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ON A CHARACTERIZATION OF SURFACES CONTAINING CYLINDERLIKE OPEN SETS

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Introduction. Let k be an algebraically closed field of characteristic zero. Let V be a nonsingular projective surface defined over k and let D be a reduced effective divisor on V. Consider the following four conditions:

(1) There exists a nonempty open set U in V-Supp (D) such that U has a structure of trivial A^1-bundle; U is called a cylinderlike open set;

(2) There exists an irreducible curve C on V such that $C \not\subset \mathrm{Supp}(D)$ and $(C \cdot D+K)<0$, where K is the canonical divisor on V;

(3) for any divisor A on V, $|A+m(D+K)|=\phi$ for all sufficiently large integer m;

(4) $|m(D+K)|=\phi$ for every positive integer m.

If D satisfies the condition that V-Supp (D) is affine and Supp (D) has only normal crossings as singularities, then the above four conditions are equivalent to each other. In effect, the equivalence of the first three conditions and the implication (3)$\Rightarrow$(4) are proved in the previous paper with Miyanishi [MS]. The implication (4)$\Rightarrow$(3) was proved by Fujita [F].

In the first part of this paper, we shall prove the following

Theorem. *With the notations as above, assume that the following conditions are satisfied:*

(i) V-Supp (D) *contains no excpetional curve of the first kind and* Supp (D) *is connected*;

(ii) Supp (D) *has only normal crossings as singularities*;

(iii) *write* $D=\sum_{i=1}^{r} C_i$, *where* C_i *is an irreducible component*; *then the* $(r\times r)$—*matrix* $((C_i\cdot C_j)_{1\leq i,j\leq r})$, *which we call simply the intersection matrix of* D, *is not negative definite. Then the above four conditions are equivalent to each other.*

This theorem does not hold if one drops off the condition that Supp (D) is connected. In the second part, we shall show this by constructing a counter-example.

We retain in this article the terminology and notations of the previous paper [MS].

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1. Proof of Theorem

The theorem is proved by following the arguments of the previous paper [*MS*] and by making necessary modifications. Thus our proof consists in pointing out the parts to be modified and indicating how these parts are modified.

1. **Lemma.** *Under the above notations, the conditions* (3) *and* (4) *are equivalent to each other. If D has only normal crossings as singularities, then we have the implications*: (1)$\Rightarrow$(2) *and* (1)$\Rightarrow$(3). *If V-*Supp(D) *does not contain an exceptional curve of the first kind, then we have the implication*: (2)$\Rightarrow$(3).

Proof. The equivalence of the conditions (3) and (4) is proved in Fujita [*F*]. (2)$\Rightarrow$(3): Since $C\not\subset \mathrm{Supp}(D)$ and $(C\cdot D+K)<0$, we have $(C\cdot K)<0$. If $(C^2)<0$, C must be an exceptional curve of the first kind. By the assumption we have $(C\cdot D)>0$ and hence $(C\cdot K)\leqq -2$, which is a contradiction. Therefore, $(C^2)\geqq 0$. Since $(A+m(D+K)\cdot C)<0$ if $m>-(A\cdot C)/(D+K\cdot C)$, we know that $|A+m(D+K)|=\phi$ if $m>-(A\cdot C)/(D+K\cdot C)$.

(1)$\Rightarrow$(2): With the notations of Lemma 1.3 and its proof in [MS], let Λ be the linear pencil on V defined by the fibration of a cylinderlike open set U. Let C be a general member of Λ. We may assume that either Λ has a base point or $C\cap D\neq\phi$. In effect, otherwise, $p_a(C)=(C^2)=0$ and $(C\cdot D)=0$, whence $(C\cdot D+K)<0$. Let P be the unique base point of Λ in the first case, and let $P:=C\cap D$ in the second case. Then the proof in 1.3 of [MS] holds without change by neglecting the condition that V-Supp(D) is affine. Also, note that, in this proof, we actually found an irreducible curve C on V such that $C\not\subset$ Supp (D), $(C\cdot D+K)<0$ and $(C^2)\geqq 0$. Then we can prove the implication (1)$\Rightarrow$(3) in the same fashion as in the proof of the implication (2)$\Rightarrow$(3). Q.E.D.

2. As for the implication (3)$\Rightarrow$(1), we have following:

Proposition. *Let V be a nonsingular projective surface and let D be a reduced effective divisor on V. Assume that* Supp(D) *is connected and that the intersection matrix of D is not negative definite. Then the condition* (3) *implies the condition* (1).

Our proof consists of several subparagraphs below.

2.1. In the case where V-Supp(D) is affine, the proposition follows from Theorems 2.1 and 2.2 (the case where V is irrational) and Theorem 6.3 (the case where V is rational) in the previous paper [MS]. To prove Theorem 6.3, we used, roughly speaking, all results from the first up to the paragraph 5.6.

Our claim is that the proposition holds even if the condition that V-Supp (D) is affine is replaced by weaker conditions:

(a) Supp (D) is connected, and

(b) the intersection matrix of D is not negative definite.

Our proof of the proposition in the present situation consists mainly in indicating necessary changes of proofs when the above relaxation of the condition is made.

