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REGULAR SUBRING OF A POLYNOMIAL RING

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Introduction. The purpose of this article is to prove the following two theorems:

Theorem 1. *Let k be an algebraically closed field of characteristic zero, and let A be a k-subalgebra of a polynomial ring $B:=k[x, y]$ such that B is a flat A-module of finite type. Then A is a polynomial ring in two variables over k.*

Theorem 2. *Let k be an algebraically closed field of characteristic zero, and let $B:=k[x, y, z]$ be a polynomial ring in three variables over k. Assume that there is given a nontrivial action of the additive group G_a on the affine 3-space $A_k^3:=\mathrm{Spec}(B)$ over k. Let A be the subring of G_a-invariant elements in B. Assume that A is regular. Then A is a polynomial ring in two variables over k.*

Theorem 1 was formerly proved in part under one of the following additional conditions (cf. [7; pp. 139–142]):

(1) B is etale over A,

(2) A is the invariant subring in B with respect to an action of a finite group.

In proofs of both theorems, substantial roles will be played by the following theorem, which is a consequence of the results obtained in Fujita [1], Miyanishi-Sugie [8] and Miyanishi [6]:

Theorem 0. *Let k be an algebraically closed field of characteristic zero, and let $X=\mathrm{Spec}(A)$ be a nonsingular affine surface defined over k. Then the following assertions hold true:*

(1) *X contains a nonempty cylinderlike open set, i.e., there exists a dominant morphism $\rho: X \rightarrow C$ from X to a nonsingular curve C whose general fibers are isomorphic to the affine line A_k^1, if and only if X has the logarithmic Kodaira dimension $\bar{\kappa}(X)=-\infty$.*

(2) *X is isomorphic to the affiine plane A_k^2 if and only if X has the logarithmic Kodaira dimension $\bar{\kappa}(X)=-\infty$, A is a unique factorization domain, and $A^*=k^*$, where A^* is the set of invertible elements in A and $k^*=k-(0)$.*

In this article, the ground field k is always assumed to be an algebraically

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closed field of characteristic zero. For the definition and relevant results on logarithmic pluri-genera and the logarithmic Kodaira dimension of an algebraic variety, we refer to Iitaka [3]. An algebraic action of the additive group G_a on an affine scheme $\mathrm{Spec}(B)$ over k can be interpreted in terms of a locally nilpotent k-derivation Δ on B. In particular, the subring A of G_a-invariant elements in B is identified with the set of elements b of B such that $\Delta(b)=0$. For results on G_a-actions necessary in the subsequent arguments, we refer to [7; pp. 14–24]. The Picard group, i.e., the divisor class group, of a nonsingular variety V over k is denoted by $\mathrm{Pic}(V)$; for a k-algebra A, A^* denotes the multiplicative group of all invertible elements of A; the affine n-space over k is denoted by $\boldsymbol{A}_k^n$; $\boldsymbol{Q}$ (resp. $\boldsymbol{Z}$) denotes the field of rational numbers (resp. the ring of rational integers).

1. Proof of Theorem 1

1.1. We shall begin with

Lemma. *Let Y be a nonsingular, rational, affine surface, and let $f\colon Y\to C$ be a surjective morphism from Y onto a nonsingular rational curve whose general fibers are isomorphic to the affine line $\boldsymbol{A}_k^1$. Then we have:*

(1) *Let F be a fiber of f. If F is irreducible and reduced, then F is isomorphic to the affine line $\boldsymbol{A}_k^1$. If F is singular, i.e., $F\not\cong\boldsymbol{A}_k^1$, then F_{red} is a disjoint union of the affine lines.*

(2) *For a point $P\in C$, we denote by μ_P the number of irreducible components of the fiber $f^{-1}(P)$. If C is isomorphic to the projective line $\boldsymbol{P}_k^1$, we have*

$$\mathrm{rank}_{\boldsymbol{Q}}\,\mathrm{Pic}\,(Y)\underset{\boldsymbol{Z}}{\otimes}\boldsymbol{Q}=1+\sum_{P\in C}(\mu_P-1)\,.$$

