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ON THE SPECTRUM OF A RIEMANNIAN MANIFOLD OF POSITIVE CONSTANT CURVATURE

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Introduction. Let M be a compact connected riemannian manifold and Δ the Laplacian acting on the space of C^∞-functions on M. The operator Δ has a discrete spectrum consisting nonnegative eigenvalues with finite multiplicities. We denote by Spec M the spectrum of the Δ. Two compact connected riemannian manifolds M and N are said to be *isospectral* to each other if Spec $M=\mathrm{Spec}\,N$. The spectrum of a riemannian manifold gives a lot of information about its riemannian structure, but it does not completely determine the riemannian structure in general. In fact, there exist two flat tori which are isospectral but not isometric (by J. Milnor, see [1]). On the other hand some distinguished riemannian manifolds are completely characterized by their spectra as riemannian manifolds. The n-dimensional sphere S^n with the canonical metric and the real projective space $P^n(\boldsymbol{R})$ with the canonical metric are completely characterized by their spectra as riemannian manifolds if $n\leqq 6$ (see [1], [8]). Recently, it has been shown successively that a 3-dimensional lens space M is completely determined by its spectrum as a riemannian manifold; first by M. Tanaka [7] in the case the order $|\pi_1(M)|$ of the fundamental group of M is odd prime or 2-times odd prime, then by the author and Y. Yamamoto [4] in a more general case, and finally by Y. Yamamoto [11] without any restriction. These examples are riemannian manifolds of positive constant curvature.

A connected complete riemannian manifold M ($\dim M\geqq 2$) *of positive constant curvature* 1 *is called a spherical space form.* Now, we consider the problem;

(0.1) *Is a spherical space form characterized by its spectrum among all spherical space forms ?*

In this paper, we shall adapt one method to solve the problem and show affirmative results for the problem in the cases where spherical space forms are 3-dimensional and where spherical space forms are homogeneous.

Let S^n ($n\geqq 2$) be the n-dimensional sphere of constant curvature 1. If M is an n-dimensional spherical space form, then there exists a finite group G of fixed point free isometries on S^n such that M is isometric to S^n/G. Let E_k ($k\geqq 0$)

be the eigenvalue $k(k+n-1)$ of the Laplacian on S^n/G. The generating function $F_G(z)$ associated to the spectrum of the Laplacian on S^n/G is defined by

$$F_G(z)=\sum_{k=0}^{\infty}(\dim E_k)z^k .$$

By the definition, the spectrum of the spherical space form determines the generating function and the converse is also true. Clearly, for a lens space, the generating function defined in the above is identical to the one defined in [4]. In the paper [4], first we showed that the generating function of a lens space has the unique meromorphic extension to the whole complex plane $\boldsymbol{C}$. And by the investigation of the Laurent expansion of the generating function at poles, we proved the Main Theorem stated in [4]. The generating function $F_G(z)$ of a spherical space form S^n/G has also the unique meromorphic extension to $\boldsymbol{C}$. Their poles and the principal part of the Laurent expansion at poles are closely related to the sets $E(g)$ $(g\in G)$ consisting of eigenvalues of g, with multiplicity counted, and to the set $\sigma(G)$ consisting of orders of elements in G. In 2, we investigate the positions, the orders of poles of $F_G(z)$ and the sets $E(g)$, $\sigma(g)$ and their relations.

Combining the results in **2** and the classification theorem of 3-dimensional spherical space forms due to W. Threlfall and H. Seifert [9], we shall show in **4**.

Theorem I. *If two 3-dimensional spherical space forms are isospectral, then they are isomenric.*

An n-dimensional $(n\geqq 2)$ *compact riemannian manifold is of constant curvature* K *if* $n\leqq 5$ *and the manifold is isospectral to a compact riemannian manifold of constant curvature* K. (See [1], [8]). Together this with Theorem I, we see

Theorem II. *A 3-dimensional spherical space forms are completely characterized by its spectrum as a riemannian manifold.*

Homogeneous spherical space forms are completely classified by J.A. Wolf [10]. Using his results, we shall investigate properties of the generating function of a homogeneous spherical space form in more details. Then we shall obtain in **5**.