2.2. A useful remark is the following

Lemma. *Let V be a nonsingular projective surface and let $D=\sum_{i=1}^{n} C_i$ be a reduced effective divisor on V such that the intesection matrix $((C_i\cdot C_j)_{1\leqq i,j\leqq n})$ is not negative definite, where C_i's are irreducible components of D. Let E be an exceptional curve of the first kind on V, and let $\sigma\colon V\to\bar{V}$ be the contraction of E. Let $\bar{D}=\sigma_*(D)$. Then the intersection matrix of $\bar{D}$ is not negative definite.*

Proof. By the assumption, there exists a divisor $A=\sum_{i=1}^{n} a_iC_i$ such that $(A^2)\geqq 0$, where $a_i\in Z$. Since

$$0\leqq(A^2)\leqq\sum_{i,j}|a_i|\cdot|a_j|(C_i\cdot C_j)\,,$$

we may assume that every $a_i\geqq 0$. If $E\not\subset\operatorname{Supp}(D)$, set $\bar{C}_i=\sigma(C_i)$ and $\bar{A}=\sum_{i=1}^{n}a_i\bar{C}_i$. Then we have

$$(\bar{A}^2)=\sum_{i,j=1}^{n}a_ia_j(\bar{C}_i\cdot\bar{C}_j)\geqq\sum_{i,j=1}^{n}a_ia_j(C_i\cdot C_j)=(A^2)\geqq 0\,.$$

Hence the intersection matrix of $\bar{D}$ is not negative definite. If $E\subset\operatorname{Supp}(D)$, we may assume $E=C_1$. Set $\bar{C}_i=\sigma(C_i)$ for $2\leqq i\leqq n$ and $\bar{A}=\sum_{i=2}^{n}a_i\bar{C}_i$. Then we have

$$\begin{aligned}
&(\bar{A}^2)-(A^2)\\
&=\sum_{i=2}^{n}a_i^2\{(C_i^2)+(E\cdot C_i)^2\}+2\sum_{2\leqq i<j\leqq n}a_ia_j\{(C_i\cdot C_j)+(E\cdot C_i)(E\cdot C_j)\}\\
&\quad-\{a_1^2(E^2)+2\sum_{i=2}^{n}a_1a_i(E\cdot C_i)+(\sum_{i=2}^{n}a_iC_i\cdot\sum_{i=2}^{n}a_iC_i)\}\\
&=a_1^2+\sum_{i=2}^{n}a_i^2(E\cdot C_i)^2+2\sum_{2\leqq i<j\leqq n}a_ia_j(E\cdot C_i)(E\cdot C_j)-2a_1\sum_{i=2}^{n}a_i(E\cdot C_i)\\
&=\{\sum_{i=2}^{n}a_i(E\cdot C_i)-a_1\}^2\geqq 0\,.
\end{aligned}$$

Hence the intersection matrix of $\bar{D}$ is not negative definite. Q.E.D.

2.3. In order to prove the proposition, we may assume an additional condition:

(c) V-Supp(D) *contains no exceptional curve of the first kind.*

In effect, if E is an exceptional curve of the first kind contained in V-Supp (D), let σ: $V\to\bar{V}$ be the contraction of E and let $\bar{D}=\sigma_*(D)$. Then Supp $(\bar{D})$ is connected, and the intersection matrix of $\bar{D}$ is not negative definite by virtue of Lemma 2.2. Moreover, since $D+K_V=\sigma^*(\bar{D}+K_{\bar{V}})+E$, the condition (3) for V and D implies the condition (3) for $\bar{V}$ and $\bar{D}$. If $\bar{V}$-Supp $(\bar{D})$ contains a cylinderlike open set, V-Supp (D) clearly contains a cylinderlike open set. Therefore, we may assume that the additional condition (c) holds on V.

2.4. Theorems 2.1 and 2.2 of [MS] hold true under the present assumptions; in effect, we did not assume that V-Supp(D) is affine. Hence the proposition holds in the case where V is an irrational ruled surface. Therefore, we assume from now on that V is rational.

2.5. Among Lemmas 3.1~3.4 of [MS], Lemma 3.1 holds without any change. As for Lemma 3.2, we need to modify the proof a little. In the paragraph 3.2.1, if $(C\cdot D)>0$, then $(C\cdot D)=1$ by Lemma 3.1 for D is connected. However, $(C\cdot D)$ might be zero, and we must consdier this case separately in the paragraph 3.2.4. In both of the cases A and B there, D is contained in a member of $|C|$. In the case A, V-Supp (D) clearly contains a cylinderlike open set. In the case B, let C_0 be a member of $|C|$ such that Supp $(D)\subset$Supp (C_0) and let C be a general member of $|C|$. Since $(C\cdot C_0)=1$, let $P:=C\cap C_0$. Then $P\notin$Supp (D). consider the linear pencil $L:=|C|-P$ and make the same arguments as in [MS]. Then we find easily a cylinderlike open set in V-Supp (D). Then proof of Lemma 3.3 has to be modified a bit as well. In the proof, it may occur that $n=0$ and $C=D$. In this case, $|C|$ is a linear pencil without base points whose general members are nonsingular rational curves. Then V-Supp (D) contains evidently a cylinderlike open set. We can also prove Lemma 3.4 just by the same way as in [MS], using the modified Lemma 3.3.

2.6. In the section 4 of [MS], Lemma 4.1 holds without any change. But, we shall note that $p_a(D)=0$ or equivalently saying, $(D\cdot D+K)=-2$ because $|D+K_V|=\phi$ and D is connected (cf. Lemma 1.2, ibid.). As for Theorem 4.2, an additional consideration is needed in the paragraphs 4.2.3.2 and 4.2.4. In the paragraph 4.2.3.2 we concluded that $D=M+D'$ in the case $n=1$ by making use of the assumption that V-Supp (D) is affine. In the present situation, we assume instead that V-Supp (D) contains no exceptional curve of the first kind. If M is not a component of D, then M would be contained in V-

Supp (D), which is a contradiction because $(M^2)=-n=-1$. In the paragraph 4.2.4, we concluded that $(D\cdot E)>0$ when $E\not\subset \text{Supp}(D)$ by using the assumption that V-Supp (D) is affine. We obtain the same conclusion by a similar fashion as above. Corollaries 4.3 and 4.5 hold with due modifications in the statements.

2.7. In the proof of Theorem 5.1 of [MS], the paragraphs 5.2 and 5.3 are valid with due modifications. In the stated assertion of the paragraph 5.4, we must replace the condition (1) by the following condition:

(1′) There is no nonsingular rational curve F (other than E if $E\not\subset \text{Supp}$ (D)) on V such that $F\not\subset \text{Supp}(D)$, $(F\cdot D)>0$ and $(F^2)<0$.