If C is isomorphic to the affine line $\boldsymbol{A}_k^1$, we have

$$\mathrm{rank}_{\boldsymbol{Q}}\,\mathrm{Pic}\,(Y)\underset{\boldsymbol{Z}}{\otimes}\boldsymbol{Q}=\sum_{P\in C}(\mu_P-1)\,.$$

Proof. There exists a nonsingular projective surface W such that W contains Y as a dense open set and that the boundary curve $E:=W-Y$ has only normal crossings as singularities. The surjective morphism $f\colon Y\to C$ defines an irreducible linear pencil Λ on W. By eliminating base points of Λ by a succession of quadratic transformations, we may assume that Λ is free from base points. Then the general members of Λ are nonsingular rational curves, and the pencil Λ defines a surjective morphism $\varphi\colon W\to\boldsymbol{P}_k^1$. The boundary curve E contains a unique irreducible component E_0 which is a cross-section of φ, and the other components of E are contained in the fibers of φ. We may assume that a fiber of φ lying outside of Y is irreducible. Let S be a fiber of φ such that $S\cap Y\neq\phi$. If S is irreducible then $S\cong\boldsymbol{P}_k^1$ and $S\cap Y=S-S\cap E_0\cong\boldsymbol{A}_k^1$.

Suppose S is reducible. If $S \cap Y$ is irreducible and reduced, we may contract all irreducible components of S lying outside of Y without losing generalities (cf. [7; Lemma 2.2, p. 115]). Hence, $S \cap Y \cong A_k^1$. We may apply Lemma 1.3 of Kambayashi-Miyanishi [5] to obtain the same conclusion. Assume that $S \cap Y$ is singular. Then S_{red} has the following decomposition into irreducible components,

$$S_{\text{red}} = \sum_i T_i + \sum_j Z_j ,$$

where $T_i \cap Y \neq \phi$ and $Z_j \cap Y = \phi$. By virtue of [7; ibid.], every component of S_{red} is a nonsingular rational curve, S_{red} is a connected curve, and the dual graph of S_{red} contains no circular chains. On the other hand, since Y is affine, Y does not contain any complete curve and E is connected, whence we know that if some T_i meets the cross-section E_0 then S_{red} has no components lying outside of Y. Suppose $S \cap Y$ is irreducible. Then $S_{\text{red}} \cap Y \cong A_k^1$, for, if otherwise, the dual graph of S_{red} would contain a circular chain. Suppose $S \cap Y$ is reducible. If either $T_i \cap Y \ncong A_k^1$ for some component T_i or $T_i \cap T_l \cap Y \neq \phi$ for distinct components T_i and T_l, the dual graph of S_{red} would contain a circular chain. Therefore, every irreducible component of $S \cap Y$ is a connected component, and isomorphic to A_k^1. This proves the first assertion.

Next, we shall prove the second assertion. Let L be an irreducible fiber of φ. Then the $\boldsymbol{Q}$-vector space $\operatorname{Pic}(W) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ has a basis of the divisor classes of the following curves:

(i) E_0, (ii) L, (iii) all irreducible components of a singular fiber S of φ except one component meeting Y, where S ranges over all singular fibers of φ. The $\boldsymbol{Q}$-vector space $\operatorname{Pic}(Y) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ is generated by a part of the above basis consisting of all classes of curves which meet Y; when $C \cong A_k^1$, we take L to be the unique irreducible fiber lying outside of Y. Then we obtain immediately the equalities in the second assertion. Q.E.D.

