definition of g_t, we have

$$\begin{aligned}\Delta_g &= -\hat{\lambda}(t^{-(n-1)}X_1^2+t\textstyle\sum_{i=2}^n X_i^2)\\ &= (t^{-(n-1)}-t)(-\hat{\lambda}(X_1^2))-t\hat{\lambda}(\textstyle\sum_{i=1}^n X_i^2)\\ &= (t^{-(n-1)}-t)(-\hat{\lambda}(X_1^2))+t\Delta_{g_1}\end{aligned}$$

Q.E.D.

3. Proof of Main Theorem. (II)

3.1. In this section, we preserve the situations in §§1, 2. In this part 3.1, we prepare, (cf. [6]), the Peter-Weyl theorem for a compact homogeneous space $M=G/K$, where G is a compact connected Lie group and K is a closed connected subgroup of G. We do not necessarily assume the condition (C′).

Let $\boldsymbol{D}(G)$ be a complete set of finite dimensional inequivalent unitary representations of G. For a representation (ρ, V_ρ) belonging to $\boldsymbol{D}(G)$, put $d_\rho=\dim V_\rho$ and $V_\rho{}^K=\{w\in V_\rho;\ \rho(k)\,w=w$ for every $k\in K\}$.

DEFINITION (cf. [6]). A representation $(\rho, V_\rho)\in\boldsymbol{D}(G)$ is called to be a *spherical representation for a pair* (G, K) if $V_\rho{}^K\neq(0)$.

Let $\boldsymbol{D}(G, K)$ be the set of all spherical representations in $\boldsymbol{D}(G)$ for a pair (G, K). For $\rho\in\boldsymbol{D}(G, K)$, let $((\ ,\ ))$ be an $\rho(G)$-invariant inner product on V_ρ and put $m_\rho=\dim V_\rho{}^K$. We choose an orthonormal basis $\{V_i\}_{i=1}^{d_\rho}$ of V_ρ such that $\{v_i\}_{i=1}^{m_\rho}$ is a basis of $V_\rho{}^K$. Let $\rho_{ij}(x)=((\rho(x)v_j, v_i))$, $x\in G$ and let $\bar{\rho}_{ij}(x)$ be the complex conjugate of $\rho_{ij}(x)$. Since $\bar{\rho}_{ij}\,(1\leqq i\leqq d_\rho,\ 1\leqq j\leqq m_\rho)$ belongs to $C^\infty(G, K)$, it can be regarded as a function on M. We denote it by the same letter $\bar{\rho}_{ij}$. As in §2, let B be an $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{m}$, g_1 be the corresponding G-invariant Riemannian metric on M, and v_{g_1} the volume element of (M, g_1). We define a hermitian inner product $((\ ,\ ))$ on $C^\infty(M)$ by

$$((f_1, f_2)) = \mathrm{vol}\,(M, g_1)^{-1} \int_M f_1(x\cdot o)\overline{f_2(x\cdot o)} v_{g_1}(x\cdot o)\,.$$

Then we have (cf. [6])

Theorem (Peter-Weyl). 1) *For every* $\rho \in \boldsymbol{D}(G, K)$, $\{\sqrt{d_\rho}\bar{\rho}_{ij};\ 1 \leqq i \leqq d_\rho,\ 1 \leqq j \leqq m_\rho\}$ *is an orthonormal system of* $C^\infty(M)$ *with respect to* $((\ ,\ ))$. *Let* $\theta_\rho(M)$ *be the subspace of* $C^\infty(M)$ *spanned by* $\{\sqrt{d_\rho}\bar{\rho}_{ij};\ 1 \leqq i \leqq d_\rho,\ 1 \leqq j \leqq m_\rho\}$ *over* $\boldsymbol{C}$. 2) *If* $\rho, \rho' \in \boldsymbol{D}(G, K)$ *are mutually inequivalent, then* $\theta_\rho(M)$ *and* $\theta_{\rho'}(M)$ *are mutually orthogonal with respect to* $((\ ,\ ))$. *Moreover we have the following decomposition*: $C^\infty(M) = \sum_{\rho \in \boldsymbol{D}(G,K)} \theta_\rho(M)$, *that is, each* $f \in C^\infty(M)$ *can be expanned by*

$$f = \sum_{\rho \in \boldsymbol{D}(G,K)} d_\rho \sum_{\substack{1 \leqq i \leqq d_\rho \\ 1 \leqq j \leqq m_\rho}} ((f, \bar{\rho}_{ij}))\bar{\rho}_{ij}\,,$$

in the sense of the uniform convergence on M *or the* L^2*-convergence with respect to* $((\ ,\ ))$.