The modified assertion can be proved in a similar fashion as for the original one. In the paragraph 5.5 (for the proof of Theorem 5.1) we have to use essentially, in the case $E\not\subset \text{Supp}(D)$, the assumption that the intersection matrix of D is not negative definite, which is a well-known property of D if V-Supp (D) is affine. In the case $E\subset \text{Supp}(D)$, E is the unique exceptional curve of the first kind. Indeed, if F is an exceptional curve of the first kind other than E, either $F\subset V$-Supp (D) or $(F\cdot D)>0$. The first case does not occur because of the assumption that V-Supp (D) contains no exceptional curve of the first kind. The second case does not occur, either, because of the above condition (1′). In the remaining parts of the proof, we have to modify only the following points. Namely, in the assertion (3) of Lemma 5.6, delete the condition that V-C is affine. It is easy to check that Lemma 5.6 still holds without this condition. Therefore, we need not show that C^* is ample (cf. the paragraph 5.5, ibid.).

2.8. Now we can prove the proposition in the following way. We shall proceed by induction on $-(K^2)$, where $-(K^2)\geqq -8$ or -9. If V is relatively minimal, the proposition follows from the modified Corollary 4.5. Therefore we shall assume that V is not relatively minimal. If $((D+K)^2)\geqq -1$, the proposition follows from Lemma 4.1, Corollary 4.3 and Theorem 5.1 in their modified versions. Hence we have only to consider the case where $((D+K)^2)\leqq -2$. Since V is not relatively minimal, there exists an exceptional curve E of the first kind on V. Consider a linear system $|E+D+K|$. If $|E+D+K|=\phi$, then $(D\cdot E)=0$ or 1 because D is connected. Let $\sigma\colon V\to\bar{V}$, be the contraction of E and let $\bar{D}=\sigma_*(D)$. Then $\bar{V}$ and $\bar{D}$ satisfy the conditions (a), (b) and (c) in the paragraphs 2.1 and 2.3. Moreover the conditions (3) (cf. Introduction) is satisfied by $\bar{V}$ and $\bar{D}$ as shown in the same fashion as in the proof of Assertion B in the paragraph 6.3 [MS]. Since $-(K_{\bar{V}}{}^2)=-(K_V{}^2)-1$, we are done by induction.

Assume now that $|E+D+K|\neq\phi$. Then, by the condition (3), there exists an integer $n \geqq 2$ such that

$$|E+(n-1)(D+K)| \neq \phi \text{ and } |E+n(D+K)| = \phi .$$

Let $\sum_i n_i C_i$ be a member of $|E+(n-1)(D+K)|$. Since $((D+K)^2) \leqq -2$ we have $(D+K \cdot K) \leqq 0$. Hence we have $(E+(n-1)(D+K) \cdot K) \leqq -1$ which implies that $\sum_i n_i C_i \not\sim 0$. Then C_i is a nonsingular rational curve such that $|C_i+D+K| = \phi$ for every i, because $|E+n(D+K)| = \phi$. If $(C_i^2) \geqq 0$ for some i, we are done by virtue of the modified Lemma 3.2. Thus we have only to consider the case where $(C_i^2) < 0$ for every i. Note that we have

$$(K \cdot E+(n-1)(D+K)) = \sum n_i (C_i \cdot K) < 0 .$$

Thence $(C_i \cdot K) < 0$ for some i, say $i=1$. Then C_1 is an exceptional curve of the first kind such that $|C_1+D+K| = \phi$. Then we are done by the same arguments as above, where E is replaced by C. This completes the proof of the proposition. Q.E.D.

Now combining Lemma 1 and Proposition 2, we get an equivalence of the four conditions (1), (2), (3), (4) in the Introduction under the assumptions (i), (ii), (iii) of the Theorem. Thus we complete the proof of the Theorem.

2. A counter-example

1. We shall now construct an example of a nonsingular projective surface V and a reduced effective divisor D on V such that V-Supp(D) does not contain a cylinderlike open set, although $|m(D+K_V)| = \phi$ for every positive interger m. A counter-example we shall present below is the one in which the number m of connected components of D equals 2 and the intersection matrix of D is not negative definite.

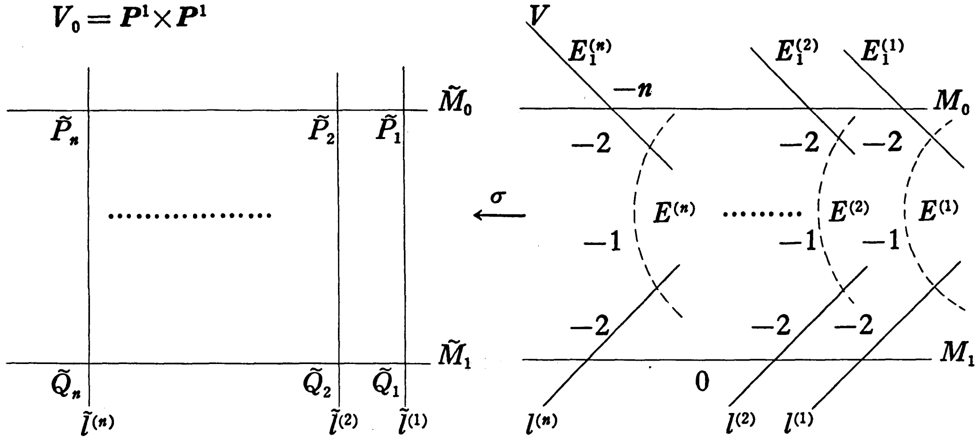
Let $V_0 = \boldsymbol{P}^1 \times \boldsymbol{P}^1$ and let $\pi : V_0 \to \boldsymbol{P}^1$ be the projection onto the second factor. A fiber of π is denoted by $\tilde{l}$ and a cross-section of π is denoted by $\tilde{M}$, where the cross-sections are understood to be fibers of the first projection $pr_1 : V_0 \to \boldsymbol{P}^1$.