1.2. With the notations in Theorem 1, set $V := \operatorname{Spec}(A)$ and $W := \operatorname{Spec}(B)$. The inclusion $A \hookrightarrow B$ induces a finite flat morphism $\pi: W \to V$. Since π is finite, π is faithfully flat. Therefore, V is a nonsingular, rational affine surface. We have the following

Lemma. (1) *V has the logarithmic Kodaira dimension $\bar{\kappa}(V) = -\infty$.* (2) $A^* = k^*$.

Proof. The second assertion is clear. As for the first assertion, note that $\pi: W \to V$ is a dominant morphism. Denote by $\bar{P}_m(V)$ the logarithmic m-th genus of V for an integer $m \geqq 0$. Then we have

$$0=\bar{P}_m(W)\geqq\bar{P}_m(V)\quad\text{for every}\quad m\gg 0\,,$$

(cf. Iitaka [3]). Hence, $\bar{\kappa}(V)=-\infty$. Q.E.D.

1.3. We shall prove the following

Lemma. *A is a polynomial ring in two variables over k.*

Proof. Our proof consists of the paragraphs 1.3.1~1.3.5.

1.3.1. By virtue of Theorem 0 and Lemma 1.2, V contains a nonempty cylinder-like open set. Namely, there exists a dominant morphism $\rho: V\to\boldsymbol{P}_k^1$ such that general fibers of ρ are isomorphic to $\boldsymbol{A}_k^1$.

1.3.2. Let $n:=\deg\pi$. Then we claim:

For every divisor D on V, nD is linearly equivalent to 0, i.e., $nD\sim 0$.

In effect, $\pi^*(D)\sim 0$ because $\mathrm{Pic}(W)=0$. Since π is proper, we have $\pi_*\pi^*(D)=nD\sim 0$ by the projection formula.

1.3.3. We claim that:

(1) $\rho(V)\cong\boldsymbol{A}_k^1$;

(2) *If $\rho^{-1}(P)$ is a singular fiber of ρ, it is of the form $\rho^{-1}(P)=n_PC_P$, where $n_P\geqq 2$, $n_P\,|\,n$ and $C_P\cong\boldsymbol{A}_k^1$.*

Proof. (1) By 1.3.2, we have $\mathrm{rank}_{\boldsymbol{Q}}\,\mathrm{Pic}\,(V)\underset{\boldsymbol{Z}}{\otimes}\boldsymbol{Q}=0$. Suppose $\rho(V)=\boldsymbol{P}_k^1$. Then, Lemma 1.1 implies that

$$\mathrm{rank}_{\boldsymbol{Q}}\,\mathrm{Pic}\,(V)\underset{\boldsymbol{Z}}{\otimes}\boldsymbol{Q}=1+\sum_{P\in\boldsymbol{P}_k^1}(\mu_P-1)\,.$$

Since $\mu_P\geqq 1$, we have $\mathrm{rank}_{\boldsymbol{Q}}\mathrm{Pic}(V)\underset{\boldsymbol{Z}}{\otimes}\boldsymbol{Q}>0$, which is a contradiction. Hence $\rho(V)$ is an affine open set of $\boldsymbol{P}_k^1$. Since $A^*=k^*$, $\rho(V)$ must be isomorphic to $\boldsymbol{A}_k^1$.

(2) By Lemma 1.1, we have

$$\mathrm{rank}_{\boldsymbol{Q}}\,\mathrm{Pic}\,(V)\underset{\boldsymbol{Z}}{\otimes}\boldsymbol{Q}=\sum_{P\in\rho(V)}(\mu_P-1)=0\,.$$

Hence $\mu_P=1$ for all points $P\in\rho(V)$. This implies that a singular fiber of ρ (if it exists at all) is of the form

$$\rho^{-1}(P)=n_PC_P\,,\quad\text{where}\quad n_P\geqq 2\quad\text{and}\quad C_P\cong\boldsymbol{A}_k^1\,.$$