3.2. In this part, we assume the condition (C′) and let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{g}$ in Lemma 2.1.

Let Δ be the root system of the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$ with respect to $\mathfrak{t}$, that is, the set of non-zero elements α of the dual space $\mathfrak{t}^*$ of $\mathfrak{t}$ such that $\mathfrak{g}^C_\alpha = \{E \in \mathfrak{g}^C;\ [H, E] = \sqrt{-1}\alpha(H)E$, for any $H \in \mathfrak{t}\}$ is not zero. We introduce a lexicographic order $>$ on $\mathfrak{t}^*$ and fix it once and for all. Let Δ^+ be the set of all positive roots with respect to this order. Let p be the dimension of the commutator subalgebra of $\mathfrak{g}$. Let $\Pi = \{\alpha_1, \cdots, \alpha_p\}$ be the fundamental system of Δ with respect to the order $>$. For $\lambda \in \mathfrak{t}^*$, let H_λ be an element in $\mathfrak{t}$ defined by $B(H, H_\lambda) = \lambda(H)$ for all $H \in \mathfrak{t}$. Here the inner product B is an $\mathrm{ad}(G)$-invariant inner product on $\mathfrak{g}$ as in §2. We define an inner product $(\ ,\ )_0$ on $\mathfrak{t}^*$ by $(\lambda, \lambda')_0 = B(H_\lambda, H_{\lambda'})$ for $\lambda, \lambda' \in \mathfrak{t}^*$. Let $\Gamma = \{H \in \mathfrak{t};\ \exp(H) = e\}$. Let

$$I = \{\lambda \in \mathfrak{t}^*;\ \lambda(H) \in 2\pi\boldsymbol{Z} \quad \text{for all} \quad H \in \Gamma\}\,.$$

Put

$$D(G) = \{\lambda \in I;\ (\lambda, \alpha_i)_0 \geqq 0 \quad (1 \leqq i \leqq p)\}\,.$$

Then the set I coincides with the set of all the weights of the representations of G. The maximal element among the weights of a representation (ρ, V_ρ) in the order $>$ in $\mathfrak{t}^*$ is called the highest weight of (ρ, V_ρ). The set $D(G)$ coincides with the set of all highest weights of the representations of G. There exists a bijection (cf. [6]) from $D(G)$ onto $\boldsymbol{D}(G)$.

3.3. For $X_1 \in \mathfrak{m}_1$ as in §2, the polynomials X_1 and X_1^2 belong to $S(\mathfrak{m})_K{}^C$, so we have

$$(3.1) \qquad \hat{\lambda}(X_1^2)\bar{\rho}_{ij}(x\cdot o) = \overline{((\rho(x)\rho(X_1)^2 v_j, v_i))}\,,$$

for every $x \in G$ and $\rho \in \boldsymbol{D}(G, K)$. Let $D_1 = \rho(X_1)^2$ be an endomorphism of V_ρ for every $\rho \in \boldsymbol{D}(G, K)$. Then we have

Lemma 3.1. *The endomorphism D_1 of V_ρ has the following properties:*
(1) $D_1 V_\rho^K \subset V_\rho^K$,
(2) *D_1 is self-adjoint on V_ρ, that is, $((D_1 u, v)) = ((u, D_1 v))$ for every $u, v \in V_\rho$, and*
(3) $D_1(V_\rho^K)^\perp \subset (V_\rho^K)^\perp$,
where $(V_\rho^K)^\perp$ is the orthocomplement of V_ρ^K in V_ρ with respect to $((\ ,\ ))$.

Proof. (1) Since $\mathrm{Ad}(k)X_1 = X_1$ $(k \in K)$, we have

$$\rho(X_1)v = \rho(\mathrm{Ad}\,(k)X_1)v = \rho(k)\rho(X_1)\rho(k^{-1})v = \rho(k)\rho(X_1)v\,,$$

for $v \in V_\rho^K$. (2) follows from the equality $((\rho(X)u, v)) + ((u, \rho(X)v)) = 0$, for all $X \in \mathfrak{g}$, u and $v \in V_\rho$. (3) is clear from (1) and (2). Q.E.D.

