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NOTE ON A PROBLEM OF DIEPENBROCK

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In this note we shall discuss a problem of Diepenbrock. Let $(\mathcal{X}, \mathcal{A})$ be the n-dimensional Euclidean Borel space and let $\mathcal{P}$ be the totality of continuous probability measures on $(\mathcal{X}, \mathcal{A})$, $\mathcal{A}'$ the weak completion of $\mathcal{A}$ and $\mathcal{P}'$ be the extension of the $\mathcal{P}$ to $\mathcal{A}'$. Then there is no measure on $\mathcal{A}$ or $\mathcal{A}'$ w.r.t. which every element in $\mathcal{P}$ or $\mathcal{P}'$ has a density.

1. Introduction

Let $\mathcal{X}$ be the n-dimensional Euclidean space and $\mathcal{A}$ be the Borel field. A probability measure P on $(\mathcal{X}, \mathcal{A})$ is said to be continuous if $P(\{x\})=0$ holds for all points in $\mathcal{X}$. Let $\mathcal{P}$ be the totality of continuous probability measures on $(\mathcal{X}, \mathcal{A})$. We define $\mathcal{A}'=\{A\subset\mathcal{X};$ for all P in $\mathcal{P}$ there exists a set B_P in $\mathcal{A}$ and N_P in $\mathcal{A}$ such that $A\triangle B_P\subset N_P$ and $P(N_P)=0\}$. Extend each P in $\mathcal{P}$ to $\mathcal{A}'$ by defining $P'(A)=P(B_P)$.

A measure m on $(\mathcal{X}, \mathcal{A})$ is said to be a dominating measure for $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ if each P in $\mathcal{P}$ has a density w.r.t. m. $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ is said to be weakly dominated if there exists a localizable dominating measure.

Diepenbrock ([1] Section 11) showed that $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ is not weakly dominated under the continuum hypothesis and raised the following problem: Is $(\mathcal{X}, \mathcal{A}', \mathcal{P}')$, where $\mathcal{P}'=\{P'; P\in\mathcal{P}\}$, weakly dominated? The aim of the present note is to show, without any set theoretic assumption, neither of them is weakly dominated. In fact we shall show, more strongly, that $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ or $(\mathcal{X}, \mathcal{A}', \mathcal{P}')$ does not have any dominating measure.

2. The proof

In this section we shall prove the following

Theorem. *$(\mathcal{X}, \mathcal{A}, \mathcal{P})$ or $(\mathcal{X}, \mathcal{A}', \mathcal{P}')$ does not have any dominating measure.*

Lemma 1 (Kuratowski [2] p. 451). *For any uncountable Borel subset B of $\mathcal{X}$ there exists a Borel isomorphism f from B to $\mathcal{X}$ (i.e., f is one-to-one, onto and bimeasurable). So we have $f(\mathcal{A}_B)=\mathcal{A}$, where $\mathcal{A}_B=\{A\in\mathcal{A}; A\subset B\}$.*

Lemma 2. *Let B be an uncountable Borel subset of $\mathscr{X}$. Then there exists a P in $\mathscr{P}$ such that $P(B)=1$.*

Proof. Take any A in $\mathscr{A}$ with $A\subset B$. Let n be a normal distribution on $(\mathscr{X}, \mathscr{A})$. Using f in Lemma 1, we define $Q(A)=n(f(A))$. For any A in $\mathscr{A}$ put $P(A)=Q(A\cap B)$. Then it follows that $P(B)=1$ and $P(\{x\})=0$ for all x in $\mathscr{X}$. So P belongs to $\mathscr{P}$.

REMARK 1. By Lemma 2, we have $\mathscr{A}''=\mathscr{A}$, where $\mathscr{A}''=\{A\subset\mathscr{X}$; there exists a set B in $\mathscr{A}$ and a set N in $\mathscr{A}$ such that $A\triangle B\subset N$ and $P(N)=0$ for all P in $\mathscr{P}\}$.