Let m be the order of C_P, i.e., m is the least positive integer such that $mC_P\sim 0$. Since $n_PC_P\sim 0$, we have $m\,|\,n_P$. Write $n_P=sm$. Let t be an inhomogeneous coordinate of $\rho(V)$. Then t is everywhere defined on V; we may assume that $P\in\rho(V)$ is defined by $t=0$. Since $mC_P\sim 0$ and $n_PC_P=(t)$ (=the divisor defined by $t=0$ on V), there exists an element $t'\in A$ such that $t=(t')^s$. If $s>1$ then $t-\alpha=(t')^s-\alpha$ has distinct s components for every $\alpha\in k^*$. This contradicts the irreducibility of general fibers of ρ. Hence $s=1$, i.e., $m=n_P$. Since $nC_P\sim 0$, we have $n_P|n$. Q.E.D.

1.3.4. Let t be the same as defined in 1.3.3. Suppose ρ has a singular fiber $\rho^{-1}(P)$. Since $\pi^*C_P\sim 0$, there exists an element $\tau\in B$ such that $t=\tau^{n_P}$. Let $A'=A\otimes_{k[t]}k[\tau]$. Since A is flat over $k[t]$ (cf. [EGA (IV, 15.4.2)]), A' is identified with a k-subalgebra $A[\tau]$ of B. Let $\tilde{A}$ be the normalization of A' is B. Let $\tilde{V}=\mathrm{Spec}\,(\tilde{A})$. Then the morphism $\pi: W\to V$ factors as

$$\pi: W\xrightarrow{\pi_1}\tilde{V}\xrightarrow{\pi_2}V.$$

Let $n_1=\deg\pi_1$ and $n_2=\deg\pi_2$. Then $n=n_1\cdot n_2$, and $n_1\tilde{D}\sim 0$ for every divisor $\tilde{D}$ on $\tilde{V}$. We claim that:

(1) *$\tilde{V}$ is a nonsingular, rational, affine surface endowed with a dominant morphism $\tilde{\rho}$: $\tilde{V}\to \boldsymbol{A}^1_k:=\mathrm{Spec}\,(k[\tau])$, which is induced by ρ. Hence, general fibers of $\tilde{\rho}$ are isomorphic to $\boldsymbol{A}^1_k$.*

(2) *The fibration $\tilde{\rho}$ has a singular fiber with two or more irreducible components.*

Proof. Let Q be a point on C_P. There exist local parameters ξ, η of V at Q such that C_P is defined locally by $\xi=0$. Then $t=u\xi^{n_P}$ for an invertible element u of $\mathcal{O}_{Q,V}$. Let $\theta:=\tau/\xi$. Then $\tilde{V}$ is analytically isomorphic to a hypersurface $\theta^{n_P}=u$ in the (θ, ξ, η)-space in a neighborhood of $\pi_2^{-1}(Q)$, and $\tilde{V}$ is smooth at every point of $\pi_2^{-1}(Q)$ by the Jacobian criterion. Since Q is arbitrary on C_P, $\tilde{V}$ is smooth along $(\rho\cdot\pi_2)^{-1}(P)$. Let $\rho^{-1}(P')$ be another fiber of ρ. Then $\rho^{-1}(P')=n_{P'}C_{P'}$, where $n_{P'}\geqq 1$ and $C_{P'}\cong\boldsymbol{A}^1_k$. Let Q' be a point on $C_{P'}$. Then there exist local parameters ξ', η' such that $C_{P'}$ is defined locally by $\xi'=0$ and $t-\alpha=u'(\xi')^{n_{P'}}$ where $\alpha\in k^*$ and u' is an invertible element of $\mathcal{O}_{Q',V}$. Then $\tilde{V}$ is analytically isomorphic to a hypersurface $\tau^{n_P}-\alpha=u'(\xi')^{n_{P'}}$ in the (τ, ξ', η')-space. By the Jacobian criterion, $\tilde{V}$ is smooth at every point of $\pi_2^{-1}(Q')$. Since Q' is arbitrary on $C_{P'}$, $\tilde{V}$ is smooth along $(\rho\cdot\pi_2)^{-1}(P')$. Thus we know that $\tilde{V}$ is smooth.