Theorem III. *Let* M, N *be spherical space forms. Suppose* M *is homogeneous and isospectral to* N. *Then* N *is isometric to* M.

By the same reason as we have obtained the Theorem II, we see

Theorem IV. *Homogeneous 5-dimensional lens spaces are completely characterized by their spectra as riemannian manifolds.*

1. Spherical space forms and their spectra

Let S^n ($n \geqq 2$) be the unit sphere centered at the origin in the $(n+1)$-dimensional Euclidean space R^{n+1}. We denote by $O(n+1)$ the orthogonal group acting on R^{n+1}. A finite group G of $O(n+1)$ is said to be *a fixed point free subgroup* of $O(n+1)$ if for any $g \in G$ with $g \neq 1_{n+1}$ (the unit matrix in $O(n+1)$), 1 is not an eigenvalue of g. By the definition, a fixed point free finite subgroup G of $O(n+1)$ acts on S^n fixed point freely, so that the quotient space S^n/G becomes a riemannian manifold of positive constant curvature 1 by the natural covering projection π of S^n onto S^n/G,

$$\pi : S^n \to S^n/G\,.$$

Conversely, it is well known that any compact connected riemannian manifold of positive constant curvature 1 is isometric to a quotient riemannian manifold S^n/G ($n \geqq 2$) with some fixed point free finite subgroup G of $O(n+1)$. A compact connected riemannian manifold of constant curvature 1 is called a *spherical space form.*

EXAMPLES. i) Let G be a fixed point free finite subgroup of $O(n+1)$. Suppose the order of G is one or two. Then G is $\{1_{n+1}\}$ or $\{\pm 1_{n+1}\}$, respectively. These yield the sphere S^n and the real projective space $P^n(\boldsymbol{R})$.
ii) Let G be a fixed point free finite subgroup of $O(2n)$ ($n \geqq 2$). Suppose G is a cyclic subgroup of order q. Take a generator g of G. Then g is conjugate in $O(2n)$ to the element

$$g' = \begin{pmatrix} R\left(\frac{p_1}{q}\right) & & 0 \\ & \ddots & \\ 0 & & R\left(\frac{p_n}{q}\right) \end{pmatrix},$$

where $p_1, \cdots, p_n$ are integers prime to q such that the eigenvalues of g are $\exp 2\pi\sqrt{-1}\frac{p_i}{q}$ and $\exp 2\pi\sqrt{-1}\frac{-p_i}{q}$ $(1 \leqq i \leqq n)$, and $R(\theta) = \begin{pmatrix} \cos 2\pi\theta & \sin 2\pi\theta \\ -\sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}$. Since G is conjugate in $O(2n)$ to the fixed point free finite subgroup $G' = \{g'^k\}_{k=0}^{q-1}$, tha spsce S^{2n-1}/G is isometric to the lens space $L(q: p_1, \cdots, p_n)$ (see [4]) as is seen by Lemma 1.2 below

The following two lemmas give fundamental properties for spherical space forms.

Lemma 1.1. i) *Even dimensional spherical space forms are only the spheres and the real projective spaces,* ii) *A finite fixed point free subgroup G of $O(2n)$ is contained in the special orthogonal group $SO(2n)$.*

Proof. i) Let G be a finite fixed point free subgroup of $O(2n+1)$ ($n \geqq 1$). Then for any $g \in G$, $\det g^2 = 1$. Since $2n+1$ is odd, g^2 has an eigenvalue 1. Hence, $g^2 = 1_{2n+1}$ which implies $g = 1_{2n+1}$ or $g = -1_{2n+1}$. Therefore $G = \{1_{2n+1}\}$ or $\{\pm 1_{2n+1}\}$.
ii) Let $g \in G$ with $g \neq 1_{2n}$. Then the eigenvalues of g are $\gamma_1, \bar{\gamma}_1, \cdots, \gamma_k, \bar{\gamma}_k$ and -1 with multiplicity $2(n-k)$, where $\gamma_1, \cdots, \gamma_k$ are unimodular complex numbers ($\neq \pm 1$), since G is fixed point free. Thus we have $\det g = (-1)^{2(n-k)} \gamma_1 \cdot \bar{\gamma}_1 \cdots \gamma_k \cdot \bar{\gamma}_k = 1$. q.e.d.