Lemma 3. *Let B be an uncountable Borel subset of $\mathscr{X}$. Then there exists a family $\{B_i; i\in I\}$ of Borel sets such that $B_i\subset B$, $B_i\neq\phi$, $B_i\cap B_j=\phi$ $(i\neq j)$, I is uncountable and B_i is uncountable for each i.*

Proof. By Lemma 1, there exists a Borel isomorphism f between B and $\mathscr{X}$.

Case 1. $n>1$: For all $i\in I\equiv R$, put $B_i=\{f^{-1}(x)$; the last coordinate of x is equal to $i\}$.

Case 2. $n=1$: For all $i\in I\equiv(0, 1)$ let $0.\ i_1i_2i_3\cdots$ be the non terminating decimal expansion of i. Put $B_i=\{f^{-1}(x); x\in(0, 1), (x_2, x_4, x_6, \cdots)=(i_1, i_2, i_3, \cdots)\}$.

We proceed to prove that $(\mathscr{X}, \mathscr{A}, \mathscr{P})$ does not have any dominating measure. Assume that there exists a dominating measure m for $(\mathscr{X}, \mathscr{A}, \mathscr{P})$. For any uncountable Borel set B we have $m(B)>0$ by Lemma 2. Let $\{B_i; i\in I\}$ be an uncountable family given in Lemma 3. Each B_i is uncountable, so we have $m(B_i)>0$. $B_i\subset B$ and $B_i\cap B_j=\phi$ $(i\neq j)$ imply that $m(B)=\infty$. Therefore m is not sigma-finite on B.

If P in $\mathscr{P}$ has a density g w.r.t. m, then $[g>0]$ is a sigma-finite set w.r.t. m. Because of continuity of P, $[g>0]$ must be uncountable. So m is not sigma-finite on $[g>0]$ by the above discussion. But this is a contradiction.

Next we shall prove that $(\mathscr{X}, \mathscr{A}', \mathscr{P}')$ does not have any dominating measure. Again assume that there exists a dominating measure m for $(\mathscr{X}, \mathscr{A}', \mathscr{P}')$.

Step 1: Let B be any uncountable Borel subset of $\mathscr{X}$. For any A in $\mathscr{A}'$ such that $A\supset B$, we have $m(A)=\infty$ and in fact m is not sigma-finite on A.

Step 2: For any P' put $g=dP'/dm$. Then we can write $[g(x)>0]=\bigcup_{i=1}^{\infty}A_i$, $0<m(A_i)<\infty$, $A_i\in\mathscr{A}'$ because $[g(x)>0]$ is sigma-finite w.r.t. m. For each $i\geqq1$ we define a finite non-zero measure m_i on $(\mathscr{X}, \mathscr{A}')$ by $m_i(A)=m(A_i\cap A)$. If there exists a point x in $\mathscr{X}$ such that $m_i(\{x\})>0$, x must belong to A_i. P' is an extension of P and P is continuous, so we have

$$0=P(\{x\})=\int_{\{x\}}gdm=g(x)m(\{x\}).$$

Therefore we have $g(x)=0$. But $x \in A_i \subset [g(x)>0]$. This is a contradiction. Hence m_i is continuous.

Since m_i is a continuous finite measure on $(\mathcal{X}, \mathcal{A}')$, we can easily show that the completion of $\mathcal{A}$ by $m_i | \mathcal{A}$ contains $\mathcal{A}'$. Therefore there exists a set $B \in \mathcal{A}$ contained in A_i and a set N in $\mathcal{A}$ such that $A_i - B \subset N$ and $m_i | \mathcal{A}(N)=0$. Hence $m_i(B)=m_i(A_i)$.

Step 3: A_i is uncountable, because $m_i(A_i)>0$ holds and m_i is continuous. B is also uncountable by the same reason. Since B is a Borel set, by step 1, we have $m(A_i)=\infty$. But this is a contradiction.

REMARK 2. To show that $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ does not have a dominating measure it is not necessary to take the totality of all continuous probability measures as $\mathcal{P}$. It is sufficient to assume that $\mathcal{P}$ satisfies the following: For every uncountable $B \in \mathcal{A}$ there is $P_B \in \mathcal{P}$ with $P_B(B)=1$. For example $\mathcal{P}$ can be taken to be all probability measures which are continuous and singular relative to Lebesgue measure.

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