Let $\tilde{\rho}$: $\tilde{V}\to\boldsymbol{A}^1_k:=\mathrm{Spec}(k[\tau])$ be the canonical morphism induced by ρ. The generic fibers of ρ and $\tilde{\rho}$ are isomorphic to $\mathrm{Spec}(A\otimes_{k[t]}k(t))$ and $\mathrm{Spec}(A\otimes_{k[t]}k(\tau))$, respectively. Since $\mathrm{Spec}(A\otimes_{k[t]}k(t))\cong\boldsymbol{A}^1_{k(t)}$, we know that $\mathrm{Spec}(A\otimes_{k[t]}k(\tau))\cong\boldsymbol{A}^1_{k(\tau)}$. Hence, general fibers of $\tilde{\rho}$ are isomorphic to $\boldsymbol{A}^1_k$.

Let P be the point on $\mathrm{Spec}(k[\tau])$ lying above P on $\mathrm{Spec}(k[t])$. Then the fiber $\tilde{\rho}^{-1}(\bar{P})$ has n_P analytic branches over any point Q of C_P, for $\tilde{\rho}^{-1}(\tilde{P})$ is analytically defined by $\theta^{n_P}=u$ as shown in the proof of the first assertion.

Hence, by Lemma 1.1, $\tilde{\rho}^{-1}(\tilde{P})$ has n_P connected components, each of which is isomorphic to $\boldsymbol{A}_k^1$. Q.E.D.

1.3.5. As remarked in 1.3.4, $\mathrm{Pic}(\tilde{V})\otimes_{\boldsymbol{Z}}\boldsymbol{Q}=0$. However, if $\rho\colon V\to\boldsymbol{A}_k^1$ has a singular fiber, Lemma 1.1 implies that $\mathrm{rank}_{\boldsymbol{Q}}\mathrm{Pic}(\tilde{V})\otimes_{\boldsymbol{Z}}\boldsymbol{Q}>0$. This is a contradiction. Therefore, ρ has no singular fibers. Then, by virtue of [5; Th. 1], V is an $\boldsymbol{A}^1$-bundle over $\rho(V)=\boldsymbol{A}_k^1$. Hence, we know that $V\cong\boldsymbol{A}_k^2$. Namely, A is a polynomial ring in two variables over k. This completes a proof of Lemma 1.3 as well as a proof of Theorem 1.

1.4. Remark. Theorem 1 is a generalization of the following result in the case of dimension 1:

Let k be a field, and let A be a normal, 1-dimensional k-subalgebra of a polynomial ring over k. Then A is a polynomial ring over k.

2. Proof of Theorem 2

2.1. We retain the notations and the assumptions of Theorem 2. Let $L:=k(x,y,z)$, and let K be the invariant subfield of L with respect to the induced G_a-action on L. Then, $A=B\cap K$. Since B is normal and $\mathrm{trans.deg}_k K=2$, we know by Zariski's Theorem (cf. Nagata [9; Th. 4, p. 52]) that A is finitely generated over k. By assumption, A is regular. By virtue of [7; Lemma 1.3.1, p. 16], we know that A is a unique factorization domain and $A^*=k^*$.

2.2. Let $W=\mathrm{Spec}(B)$, let $V=\mathrm{Spec}(A)$, and let $\pi\colon W\to V$ be the dominant morphism induced by the injection $A\hookrightarrow B$. We shall prove

Lemma. *$\pi\colon W\to V$ is a faithfully flat, equi-dimensional morphism of dimension* 1.