Lemma 1.2. *Let S^n/G and S^n/G' be spherical space forms. Then S^n/G is isometric to S^n/G' if and only if G is conjugate to G' in $O(n+1)$.*

Proof. Let g be an isometry of S^n/G onto S^n/G'. Then there exists an isometry $\tilde{g}$ of S^n onto istself which covers the isometry g. Then isometry $\tilde{g}$ can be considered as an element of $O(n+1)$ and yields an conjugation between G and G'. Conversely let $\tilde{g} \in O(n+1)$ such that $\tilde{g} G \tilde{g}^{-1} = G'$. Then $\tilde{g}$ induces the isometry g of S^n/G onto S^n/G' such that $g\pi(x) = \pi(\tilde{g}x)$ for any $x \in S^n$. q.e.d.

For a differentiable manifold M, we denote by $C^\infty(M)$ *the space of complex valued C^∞-functions on M.* Let Δ_0 be the Laplacian on R^{n+1} and $(x_1, \cdots, x_{n+1})$ the standard coordinate system on R^{n+1}. Set $r^2 = \sum_{i=1}^{n+1} x_i^2$. For $k \geqq 0$, let P^k be the space of complex valued homogeneous polynomials of degree k on R^{n+1} and H^k the subspace of P^k consisting of harmonic polynomials on R^{n+1},

$$H^k = \{f \in P^k : \Delta_0 f = 0\} .$$

The group $O(n+1)$ acts on P^k. The subspace H^k is a G-invariant subspace and r^2 is a G-invariant element of P^2. We have a direct sum decomposition (see [4]),

$$P^k = H^k \oplus r^2 P^{k-2} , \tag{1.1}$$

where we put $P^{-1} = P^{-2} = \{0\}$.

The natural inclusion map i of S^n into R^{n+1} induces the restriction map i^* of $C^\infty(R^{n+1})$ into $C^\infty(S^n)$,

$$i^* : C^\infty(R^{n+1}) \to C^\infty(S^n) .$$

The group $O(n+1)$ acts naturally on the spaces $C^\infty(R^{n+1})$, $C^\infty(S^n)$ and its actions commute with the map i^*.

Lemma 1.3. (See [4]). *Let $\mathcal{H}^k$ be the eigenspace with eigenvalue $k(k+n-1)$ of the Laplacian on S^n. Then the map i^* gives an $O(n+1)$-isomorphism H^k onto $\mathcal{H}^k$,*

$$i^* : H^k \cong \mathcal{H}^k .$$

Let S^n/G be a spherical space form. Then the natural projection π induces the injective map π^* of $C^\infty(S^n/G)$ into $C^\infty(S^n)$,

$$\pi^* : C^\infty(S^n/G) \to C^\infty(S^n).$$

We see easily

Lemma 1.4. *The subspace $\pi^*(C^\infty(S^n/G))$ of $C^\infty(S^n)$ consists of all G-invariant functions on S^n.*

Let $\tilde{\Delta}$ and Δ be the Laplacians on S^n and S^n/G, respectively. Then we have for any $f \in C^\infty(S^n/G)$,

$$\tilde{\Delta}\pi^* f = \pi^*\Delta f, \tag{1.3}$$

since π is a locally isometric mapping.