Let $\tilde{M}_0$ and $\tilde{M}_1$ be cross-sections of V_0 and let $\tilde{l}^{(1)}, \tilde{l}^{(2)}, \cdots, \tilde{l}^{(n)}$ be n distinct fibers of π. Let $\tilde{P}_i := \tilde{M}_0 \cap \tilde{l}^{(i)}$ and $\tilde{Q}_i := \tilde{M}_1 \cap \tilde{l}^{(i)}$. Let $\tilde{P}_i'$ be the infinitely near point of order one of $\tilde{P}_i$ on $\tilde{l}^{(i)}$. Let $\sigma : V \to V_0$ be the composition of quadratic transformations with centers $\tilde{P}_i$ and $\tilde{P}_i'$ for $1 \leqq i \leqq n$, let $M_0 = \sigma'(\tilde{M}_0)$ and $M_1 = \sigma'(\tilde{M}_1)$, let $\tilde{l}^{(i)} = \sigma'(\tilde{l}^{(i)})$ and let $E_1^{(i)}$ and $E^{(i)}$ be the proper transforms of the irreducible exceptional curves obtained by the quadratic transformations with centers $\tilde{P}_i$ and $\tilde{P}_i'$ for $1 \leqq i \leqq n$. Put $P_i := E_1^{(i)} \cap M_0$, $Q_i := l^{(i)} \cap M_1$. Then we have,

$$(M_0^2) = -n, \ (M_1^2) = 0, \ ((l^{(i)})^2) = -2 ,$$

$$((E_1^{(i)})^2) = -2, \text{ and } ((E^{(i)})^2) = -1 .$$

We have the following configuration:



2. We define a reduced effective divisor D on V by $D := M_0 + M_1 + \sum_{i=1}^{n} (E_1^{(i)} + l^{(i)})$. Then Supp (D) is not connected; $M_0 + \sum_{i=1}^{n} E_1^{(i)}$ and $M_1 + \sum_{i=1}^{n} l^{(i)}$ are its connected components. It is easy to show the followings:

$$\sigma^*(\tilde{l}) \sim l^{(i)} + E_1^{(i)} + 2E^{(i)} \quad (1 \leqq i \leqq n)$$
$$\sigma^*(\tilde{M}_0) \sim M_0 + \sum_{i=1}^{n} (E_1^{(i)} + E^{(i)}) \sim M_1$$
$$K_V \sim \sigma^*(K_{V_0}) + \sum_{i=1}^{n} (E_1^{(i)} + 2E^{(i)})$$

and

$$\begin{aligned} D + K_V &\sim M_0 + M_1 + \sigma^*(K_{V_0}) + \sum_{i=1}^{n} E_1^{(i)} + \sigma^*(n\tilde{l}) \\ &\sim \sigma^*(2\tilde{M}) + \sigma^*(K_{V_0}) + n\sigma^*(\tilde{l}) - \sum_{i=1}^{n} E^{(i)} \\ &\sim \sigma^*((n-2)\tilde{l}) - \sum_{i=1}^{n} E^{(i)} . \end{aligned}$$

3. We have then the following:

Lemma. (1) *If $n \geqq 5$, then $\kappa(D+K_V)=1$;*
(2) *If $n=4$, then $\kappa(D+K_V)=0$,*
(3) *If $n \leqq 3$, then $\kappa(D+K_V)=-\infty$,*

Proof. (1) From the above formulas, we have

$$2(D+K_V) \sim (n-4)\sigma^*(\tilde{l}) + \sum_{i=1}^{n} (l^{(i)} + E_1^{(i)}) .$$

Since $(K+D \cdot \sigma^*(\tilde{l}))=0$, any irreducible component of an effective divisor of $|m(D+K_V)|$ (if it is non-empty for some $m>0$) is a component of a divisor

of the form $\sigma^*(\tilde{l})$. Hence $\kappa(D+K_V)=1$ if $n\geqq 5$ and $\kappa(D+K_V)=0$ if $n=4$.
(ii) Suppose $n=3$. Then $D+K_V\sim\sigma^*(\tilde{l})-(E^{(1)}+E^{(2)}+E^{(3)})$. Suppose $|m(D+K_V)|=\phi$ for some $m>0$. Since $(m(D+K_V)\cdot\sigma^*(\tilde{l}))=0$, any effective member of $|m(D+K_V)|$ is a linear combination of $\sigma^*(\tilde{l})$, $E_1{}^{(i)}$, $E^{(i)}$ and $l^{(i)}$ for $1\leqq i\leqq n$. Then we have a relation of the form

$$r\sigma^*(\tilde{l})\sim\sum_{i=1}^{n}(\alpha_i E_1{}^{(i)}+\beta_i E^{(i)}+\gamma_i l^{(i)}),$$

where $r, \alpha_i, \beta_i, \gamma_i\in \boldsymbol{Z}$, and $\alpha_i, \beta_i, \gamma_i\geqq 0$ for $i=1, 2, 3$; if $\beta_i\geqq 2$, then we may assume that $\alpha_i\gamma_i=0$. We have then

$$\begin{aligned} 0 = (r\sigma^*(\tilde{l}))^2 &= \Big(\sum_{i=1}^{n}(\alpha_i E_1{}^{(i)}+\beta_i E^{(i)}+\gamma_i l^{(i)})\Big)^2 \\ &= \sum_{i=1}^{n}(\alpha_i E_1{}^{(i)}+\beta_i E^{(i)}+\gamma_i l^{(i)})^2 \\ &= -\frac{1}{2}\sum_{i=1}^{n}\{(2\alpha_i-\beta_i)^2+(2\gamma_i-\beta_i)^2\}, \end{aligned}$$

whence $2\alpha_i=\beta_i=2\gamma_i$ for $i=1, 2, 3$. Thus $\alpha_i=\beta_i=\gamma_i=0$ for $i=1, 2, 3$. This implies that

$$m\sigma^*(\tilde{l})-m\sum_{i=1}^{3}E^{(i)}\sim s\rho^*(\tilde{l})+\frac{m}{2}\sum_{i=1}^{3}(E_1{}^{(i)}+l^{(i)})$$

where $s+\dfrac{3}{2}m=m$; *i.e.*, $s=-\dfrac{m}{2}<0$. This is a contradiction. Hence $\kappa(D+K_V)=-\infty$. If $n\leqq 2$, it is clear that $\kappa(D+K_V)=-\infty$ because $|-(D+K_V)|=\phi$.
Q.E.D.