Proof. (1) We shall show that B is flat over A. Let $\mathfrak{q}$ be a prime ideal of B and let $\mathfrak{p}=\mathfrak{q}\cap A$. Then $B_\mathfrak{q}$ dominates $A_\mathfrak{p}$. Since $A_\mathfrak{p}$ is regular and $B_\mathfrak{q}$ is Cohen-Macaulay, $B_\mathfrak{q}$ is flat over $A_\mathfrak{p}$ by virtue of [EGA (IV, 15.4.2)]. Hence B is flat over A.

(2) We shall show that π is surjective. Suppose π is not surjective. Then there exists a maximal ideal $\mathfrak{m}$ of A such that $\mathfrak{m}B=B$. Let $(\mathfrak{O}, t\mathfrak{O})$ be a discrete valuation ring of K such that $\mathfrak{O}$ dominates $A_\mathfrak{m}$. Let $R:=B\otimes_A\mathfrak{O}$. Since B is A-flat, R is identified with a subring of L. Let Δ be a locally nilpotent derivation on B associated to the given G_a-action. Then Δ extends to a locally nilpotent $\mathfrak{O}$-derivation, and $\mathfrak{O}$ is the ring of Δ-invariants in R, i.e., $\mathfrak{O}=\{r\in R;\ \Delta(r)=0\}$. By assumption, we have $tR=R$, where t is a uniformisant of $\mathfrak{O}$. Hence $tr=1$ for some element $r\in R$. Then $t\Delta(r)=0$, whence $r\in\mathfrak{O}$. This is a contradiction.

(3) Note that general fibers of π are isomorphic to A_k^1. Hence, each irreducible component has dimension $\geqq 1$. Suppose that some component T of a fiber $\pi^{-1}(P)$ (with $P \in V$) has dimension 2. Since B is factorial, there exists an irreducible element $b \in B$ such that T is defined by $b=0$. Since T is invariant with respect to the G_a-action, we know that $b \in A$. Let $C = \mathrm{Spec}(A/bA)$. Then C is an irreducible curve and $\pi^{-1}(C) = T \subset \pi^{-1}(P)$. This is a contradiction because π is surjective. Thus, π is a faithfully flat, equi-dimensional morphism of dimension 1. Q.E.D.

2.3. Let U be the subset of all points P of V such that $\pi^{-1}(P)$ is irreducible and reduced. Then U is a dense open set of V. By virtue of [5; Th. 1], $\pi^{-1}(U) := W \times_V U$ is an A^1-bundle over U.

2.4. We shall prove the following:

Lemma. *V has the logarithmic Kodaira dimension $\bar{\kappa}(V) = -\infty$.*

Proof. We follow the arguments of Iitaka-Fujita [4]. Let X be a hyperplane in $W \cong A_k^3$ such that $X \cap \pi^{-1}(U) \neq \phi$. Suppose that $\bar{\kappa}(V) \geqq 0$. Let C be a prime divisor in X. Consider a morphism:

$$\varphi\colon\ C \times A_k^1 \hookrightarrow X \times A_k^1 = W \xrightarrow{\pi} V\,,$$

and assume that φ is a dominant morphism. Since $\dim(C \times A_k^1) = \dim V = 2$, we know by [3; Prop. 1] that

$$0 = \bar{P}_m(C \times A_k^1) \geqq \bar{P}_m(V) \geqq 1 \quad \text{for for every} \quad m \gg 0\,.$$

This is a contradiction. Hence φ is not a dominant morphism. Let D be the closure of $\varphi(C \times A_k^1)$ in V. Then $C \times A_k^1 \subset \pi^{-1}(D)$. Suppose $C \cap \pi^{-1}(U) \neq \phi$. Then the general fibers of $\pi\colon \pi^{-1}(D) \to D$ are isomorphic to A_k^1. Hence $\pi^{-1}(D)$ is irreducible and reduced. Since $\dim(C \times A_k^1) = \dim \pi^{-1}(D) = 2$, we have $C \times A_k^1 = \pi^{-1}(D)$.