By Lemma 1.3, Lemma 1.4 and (1.3), we have

Proposition 1.5. *Let $\mathcal{H}_G^k$ and H_G^k be the subspaces of $\mathcal{H}^k$ and H^k consisting of all G-invariant elements of $\mathcal{H}^k$ and H^k, respectively. Then the space $(\pi^*)^{-1}\mathcal{H}_G^k$ is the eigenspace with eigenvalue $k(k+n-1)$ of the Laplacian Δ on S^n/G and isomorphic to H_G^k. Further, every eigenspace of Δ on S^n/G obtained in this way.*

2. The generating function associated to the spectrum of a spherical space form

It is well known that the sphere S^n is not isospectral to the real projective space $P^n(\boldsymbol{R})$ (cf.[1]). Together this fact with i) in Lemma 1.1, in case of even dimensional spherical space forms, our isospectral problem is solved.

In [4], we defined the generating function associated to the spectrum of a lens space. This generating function played an important role in our isospectral problem for lens spaces. In this section, we also define the generating function associated to the spectrum of a spherical space form and study its properties.

Let $M=S^{2n-1}/G$ $(n \geqq 2)$ be a spherical space form with a fixed point free finite subgroup G of $SO(2n)$.

DEFINITION. The generating function $F_G(z)$ associated to the spectrum of the Laplacian on M is *the generating function associated to the infinite sequence* $\{\dim H_G^k\}_{k=0}^\infty$, *i.e.*,

$$F_G(z) = \sum_{k=0}^\infty (\dim H_G^k) z^k.$$

By the definition and Proposition 1.5, we have

Proposition 2.1. *Let S^{2n-1}/G and S^{2n-1}/G' be spherical space forms. Then S^{2n-1}/G is isospectral to S^{2n-1}/G' if and only if*

$$F_G(z) = F_{G'}(z)\,.$$

Theorem 2.2. *Let S^{2n-1}/G be a spherical space form. On the domain $\{z\in C\colon |z|<1\}$, $F_G(z)$ converges to the function*

$$F_G(z) = \frac{1}{|G|}\sum_{g\in G}\frac{1-z^2}{\det(1_{2n}-gz)}\,,$$

where $|G|$ denotes the order of G.

Proof. Let χ_k, $\tilde{\chi}_k$, be the characters of the natural representations of $SO(2n)$ on H^k and P^k, respectively. Then we have

$$\dim H^k_G = \frac{1}{|G|}\sum_{g\in G}\chi_k(g)\,, \tag{2.1}$$

and also by (1.1)

$$\chi_k(g) = \tilde{\chi}_k(g) - \tilde{\chi}_{k-2}(g)\,, \tag{2.2}$$

where we put $\tilde{\chi}_{-t}=0$ for $t>0$.

If an element $g\in SO(2n)$ is conjugate to an element $g'\in SO(2n)$ in $O(2n)$, then

$$\tilde{\chi}_k(g') = \tilde{\chi}_k(g) \qquad k\geqq 0\,. \tag{2.3}$$

Let g be an element in G of order q. Set $\gamma=\exp 2\pi\sqrt{-1}/q$. And let γ^{p_1}, $\bar{\gamma}^{p_1}, \cdots, \gamma^{p_n}, \bar{\gamma}^{p_n}$ be the eigenvalues of g, where $p_1, p_2, \cdots, p_n$ are integers prime to q. Then g is conjugate to the element

$$g' = \begin{pmatrix} R\left(\frac{p_1}{q}\right) & & 0 \\ & \ddots & \\ 0 & & R\left(\frac{p_n}{q}\right)\end{pmatrix} \qquad \text{in } SO(2n)\,.$$