4. First of all we have the following

Lemma. V-Supp (D) *does not contain an exceptional curve of the first kind.*

Proof. If V-Supp (D) contains an exceptional curve F of the first kind, then we have $-1=(F\cdot K_V)=(\sigma_*(F)\cdot K_{V_0})+2\sum_{i=1}^{n}(F\cdot E^{(i)})$. Since $(\sigma_*(F)\cdot K_{V_0})\equiv 0$ (mod 2), we have a contradiction. Q.E.D.

5. Hereafter, we shall only consider the case $n=3$. Our objective is to show that V-Supp (D) contains no cylinderlike open set. Suppose V-Supp (D) contains a cylinderlike open set $U\cong \boldsymbol{A}^1\times U_0$, where U_0 is an open set of a parameter curve. The fibers C_0 of the fibration $U\to U_0$ define a linear pencil L on V whose general members C are the closures of general fibers C_0 of the fibration $U\to U_0$. Then C satisfies the following properties:

1) The geometric genus $g(C)$ of C is 0.
2) $C-C_0$ consists of a one-place point.
3) $(C^2)\geqq 0$ and $(C\cdot D+K_V)<0$.

4) $C \sim aM_0 + \sum_{i=1}^{3} (\alpha_i E_1^{(i)} + \beta_i E^{(i)} + \gamma_i l^{(i)})$ where $a, \alpha_i, \beta_i, \gamma_i \in \boldsymbol{Z}$.

5) $(C \cdot \sigma^*(\tilde{l})) = a$, $(C \cdot E_1^{(i)}) = a - 2\alpha_i + \beta_i$,

$$(C \cdot E^{(i)}) = \alpha_i - \beta_i + \gamma_i,\ (C \cdot l^{(i)}) = \beta_i - 2\gamma_i\,,$$
$$(C \cdot M_0) = -3a + \alpha_1 + \alpha_2 + \alpha_3\,,$$
$$(C \cdot M_1) = \gamma_1 + \gamma_2 + \gamma_3\,,$$
$$(C \cdot D + K_V) = a - \sum_{i=1}^{3} (\alpha_i - \beta_i + \gamma_i)\,,$$
$$(C^2) = -3a^2 + 2a\Sigma\alpha_i - 2\Sigma\alpha_i^2 - \Sigma\beta_i^2 - 2\Sigma\gamma_i^2 + 2\Sigma\alpha_i\beta_i + 2\Sigma\beta_i\gamma_i$$
$$= -3a^2 + 2a\sum_{i=1}^{3} \alpha_i - \frac{1}{2}\sum_{i=1}^{3} \{(2\alpha_i - \beta_i)^2 + (2\gamma_i - \beta_i)^2\}\,.$$

6) $\sigma_*(C) = \sigma(C)$ is an irreducible curve on V_0 such that $(\sigma_*(C) \cdot \tilde{M}_0) = (\sigma_*(C) \cdot \tilde{M}_1) > 0$; indeed, if $(\sigma_*(C) \cdot \tilde{M}_0) = 0$, then $\sigma_*(C) \sim \tilde{M}_0$; since C is a general member of Λ, $\sigma_*(C) \neq \tilde{M}_0$ and $\tilde{M}_1$; then C meets $l^{(1)}$, $l^{(2)}$ and $l^{(3)}$ whence $C - C_0$ consists of at least three points, which is a contradiction.

7) Since $(\sigma_*(C) \cdot \tilde{M}_1) = (C \cdot M_1) > 0$, C does not meet any one of M_0 and $E_1^{(i)}$ for $i = 1, 2, 3$. Hence we have:

$$\beta_i = 2\alpha_i - a \quad \text{and} \quad \alpha_1 + \alpha_2 + \alpha_3 = 3a\,.$$

Let $\tilde{Q} = \sigma_*(C) \cap \bar{M}_1$; $\tilde{Q}$ is a one-place point of $\sigma_*(C)$.

6. Now we need the following lemmas about linear pencils on surfaces.

6.1. **Lemma** (cf. [MS-2], Lemma 1.2). *Let V be a nonsingular projective surface and let Λ be an irreducible linear pencil on V such that general members of Λ are rational curves. Let B be the set of points of V which are base points of Λ. Let $F := n_1C_1 + n_2C_2 + \cdots + n_rC_r$ be a reducible member of Λ such that $r \geqq 2$, where C_i is an irreducible component, $C_i \neq C_j$ if $i \neq j$, and $n_i > 0$. Then the following assertions hold true:*

(1) *If $C_i \cap B = \phi$, then C_i is isomorphic to $\boldsymbol{P}_k^1$ and $(C_i^2) < 0$.*

(2) *If $C_i \cap C_j \neq \phi$ for $i \neq j$ and $C_i \cap C_j \cap B = \phi$ then $C_i \cap C_j$ consists of a single point where C_i and C_j intersect each other transversally.*

(3) *For three distinct indices i, j, l, $C_i \cap C_j \cap C_l \cap B = \phi$, then $C_i \cap C_j \cap C_l = \phi$.*

(4) *If* Supp (F) *contains a loop F', then* Supp (F') *must contain base points of Λ.*

(5) *Assume that $(C_i^2) < 0$ whenever $C_i \cap B \neq \phi$. Then the set $S := \{C_i : C_i$ is an irreducible component of F such that $C_i \cap B = \phi\}$ is nonempty and there is an exceptional component in the set S.*

6.2. **Lemma.** *Let V and Λ be as in the above lemma. Moreover, we assume* *that* Λ *has a single base point* P *and* P *is a one-place point for a general member of* Λ. *Let* $F:=n_1C_1+n_2C_2+\cdots+n_rC_r$ *be a member of* Λ; *we only assume* $r\geqq 1$ *in this lemma. Then* Supp (F) *does not contain a loop.*

Proof. Let ρ: $\tilde{V}\to V$ be the shortest succession of quadratic transformations with centers at base points (including infinitely near base points) of Λ such that the proper transform $\tilde{\Lambda}$ of Λ by ρ has no base points. Let $\tilde{F}$ be the member of $\tilde{\Lambda}$ corresponding to F and let E be the exceptional curve obtained by the last quadratic transformation. Then E is a cross-section of $\tilde{\Lambda}$ and other exceptional curves appear in the process ρ are contained in several members of $\tilde{\Lambda}$ since P is a one-place point for a general member of Λ.