Thus, due to Lemma 3.1, there exists an orthonormal basis $\{u_j\}_{j=1}^{d_\rho}$ of V_ρ with respect to $((\ ,\ ))$ such that $\{u_j\}_{j=1}^{m_\rho}$ is a basis of V_ρ^K and

$$(3.2) \qquad D_1 u_j = \mu_j u_j \qquad (j = 1, \cdots, d_\rho)$$

for some real numbers μ_j $(j=1, \cdots, d_\rho)$. For each $\rho \in \boldsymbol{D}(G, K)$, choose such a basis on V_ρ and let $\bar{\rho}_{ij}$ be a function in $\theta_\rho(M)$, as in 3.1 with respect to this basis. Then, for each $\rho \in \boldsymbol{D}(G, K)$, we have

$$(3.3) \qquad \Delta_{g_1}\bar{\rho}_{ij} = (\mu_\rho + 2\delta, \mu_\rho)_0 \bar{\rho}_{ij}\,,$$

where $\delta = \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$ and $\mu_\rho \in D(G)$ is the highest weight for $\rho \in \boldsymbol{D}(G, K) \subset \boldsymbol{D}(G)$. Because

$$\begin{aligned}\Delta_{g_1}\bar{\rho}_{ij} &= -\hat{\lambda}(\textstyle\sum_{k=1}^n X_k^2)\bar{\rho}_{ij} && \text{(by Corollary 1)},\\ &= -\hat{\lambda}(C)\bar{\rho}_{ij}\,,\\ &= (\mu_\rho + 2\delta, \mu_\rho)_0 \bar{\rho}_{ij} && \text{(cf. [6])},\end{aligned}$$

where the operator $\hat{\lambda}(C)$ is the Casimir operator (cf. [6]) of $\mathfrak{g}$ with respect to the $\mathrm{Ad}(G)$-invariant inner product B on $\mathfrak{g}$ for $\rho \in \boldsymbol{D}(G, K) \subset \boldsymbol{D}(G)$. The second equality follows from that $\bar{\rho}_{ij} \in C^\infty(M)$, the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ is orthogonal, and $\{X_k\}_{k=1}^n$ is an orthonormal basis of $\mathfrak{m}$ with respect to B. Therefore we have

$$(3.4) \qquad \Delta_{g_t}\bar{\rho}_{ij} = [(t^{-(n-1)} - t)(-\mu_j) + t(\mu_\rho + 2\delta, \mu_\rho)_0]\bar{\rho}_{ij}\,,$$

due to Lemma 2.4, (3.1), (3.2) and (3.3). Also we have

$$\Delta_{g_t}\theta_\rho(M) \subset \theta_\rho(M)\,,$$

for each $\rho\in \boldsymbol{D}(G,K)$ and $0<t<\infty$. So we put $\lambda_1(g_t,\rho)$ the least positive eigenvalue of Δ_{g_t} on $\theta_\rho(M)\,(0<t<\infty)$. Then we have

$$\lambda_1(g_t)=\min_{\rho\in \boldsymbol{D}(G,K)-(0)}\lambda_1(g_t,\rho)\,, \tag{3.5}$$

by the Peter-Weyl theorem. Moreover we have

$$\lambda_1(g_t,\rho)=\min_{1\leqq j\leqq m_\rho}[(t^{-(n-1)}-t)(-\mu_j)+t(\mu_\rho+2\delta,\mu_\rho)_0]\,, \tag{3.6}$$

by (3.4).

3.4. We will prove our Main Theorem due to the above preparations. Our claims are divided into two cases.

Case (1). $t\leqq 1$, that is, $t^{-(n-1)}-t\geqq 0$. In this case, we have

$$\lambda_1(g_t,\rho)=(t^{-(n-1)}-t)[\min_{1\leqq j\leqq m_\rho}(-\mu_j)]+t(\mu_\rho+2\delta,\mu_\rho)_0\,. \tag{3.7}$$

Case (2). $t\geqq 1$, that is $t^{-(n-1)}-t\leqq 0$. In this case, we have

$$\lambda_1(g_t,\rho)=(t^{-(n-1)}-t)[\max_{1\leqq j\leqq m_\rho}(-\mu_j)]+t(\mu_\rho+2\delta,\mu_\rho)_0\,. \tag{3.8}$$

Lemma 3.2. *We have*

(1) $\displaystyle\min_{1\leqq j\leqq m_\rho}(-\mu_j)=\min_{v\in V_\rho{}^K,((v,v))=1}((-D_1v,v)),$ *and*

(2) $\displaystyle\max_{1\leqq j\leqq m_\rho}(-\mu_j)=\max_{v\in V_\rho{}^K,((v,v))=1}((-D_1v,v))$

$$\leqq \max\ \{(\mu_1,\mu_1)_0;\ \mu \text{ is a weight of } V_\rho\}\,,$$

where μ_1 *is the restriction of* $\mu\in\mathfrak{t}^*$ *onto* $\mathfrak{m}_1$.