Let $(x_1, y_1, \cdots, x_n, y_n)$ be the standard euclidean coordinate system on R^{2n}. Put $z_i=x_i+\sqrt{-1}y_i$ $(i=1, 2, \cdots, n)$. Then the space P^k has a base consisting of all monomials of the forms

$$z^I\cdot\bar{z}^J = (z_1)^{i_1}\cdots(z_n)^{i_n}\cdot(\bar{z}_1)^{j_1}\cdots(\bar{z}_n)^{j_n}\,,$$

where $i_1, \cdots, i_n, j_1, \cdots, j_n\geqq 0$ and $i_1+\cdots+i_n+j_1+\cdots+j_n=k$. For any monomial $z^I\cdot\bar{z}^J$, we have

$$g'(z^I\cdot\bar{z}^J) = \gamma^{i_1p_1+\cdots+i_np_n-j_1p_1-\cdots-j_np_n}(z^I\cdot\bar{z}^J)\,. \tag{2.4}$$

By (2,.2), (2.3) and (2.4), we have

$$\begin{aligned}\sum_{k=0}^{\infty}\chi_k(g)z^k &= \sum_{k=0}^{\infty}(\tilde{\chi}_k(g)-\tilde{\chi}_{k-2}(g))z^k \\ &= (1-z^2)\sum_{k=0}^{\infty}\tilde{\chi}_k(g)z^k \\ &= (1-z^2)\sum_{k=0}^{\infty}\tilde{\chi}_k(g')z^k \\ &= (1-z^2)\sum_{k=0}^{\infty}\sum_{i_1+\cdots+i_n+j_1+\cdots+j_n=k}\gamma^{i_1p_1+\cdots+i_np_n-\cdots-j_np_n}z^k \\ &= (1-z^2)\sum_{k=0}^{\infty}\sum_{i_1+\cdots+i_n+j_1+\cdots+j_n=k}(\gamma^{p_1}z)^{i_1}\cdots(\gamma^{p_n}z)^{i_n}\cdot(\gamma^{-p_1}z)^{j_1}\cdots(\gamma^{-p_n}z)^{j_n} \\ &= (1-z^2)\frac{1}{(1-\gamma^{p_1}z)\cdots(1-\gamma^{p_n}z)\cdot(1-\gamma^{-p_1}z)\cdots(1-\gamma^{-p_n}z)} \\ &= \frac{1-z^2}{\det(1_{2n}-gz)}\,.\end{aligned}$$

Together this with (2.1), we obtain the theorem. q.e.d.

REMARKS. i) Let $E(g)$ be the set of eigenvalues of g, with multiplicity counted. Then we have

$$\det(1_{2n}-gz)=\prod_{\gamma\in E(g)}(1-\gamma z)=\prod_{\gamma\in E(g)}(z-\gamma)\,.$$

ii) By the theorem, the generating function can be considered as a meromorphic function on the whole complex plane $\boldsymbol{C}$ and its poles are on the unit circle $S^1=\{z\in\boldsymbol{C}\,|\,|z|=1\}$. This meromorphic extension of the generating function to $\boldsymbol{C}$ is clearly unique.

In the followings, we consider the generating function associated to the spectrum of a spherical space form S^n/G as the meromorphic function of the form in Theorem 2.2.

From i) in the above Remarks, we have

Corollary 2.3. *Let S^{2n-1}/G and S^{2n-1}/G' be spherical space forms. Assume there is a one to one mapping ϕ of G onto G' such that the set $E(g)=$the set $E(\phi(g))$ for any $g\in G$. Then S^{2n-1}/G is isospectral to S^{2n-1}/G'.*

Corollary 2.4. *Let S^{2n-1}/G and S^{2n-1}/G' be spherical space forms. Assume S^{2n-1}/G is isospectral to S^{2n-1}/G'. Then we have $|G|=|G'|$.*

Proof. The generating function $F_G(z)$ has a pole of order $2n-1$ at $z=1$ and

$$\lim_{z\to1}(1-z)^{2n-1}F_G(z)=\frac{2}{|G|}\,.$$

This proves Corollary 2.4. q.e.d.

Let G be a finite group.