Assume that F contains a loop, say $G=\{C_1, C_2, \cdots, C_l\}$. Then Supp (G) must contain the base point P by the above lemma 6.1, (4). Consider the proper transforms $C_i'=\rho'(C_i)$ of irreducible components C_i of G and set $G'=\{C_1', C_2', \cdots, C_l'\}$. Then G' contains no loops, since $\tilde{\Lambda}$ has no base points. Note that $\rho^{-1}(P)\cap \text{Supp}\,(G')$ consists of a finite number of points, at least two of which are contained in one and the same connected component of Supp (G'), because, if otherwise, G would not be a loop. Take two points P_1 and P_2 from $\rho^{-1}(P)\cap \text{Supp}\,(G')$ such that P_1 and P_2 are contained in the same connected component, say G'', of G'. Then there should exist a chain $\{A_1^{(1)}, A_1^{(2)}, \cdots, A_1^{(m)}\}$ of irreducible components of $\tilde{F}\cap\rho^{-1}(P)$ such that $(E\cdot A_1^{(1)})=(A_1^{(1)}\cdot A_1^{(2)})=\cdots=(A_1^{(m-1)}\cdot A_1^{(m)})=1$ and $P_1\in A_1^{(m)}$. Similarly, there exists a chain $\{A_2^{(1)}, A_2^{(2)}, \cdots, A_2^{(n)}\}$ of irreducible components of $\tilde{F}\cap\rho^{-1}(p)$, such that $(E\cdot A_2^{(1)})=\cdots(A_2^{(n-1)}\cdot A_2^{(n)})=1$ and $P_2\in A_2^{(n)}$. Since E is a cross-section of $\tilde{\Lambda}$, $A_1^{(1)}$ must coincide $A_2^{(1)}$. Then $\tilde{F}$ must contain a loop formed by some irreducible components among $\{A_1^{(1)}=A_2^{(1)}, A_1^{(2)}, \cdots, A_1^{(m)}, A_2^{(2)}, \cdots, A_2^{(n)}, C_1', \cdots, C_l'\}$. This is a contradiction. Q.E.D.

7. Retaining the foregoing notations (except in the pragraph 6), we shall derive a contradiction from the assumption that V-Supp (D) contains a cylinderlike open set.

Since the linear pencil Λ has a unique base point $Q:=C\cap M_1$ and since a general member C of Λ does not meet M_0 and $E_1^{(i)}$ for $i=1,2,3$, these four irreducible curves must be contained in one and the same member Γ_0 of Λ. These four components, of course, do not exhaust all irreducible components of Γ_0. Hence there is at least one more irreducible component of Γ_0 which passes through the base point Q of Λ. We shall prove the following

Lemma. *At least two of* $E_1^{(1)}, E_1^{(2)}, E_1^{(3)}$ *are terminal components of* Γ_0. *Here, a terminal component is an irreducible component which meets only one of the other irreducible components.*

Our proof will be done in the subsequent two paragraphs. We shall consider two cases separately.

8. Case: $Q \neq Q_i$ for $i=1,2,3$.

8.1. At first we have:

i) $(C \cdot l^{(i)}) = \beta_i - 2\gamma_i = 0$ for $i=1,2,3$;

ii) $(C \cdot \sigma^*(\tilde{l})) = (C \cdot 2E^{(i)}) = 2(\alpha_i - \beta_i + \gamma_i) = a$;

iii) $\sum_{i=1}^{3} (\alpha_i - \beta_i + \gamma_i) = (\sigma_*(C) \cdot \tilde{M}_0) = (C \cdot M_1) = \gamma_1 + \gamma_2 + \gamma_3$, because $i(\sigma_*(C), \tilde{M}_0; \tilde{P}_i) = (C \cdot E^{(i)}) = \alpha_i - \beta_i + \gamma_i$ for $i=1,2,3$. Therefore we obtain,

$$\beta_i = 2\alpha_i - a, \quad \gamma_i = \alpha_i - \frac{1}{2}a \quad \text{and}$$

$$\begin{aligned} C &\sim aM_0 + \sum_{i=1}^{3} \{\alpha_i E_1^{(i)} + (2\alpha_i - a)E^{(i)} + (\alpha_i - \frac{1}{2}a)l^{(i)}\} \\ &\sim a\{M_0 - \sum_{i=1}^{3} (E^{(i)} + \frac{1}{2}l^{(i)})\} + 3a\sigma^*(\tilde{l}) \\ &\sim a\{M_0 + \sum_{i=1}^{3} (E_1^{(i)} + E^{(i)}) + \frac{1}{2}\sum_{i=1}^{3} l^{(i)}\} \\ &\sim a(\sigma^*(\tilde{M}_0) + \frac{1}{2}\sum_{i=1}^{3} l^{(i)}) . \end{aligned}$$

8.2. Now, let $\tau\colon V \to W := F_3$ be the contraction of components $E^{(i)}$ and $l^{(i)}$ for $i=1,2,3$, where F_3 is the Hirzebruch surface of degree 3. Let $M = \tau(M_0)$ and $M' = \tau(M_1)$. Then M is the minimal section of W and $(M \cdot M') = 0$. Let $L_i = \tau(E_1^{(i)})$ for $i=1,2,3$. Denote by L a fiber of W. Then, by 8.1, we have $\tau_*(C) \sim a(M + 3L) \sim aM'$ and $\tau_* C'$s span a linear pencil $\tau_*\Lambda$ on W, whose base points are $R_i := L_i \cap M'$ $(i=1,2,3)$ and $R := \tau(Q)$.