Proof. For $v=\sum_{j=1}^{m_\rho}x_ju_j\in V_\rho{}^K$ $(x_j\in \boldsymbol{C},\ 1\leqq j\leqq m_\rho,$ and $((v,v))=\sum_{j=1}^{m_\rho}|x_j|^2=1)$, we have

$$((D_1v,v))=\sum_{j=1}^{m_\rho}\mu_j|x_j|^2\,.$$

Then we obtain

$$\begin{aligned}\min_{1\leqq j\leqq m_\rho}(-\mu_j)&=\min_{\sum_{j=1}^{m_\rho}|x_j|^2=1}\sum_{j=1}^{m_\rho}(-\mu_j)|x_j|^2\,,\\ &=\min_{v\in V_\rho{}^K,((v,v))=1}((-D_1v,v))\,.\end{aligned}$$

In the same manner, we have,

$$\begin{aligned}\max_{1\leqq j\leqq m_\rho}(-\mu_j)&=\max_{v\in V_\rho{}^K,((v,v))=1}((-D_1v,v))\,,\\ &\leqq \max_{v\in V_\rho{}^K,((v,v))=1}((-D_1v,v))\,.\end{aligned}$$

The right hand side coincides with $\max\{(\mu_1, \mu_1)_0;\ \mu$ is a weight of $V_\rho\}$. For, let $V_\rho=\sum_{\mu\in I} V_\mu$ (the decomposition of V_ρ into weight spaces). Then $\rho(H)v_\mu=\sqrt{-1}\mu(H)v_\mu$, $H\in\mathfrak{t}$, $v_\mu\in V_\mu(\mu\in I)$ and V_μ and $V_{\mu'}$ are mutually orthogonal with respect to $((\ ,\ ))$ if $\mu\neq\mu'(\mu, \mu'\in I)$. Due to Lemma 2.1, we have

$$D_1v_\mu = -\mu(X_1)^2v_\mu = -(\mu_1, \mu_1)_0v_\mu\ ,$$

by $B(X_1, X_1) = 1$. Lemma 3.2 is proved completely.

Now, firstly, we treat Case (1). In this case, due to the above Lemma 3.2, we have

$$(3.7)' \qquad \lambda_1(g_t, \rho) = (t^{-(n-1)}-t)\min_{v\in V_\rho{}^K,((v,v))=1}((-D_1v, v))+t(\mu_\rho+2\delta, \mu_\rho)_0\,.$$

Lemma 3.3. *If the one-parameter subgroup $T_1=\{\exp(sX_1);\ s\in\boldsymbol{R}\}$ is closed in G, then there exists an element $\rho_0\in D(G, K)-(0)$ such that* $\min_{v\in V_{\rho_0}{}^K,\ ((v,v))=1}((-D_1v, v))=0$.

Proof. Let $K'=\{kt;\ k\in K,\ t\in T_1\}$. Then K' is a closed Lie subgroup of G with the Lie subalgebra $\mathfrak{k}+\mathfrak{m}_1$ of $\mathfrak{g}$, due to the closedness of T_1. Moreover it includes K as a closed subgroup. Let $M'=G/K'$ be a coset space of G by K'. We can apply the Peter-Weyl theorem for this coset space. Let $V_\rho{}^{K'}=\{v\in V_\rho;\ \rho(k')v=v$ for all $k'\in K'\}$. Then we have

$$\begin{aligned} V_\rho{}^{K'} &= \{v\in V_\rho{}^K;\ \rho(t)v = v \text{ for all } t\in T_1\}\ ,\\ &= \{v\in V_\rho{}^K;\ \rho(X_1)v = 0\}\ . \end{aligned}$$