DEFINITIONS. *G_k is the subset of G consisting of all elements of order k in G. $\sigma(G)$ is the set consisting of orders of elements in G.*
We have

$$G = \bigcup_{k\in\sigma(G)} G_k \qquad \text{(disjoint union).}$$

Lemma 2.5. *Let G be a fixed point free finite subgroup of $SO(2n)$ $(n\geqq 2)$. Then the subset G_k is divided into the disjoint union of subsets $C_k^1, \cdots, C_k^{i_k}$ such that each C_k^t $(t=1, 2, \cdots, i_k)$ consists of all generic elements of some cyclic subgroup of order k in G.*

Proof. For any $g\in G_k$, we denote by A_g the cyclic subgroup of G generated by g. For $g, g'\in G_k$ the cyclic group $A_g\cap A_{g'}$ is of order k if and only if $A_g=A_{g'}$. Now the lemma follows immediately. q.e.d.

Lemma 2.6. *Let g be an element in $SO(2n)$ $(n\geqq 2)$ and of order $q (q\geqq 3)$. Set $\gamma=\exp 2\pi\sqrt{-1}/q$. Assume g has eigenvalues $\gamma, \gamma^{-1}, \gamma^{p_1}, \gamma^{-p_1}, \cdots, \gamma^{p_k}, \gamma^{-p_k}$ with multiplicities $l, l, i_1, i_1, \cdots, i_k, i_k$, respectively, where $p_1, \cdots, p_k$ are integers prime to q with $p_i\not\equiv \pm p_j$ (mod q) $(1\leqq i<j\leqq k)$, $p_i\not\equiv\pm 1$ (mod q) $(i=1, \cdots, k)$ and $l+i_1+\cdots+i_k=n$. Then the Laurent expansion of the meromorphic function $\dfrac{1-z^2}{\det(1_{2n}-gz)}$ at $z=\gamma$ is*

$$\frac{1}{(z-\gamma)^l}\frac{(\sqrt{-1})^{n+l}\gamma^l}{2^{n-l}(1-\gamma^2)^{n-1}}\prod_{j=1}^{k}\{\cot\frac{\pi}{q}(p_j+1)-\cot\frac{\pi}{q}(p_j-1)\}^{i_j} \tag{2.5}$$
$$+ \textit{ the lower order terms.}$$

Proof. We have

$$\begin{aligned}
&\lim_{z\to\gamma}(z-\gamma)^l\frac{1-z^2}{(z-\gamma)^l(z-\gamma^{-1})^l(z-\gamma^{p_1})^{i_1}\cdots(z-\gamma^{-p_k})^{i_k}}\\
&=\frac{1-\gamma^2}{(\gamma-\gamma^{-1})^l}\prod_{j=1}^{k}\frac{1}{(\gamma-\gamma^{p_j})^{i_j}(\gamma-\gamma^{-p_j})^{i_j}}\\
&=\frac{1-\gamma^2}{(\gamma-\gamma^{-1})^l}\prod_{j=1}^{k}\left\{\frac{1}{\gamma^{p_j+1}(\gamma^{-p_j+1}-1)(1-\gamma^{-p_j-1})}\right\}^{i_j}\\
&=\frac{1-\gamma^2}{(\gamma-\gamma^{-1})^l}\prod_{j=1}^{k}\left\{\frac{1}{\gamma^{p_j+1}(\gamma^{-p_j-1}-\gamma^{-p_j+1})}\left[\frac{1}{1-\gamma^{-p_j+1}}-\frac{1}{1-\gamma^{-p_j-1}}\right]\right\}^{i_j}\\
&=\frac{1-\gamma^2}{(\gamma-\gamma^{-1})^l(1-\gamma^2)^{n-l}}\prod_{j=1}^{k}\left\{\frac{1}{\gamma^{-p_j+1}}-\frac{1}{1-\gamma^{-p_j-1}}\right\}^{i_j}
\end{aligned}$$