In order to prove the lemma in the paragraph 7, assume that an irreducible component A of Γ_0 (other than M_0) intersects one of $E_1^{(i)} (i=1,2,3)$, say $E_1^{(3)}$. Then we have $(E_1^{(3)} \cdot A) = 1$, $(M_0 \cdot A) = 0$ and $(E_1^{(i)} \cdot A) = 0$ for $i=1,2$, by virtue of Lemma's 6.1 and 6.2. Let $A' = \tau(A)$. Since $A' \cap M = \phi$, A' is linearly equivalent to a divisor $\alpha(M + 3L)$, where α is a positive integer. Note that A meets at most one of $l^{(i)}$'s for $i=1,2,3$, for, if otherwise, Supp (Γ_0) would contain $l^{(i)} (i=1,2,3)$ and M_1 as its irreducible components and, consequently, contain a loop, which contradicts Lemma 6.2. We may assume that A does not meet $l^{(1)}$. Then, since

$$2(A \cdot E^{(1)}) = (A \cdot E_1^{(1)} + 2E^{(1)} + l^{(1)}) = (A \cdot \tau^*(L)) = (A' \cdot L) = \alpha ,$$

we have $\mathrm{mult}_{R_1} A' = (A \cdot E^{(1)}) = \dfrac{\alpha}{2}$. Hence $\alpha \equiv 0 \pmod 2$.

Suppose A does not met meet $l^{(3)}$. Then, since

$$\begin{aligned} (A \cdot E_1^{(3)}) + 2(A \cdot E^{(3)}) &= (A \cdot E_1^{(3)} + 2E^{(3)} + l^{(3)}) = (A \cdot \tau^*(L)) \\ &= (A' \cdot L) = \alpha \end{aligned}$$

and $(A\cdot E_1^{(3)})=1$, we have $\text{mult}_{R_3}A'=(A\cdot E^{(3)})=\frac{1}{2}(\alpha-1)$. Then $\alpha\equiv 1 \pmod 2$. This contradicts the previous relation $\alpha\equiv 0 \pmod 2$. This implies that if an irreducible component A of Γ_0 meets one of $E_1^{(i)}$'s for $i=1,2,3$, say $E_1^{(3)}$, then A must intersect $l^{(3)}$ as well and $l^{(i)}$ $(i=1,2,3)$ and M_1 are irreducible components of Γ_0. By virtue of Lemma 6.2, the number of such irreducible components A of Γ_0 is at most one. Therefore we conclude that at least two of $E_1^{(i)}$'s $(i=1,2,3)$ are terminal components of Γ_0.

9. Case: $\tilde{Q}$ is one of $\tilde{Q}_1, \tilde{Q}_2$ and $\tilde{Q}_3$. We may assume that $\tilde{Q}=\tilde{Q}_3$.

9.1. Then we have:

i) $(C\cdot l^{(1)}) = \beta_1-2\gamma_1 = 0$ and $(C\cdot l^{(2)}) = \beta_2-2\gamma_2 = 0$;
$(C\cdot E_1^{(i)}) = a-2\alpha_i+\beta_i = 0$ for $i = 1,2,3$.

ii) $(C\cdot\sigma^*(\tilde{l})) = (C\cdot 2E^{(1)}) = 2(\alpha_1-\beta_1+\gamma_1) = a$,
$(C\cdot\sigma^*(\tilde{l})) = (C\cdot 2E^{(2)}) = 2(\alpha_2-\beta_2+\gamma_2) = a$,
$(C\cdot M_0) = -3a+\sum_{i=1}^{3}\alpha_i = 0$.

iii) $(\sigma_*(C)\cdot\tilde{M}_0) = \sum_{i=1}^{3}(\alpha_i-\beta_i+\gamma_i) = (C\cdot M_1) = \sum_{i=1}^{3}\gamma_i$.

Thence we have

i)′ $\beta_i = 2\alpha_i-a$ for $i = 1,2,3$.

ii)′ $\gamma_i = -\frac{1}{2}a+\alpha_i$ for $i = 1,2$.

Hence we have

$$C\sim aM_0+\sum_{i=1}^{3}(\alpha_iE_1^{(i)}+\beta_iE^{(i)}+\gamma_il^{(i)})$$
$$\sim a(\sigma^*(\tilde{M}_0)+\frac{1}{2}l^{(1)}+\frac{1}{2}l^{(2)})+(\gamma_3-\alpha_3+a)l^{(3)}.$$

9.2. As in the former case, let $\tau: V\rightarrow W$ be the contraction of components $E^{(i)}$ and $l^{(i)}$ for $i=1,2,3$. With the same notations as before, the calculation in 9.1 shows that $\tau_*C\sim a(M+3L)\sim aM'$.

We shall prove that $E_1^{(1)}$ and $E_1^{(2)}$ are terminal components of Γ_0. Assume, to the contrary, that an irreducible component A of Γ_0 (other than M_0) intersects one of $E_1^{(1)}$ and $E_1^{(2)}$, say $E_1^{(2)}$. Let $A'=\tau(A)$. Then A' is linearly equivalent to $\alpha(M+3L)$ with $\alpha>0$. We shall consider three cases separately.

9.2.1. Case: A intersects none of $l^{(1)}$ and $l^{(2)}$.

Then, by the same computation as in 8.2, we have

$$\text{mult}_{R_1}A' = (E^{(1)}\cdot A) = \frac{1}{2}\alpha$$

and

$$\text{mult}_{R_2}A' = (E^{(2)}\cdot A) = \frac{1}{2}(\alpha-1).$$

Thus we have a contradiction.