Since $\dim(M')\geqq 1$, there exists a non-zero element ρ_0 in $\boldsymbol{D}(G, K')$ by the Peter-Weyl theorem. Since $\boldsymbol{D}(G, K')=\{\rho\in\boldsymbol{D}(G);\ V_\rho{}^{K'}\neq(0)\}$, we have a non-zero element ρ_0 in $\boldsymbol{D}(G, K)$ such that $\{v\in V_{\rho_0}{}^K;\ \rho_0(X_1)v=0\}\neq(0)$, that is, there exists a non-zero element $v_0\in V_{\rho_0}{}^K$ satisfying $\rho_0(X_1)v_0=0$. Then, by the definition of D_1, we have $((-D_1v_0, v_0))=0$, for $\rho_0\in\boldsymbol{D}(G, K)$. Since $((-D_1v, v))=((\rho_0(X_1)v, \rho_0(X_1)v))\geqq 0$ for every $v\in V_{\rho_0}{}^K$, we have the desired result. Q.E.D.

Due to Lemma 3.3, we obtain

$$\lambda_1(g_t)\leqq\lambda_1(g_t, \rho_0) = t(\mu_{\rho_0}+2\delta, \mu_{\rho_0})_0\,.$$

Thus we obtain $\lim_{t\to 0}\lambda_1(g_t)=0$.

Case (2). $t\geqq 1$, that is $(t^{-(n-1)}-t)\leqq 0$. By Lemma 3.2 and (3.8), we have

$$\lambda_1(g_t, \rho)\geqq(t^{-(n-1)}-t)\max\{(\mu_1, \mu_1)_0;\ \mu \text{ is a weight of } V_\rho\}+t(\mu_\rho+2\delta, \mu_\rho)_0\,.$$

Notice that for each weight μ of V_ρ, we have

$$(\mu_1, \mu_1)_0\leqq(\mu, \mu)_0\leqq(\mu_\rho, \mu_\rho)_0\,,$$

(cf. [8] p. 221). The first inequality follows due to the definition of μ_1 for $\mu \in \mathfrak{t}^*$. Thus we have, for each $\rho \in \boldsymbol{D}(G, K)$,

$$\lambda_1(g_t, \rho) \geqq (t^{-(n-1)} - t)(\mu_\rho, \mu_\rho)_0 + t(\mu_\rho + 2\delta, \mu_\rho)_0$$
$$= t^{-(n-1)}(\mu_\rho, \mu_\rho)_0 + t(2\delta, \mu_\rho)_0 \geqq t(2\delta, \mu_\rho)_0 .$$

Thus we have

$$\lambda_1(g_t) \geqq t \min_{\rho \in \boldsymbol{D}(G, K) - (0)} (2\delta, \mu_\rho)_0 \geqq t \min_{\rho \in \boldsymbol{D}(G) - (0)} (2\delta, \mu_\theta)_0 .$$

By the assumption of the semi-simplicity of G, we have $\min_{\rho \in \boldsymbol{D}(G) - (0)} (2\delta, \mu_\rho)_0 > 0$. Therefore we obtain $\lim_{t \to \infty} \lambda_1(g_t) = \infty$. Main Theorem is proved completely.

References

[1] M. Berger, P. Gauduchon and E. Mazet: Le spectre d'une variété riemannienne, Lecture note in Math., 194, Springer, Berlin, 1971.

[2] S. Helgason: Differential geometry and symmetric spaces, Academic Press, New York, 1962.

[3] S. Kobayashi and K. Nomizu: Foundations of differential geometry II, Interscience, New York, 1969.

[4] G.W. Lukesh: *Compact homogeneous Riemannian manifolds,* Geometriae Dedicata **7** (1978) 131–137.

[5] G.W. Lukesh: *Variations of metrics on homogeneous space* (a preprint), 1978.

[6] M. Takeuchi: Modern theory of spherical functions (in Japanese), Iwanami, Tokyo, 1975.

[7] S. Tanno: *The first eigevnalue of the Laplacian on spheres,* to appear in Tohoku Math. J.

[8] H. Urakawa: *On the least positive eigenvalue of the Laplacian for compact group manifolds,* J. Math. Soc. Japan **31** (1979), 209–226.

[9] J. Wolf: *The geometry and structure of isotropy irreducible homogeneous spaces,* Acta Math. **120** (1968), 59–148.

Hideo Muto
Department of Mathematics
Tohoku University
Sendai 980, Japan

Hajime Urakawa
Department of Mathematics
College of General Education
Tohoku University
Sendai 980, Japan