Let Q be a point on X, and let $C_1, \cdots, C_r$ be prime divisors of X such that $C_1 \cap \cdots \cap C_r = \{Q\}$ and that $C_i \cap \pi^{-1}(U) \neq \phi$ for every $1 \leqq i \leqq r$. For any point Q of X, we can find such a set of prime divisors. In effect, X is the affine plane A_k^2 and $X \cap (W - \pi^{-1}(U))$ has dimension $\leqq 1$. Thus, we have only to take a set of suitably chosen lines on X passing through Q. Let D_i be the irreducible curve which is the closure of $\varphi(C_i \times A_k^1)$ in V, where $1 \leqq i \leqq r$. Then $C_i \times A_k^1 = \pi^{-1}(D_i)$ for $1 \leqq i \leqq r$. Since we have

$$\begin{aligned}(Q) \times A_k^1 &= (C_1 \cap \cdots \cap C_r) \times A_k^1 = (C_1 \times A_k^1) \cap \cdots \cap (C_r \times A_k^1) \\ &= \pi^{-1}(D_1) \cap \cdots \cap \pi^{-1}(D_r) = \pi^{-1}(D_1 \cap \cdots \cap D_r)\,,\end{aligned}$$

we know that $D_1 \cap \cdots \cap D_r = \{P\}$, a point in V. The correspondence $Q \mapsto P$ defines a morphism $\psi: X \to V$. If ψ is a dominant morphism, we have

$$0 = \bar{P}_m(X) \geqq \bar{P}_m(V) \geqq 1 \quad \text{for every} \quad m \gg 0,$$

which is a contradiction. Hence ψ is not a dominant morphism. Let F be the closure of $\psi(X)$ in V. Then, for every point P of F, we have $\dim \psi^{-1}(P) \geqq 1$, and $\pi(\psi^{-1}(P) \times \boldsymbol{A}_k^1) = \psi(\psi^{-1}(P)) = P$. This contradicts Lemma 2.2. Therefore, $\bar{\kappa}(V) = -\infty$. Q.E.D.

2.5. As observed in the above arguments, V is a nonsingular, rational, affine surface such that $\bar{\kappa}(V) = -\infty$, A is a unique factorization domain and $A^* = k^*$. Then, by virtue of Theorem 0, V is isomorphic to the affine plane $\boldsymbol{A}_k^2$. Namely, A is a polynomial ring in two variables over k. This completes a proof of Theorem 2.

2.6. We do not know whether or not the regularity condition on A can be eliminated. As a view-point of practical use, the following result might be interesting:

Proposition. *Let the notations and the assumptions be the same as in Theorem 2. Instead of the regularity condition on A, we assume that A contains a coordinate, say z. Then A is a polynomial ring in two variables over k.*

Proof. (1) Let H_α be the hyperplane $z = \alpha$ in $W := \mathrm{Spec}(B)$ for every $\alpha \in k$. Since z is G_a-invariant, the hyperplane H_α is G_a-invariant. Set $B_\alpha := B/(z-\alpha)B \cong k[x, y]$. Let R_α be the invariant subring of B_α with respect to the induced G_a-action on H_α. Then it is clear that we have the following inclusions:

$$A_\alpha := A/(z-\alpha)A \hookrightarrow R_\alpha \hookrightarrow B_\alpha .$$

For a general element $\alpha \in k$, the induced G_a-action on H_α is nontrivial. Hence R_α is a one-parameter polynomial ring over k (cf. [7; Lemma, p. 39]).