9.2.2. Case: A intersect $l^{(1)}$.

Assume that A also intersects $l^{(3)}$. If $Q_3 \notin l^{(3)} \cap A$, $l^{(3)}$ and M_1 would be components of Γ_0, and A, $l^{(3)}$, $l^{(1)}$ and M_1 would form a loop contained in Supp (Γ_0). This contradicts Lemma 6.2. The case $Q \in l^{(3)} \cap A$ does not occur, either, by virtue of Lemma 6.2, for, if otherwise, A, M_1, $l^{(1)}$ would form a loop contained in Supp (Γ_0). Thus we know that A does not intersect $l^{(3)}$. Then, by the same computation as in 8.2, we have

$$\mathrm{mult}_{R_3} A' = (A \cdot E^{(3)}) = \frac{\alpha}{2} .$$

On the other hand, we have

$$\mathrm{mult}_{R_2} A' = (A \cdot E^{(2)}) = \frac{1}{2}(\alpha - 1) ,$$

because $(A \cdot l^{(2)}) = 0$. This is a contradiction. Therefore A does not intersect $l^{(1)}$.

9.2.3. Case: A intersects $l^{(2)}$.

Then A intersects none of $l^{(1)}$, $l^{(3)}$, $E_1^{(1)}$ and $E_1^{(3)}$, for, if otherwise, we could find a loop in Supp (Γ_0), contradicting Lemma 6.2. Hence, by the same computation as above, we have

$$\mathrm{mult}_{R_1} A' = (A \cdot E^{(1)}) = \frac{\alpha}{2} \text{ and } \mathrm{mult}_{R_3} A' = (A \cdot E^{(3)}) = \frac{\alpha}{2} .$$

Moreover, we can compute $\mathrm{mult}_{R_2} A'$ in the following way: Since $(A \cdot l^{(2)}) = (A \cdot E_1^{(2)}) = 1$ by virtue of Lemma 6.1,

$$\begin{aligned} 2 + 2(E^{(2)} \cdot A) &= (l^{(2)} + 2E^{(2)} + E_1^{(2)} \cdot A) = (\tau^* L_2 \cdot A) \\ &= (L_2 \cdot A') = \alpha . \end{aligned}$$

Since $(E^{(2)} \cdot A) = \mathrm{mult}_{R_2} A' - 1$, we have $\mathrm{mult}_{R_2} A' = \frac{\alpha}{2}$.

Now, we shall compute the intersection number $(C \cdot A)$. Firstly, we can express $\tau^* A'$ as

$$\tau^* A' = A + \frac{\alpha}{2} l^{(1)} + \alpha E^{(1)} + \frac{\alpha}{2} l^{(3)} + \alpha E^{(3)} + \frac{\alpha}{2} l^{(2)} + (\alpha - 1) E^{(2)} .$$

Then

$$\begin{aligned} &(\tau^* A' \cdot C) \\ &= (A' \cdot \tau_*(C)) = (\alpha(M + 3L) \cdot a(M + 3L)) = 3\alpha a \\ &= (A \cdot C) + \alpha(E^{(1)} \cdot C) + \frac{\alpha}{2}(\sigma^*(\tilde{l}) \cdot C) + (\alpha - 1)(E^{(2)} \cdot C) \\ &= (A \cdot C) + \frac{1}{2}\alpha a + \frac{1}{2}\alpha a + (\alpha - 1)\frac{1}{2} a. \end{aligned}$$

Thus, we have $(A\cdot C)=\frac{1}{2}(3\alpha+1)a>0$. However, since A does not pass through the base point Q_3 of Λ as one easily shows by using Lemma 6.2, the intersection number $(A\cdot C)$ should be zero. Hence we get a contradiction. This implies that A does not intersect $l^{(2)}$. By virtue of the above three cases, we conclude that $E_1^{(1)}$ and $E_1^{(2)}$ are terminal components of Γ_0. This completes the proof of the lemma in the paragraph 7.

10. Now, we may assume that $E_1^{(1)}$ and $E_1^{(2)}$ are terminal components of Γ_0. Let ϕ: $\bar{V}\rightarrow V$ be the shortest succession of quadratic transformations with centers at base points of Λ such that the proper transform $\bar{\Lambda}$ of Λ by Φ has no base points, and let $\bar{\Gamma}_0$ be the member of $\bar{\Lambda}$ corresponding to Γ_0. We shall contract, in a certain way, the components of $\bar{\Gamma}_0$, one by one, to obtain a non-degenarte member, i.e., $\boldsymbol{P}^1_k$. Put $\bar{M}_0=\Phi'(M_0)$ and $\bar{E}_1^{(i)}=\Phi'(E_1^{(i)})$ for $i=1,2,3$. Then, among the components $\bar{M}_0$, $\bar{E}_1^{(1)}$, $\bar{E}_1^{(2)}$ and $\bar{E}_1^{(3)}$, $\bar{M}_0$ is not contracted first, for, if otherwise, three components meet each other at a single point which contradicts Lemma 6.1,(3). Hence $\bar{E}_1^{(3)}$ should be contracted first. After the contraction of $\bar{E}_1^{(3)}$ (and some components other than $\bar{M}_0$, $\bar{E}_1^{(1)}$ and $\bar{E}_1^{(2)}$), the component $\bar{M}_0$ should be contracted next. Then after the contraction ρ of $\bar{E}_1^{(3)}$ and $\bar{M}_0$, $(\rho(E_1^{(i)})^2)=-1$ for $i=1,2$. When we contract $\bar{M}_0$, all components of $\bar{\Gamma}_0$ except $\bar{E}_1^{(i)}$ $(i=1,2)$ must have been already contracted, and $\rho(E_1^{(i)})'s$ $(i=1,2)$ meet each other at a single point of a cross-section of the pencil. Thıs is a contradiction. Therefore, we know that V-Supp (D) does not contain cylinderlike open sets.

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Notes added in proof: According to S. Iitaka, we may eliminate the assumption that the intersection matrix of D is not negative definite.

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