(2) Let $\mathfrak{k} = k(z)$, let $B_{\mathfrak{k}} := \mathfrak{k}[x, y]$ and let $R_{\mathfrak{k}}$ be the G_a-invariant subring of $B_{\mathfrak{k}}$ with respect to the induced G_a-action on $B_{\mathfrak{k}}$. Then it is not hard to show that $R_{\mathfrak{k}} = A \underset{k[z]}{\otimes} \mathfrak{k}$ and that $R_{\mathfrak{k}}$ is a one-parameter polynomial ring over $\mathfrak{k}$ (cf. Remark 1.4 and [7; ibid.]). Now, set $Y := \mathrm{Spec}(A)$ and let $f: Y \to \boldsymbol{A}_k^1 = \mathrm{Spec}(k[z])$ be the morphism associated to the inclusion $k[z] \hookrightarrow A$. Then the foregoing observations imply the following:

(i) The generic fiber of f is $\mathrm{Spec}(R_{\mathfrak{k}})$, which is isomorphic to $\boldsymbol{A}_{\mathfrak{k}}^1$;

(ii) For every $\alpha \in k$, the fiber $f^{-1}(\alpha)$ is geometrically integral (cf. the inclusions $A_\alpha \hookrightarrow R_\alpha \hookrightarrow B_\alpha$ in the step (1)).

Since f is clearly faithfully flat morphism, we know by [5; Th.1] that $f: Y \to \boldsymbol{A}_k^1$ is an $\boldsymbol{A}^1$-bundle over $\boldsymbol{A}_k^1$. Hence Y is isomorphic to the affine plane $\boldsymbol{A}_k^2$.

Namely, A is a polynomial ring in two variables over k. Q.E.D.

2.7. Corollary. *Let the notations and the assumptions be the same as in Theorem 2. Instead of the regularity condition on A, assume that G_a acts linearly on $\boldsymbol{A}_k^3$ via the canonical action of $GL(3, k)$ on the vector space $kx+ky+kz$. Then A is a polynomial ring in two variables over k.*

Proof. Since G_a is a unipotent group, we may assume, after a change of coordinates x, y, z, that G_a acts on $\boldsymbol{A}_k^3$ via the subgroup of upper triangular matrices in $GL(3, k)$. Then, one of coordinates, say z, is G_a-invariant. Then Proposition 2.5 implies that our assertion holds true. Q.E.D.

2.8. In the case of an algebraic action of G_a on a polynomial ring $k[x, y]$ in two variables over k, $k[x, y]$ is a one-parameter polynomial ring over the subring of G_a-invariant elements (cf. [7; p. 39]). However, this does not hold in the case of an algebraic G_a-action on a polynomial ring of dimension $\geqq 3$ over k, as is shown by the following:

EXAMPLE. *Let Δ be a k-derivation on a polynomial ring $B:=k[x, y, z]$ defined by $\Delta(x)=y$, $\Delta(y)=z$ and $\Delta(z)=0$. Then Δ is locally nilpotent, whence Δ defines an algebraic G_a-action on B. The subring A of G_a-invariant elements is $A=k[z, zx-\frac{1}{2}y^2]$. However, B is not a one-parameter polynomial ring over A.*

Proof. We shall prove only the last assertion. Suppose that $B=A[t]$ for some element t of B. Set $W:=\operatorname{Spec}(B)$ and $V:=\operatorname{Spec}(A)$. Let $f\colon W\to V$ be the morphism defined by the inclusion $A\hookrightarrow B$. Let P be a point of V such that $z=\alpha$ and $zx-\frac{1}{2}y^2=\beta$, where α, $\beta\in k$. If $\alpha\neq 0$, the fiber $f^{-1}(P)$ is isomorphic to $\boldsymbol{A}_k^1$. If $\alpha=0$ and $\beta\neq 0$, $f^{-1}(P)$ is a disjoint union of two irreducible components isomorphic to $\boldsymbol{A}_k^1$. If $\alpha=\beta=0$, $f^{-1}(P)$ is a non-reduced curve with only one irreducible component isomorphic to $\boldsymbol{A}_k^1$ counted twice. But, if $B=A[t]$, all fibers of f should be isomorphic to $\boldsymbol{A}_k^1$. Therefore, we have a contradiction. Q.E.D.

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