$$= \frac{(-1)^l \gamma^l}{(1-\gamma^2)^{n-1}} \prod_{j=1}^{k} \left\{ \frac{1}{1-\gamma^{-p_j+1}} - \frac{1}{1-\gamma^{-p_j-1}} \right\}^{i_j}$$
$$= \frac{(\sqrt{-1})^{n+l}\gamma^l}{2^{n-l}(1-\gamma^2)^{n-1}} \prod_{j=1}^{k} \left\{ \cot \frac{\pi}{q}(p_j+1) - \cot \frac{\pi}{q}(p_j-1) \right\}^{i_j},$$

where we used the formula $\frac{1}{1-e^{-2\sqrt{-1}\theta}} = \frac{1}{2}\{1-\sqrt{-1}\cot\theta\}$. q.e.d.

Proposition 2.7. *Let G be a fixed point free finite subgroup of $SO(2n)$ $(n \geqq 2)$, and let $k \in \sigma(G)$. We define a positive integer k_0 by*

$$k_0 = 2n-1 \quad \textit{if} \quad k=1 \quad \textit{or } 2,$$
$$= \max_{g \in G_k} \{\textit{max. of multiplicities of eigenvalues of } g\} \quad \textit{if} \quad k \geqq 3.$$

Then the generating function $F_G(z)$ has a pole of order k_0 at any primitive k-th root of 1.

Proof. It is easy to check that the generating function has poles of order $2n-1$ at $z=\pm 1$. Hence, we may assume $k \geqq 3$. Let $G_k, C_k^1, \cdots, C_k^{i_k}$ be as in Lemma 2.5. Then we have

$$(2.6) \qquad |G|F_G(z) = \sum_{g \in G_k} \frac{1-z^2}{\det(1_{2n}-gz)} + \sum_{g \in G-G_k} \frac{1-z^2}{\det(1_{2n}-gz)}$$
$$= \sum_{j=1}^{i_k} \sum_{g \in C_k^j} \frac{1-z^2}{\det(1_{2n}-gz)} + \sum_{g \in G-G_k} \frac{1-z^2}{\det(1_{2n}-gz)}.$$

Set $\gamma = \exp 2\pi\sqrt{-1}/k$. For any primitive k-th root γ^t of 1, where t is an integer prime to k, let

$$\frac{a_{k_0}(t)}{(z-\gamma^t)^{k_0}} + \frac{a_{k_0-1}(t)}{(z-\gamma^t)^{k_0-1}} + \cdots + \frac{a_1(t)}{(z-\gamma^t)}$$

by the principal part of the Laurent expansion of $F_G(z)$ at $z=\gamma^t$. Then each coefficient $a_i(t)$ is an element in the k-th cyclotomic field $\boldsymbol{Q}(\gamma)$ over the rational number field $\boldsymbol{Q}$. The automorphism σ_t of $\boldsymbol{Q}(\gamma)$ defined by

$$\gamma \mapsto \gamma^t$$

transforms $a_i(1)$ to $a_i(t)$ $(i=1, 2, \cdots, k_0)$ by (2.6). Hence, it is sufficient to show that the generating function $F_G(z)$ has a pole of order k_0 at $z=\gamma$, that is, to show $a_{k_0}(1) \neq 0$.
Note

$$\cot a - \cot b < 0 \qquad \text{if} \quad 0 < b < a < \pi.$$

Now, Proposition 2.7 follows immediately from (2.5) in Lemma 2.6 and (2.6). q.e.d.

From Proposition 2.7, we have

Corollary 2.8. *Let S^{2n-1}/G and S^{2n-1}/G' be spherical space forms. Assume S^{2n-1}/G is isospectral to S^{2n-1}/G'. Then, $\sigma(G)=\sigma(G')$.*

Proposition 2.9. *Let S^{2n-1}/G and S^{2n-1}/G' be as in Corollary 2.8. Then the numbers of elements of order 4 in G and G' are equal.*

Proof. If $g\in SO(2n)$ is of order 4, then g has eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ with the same multiplicities n. Hence, we have

$$\sum_{g\in G_4}\frac{1-z^2}{\det(1_{2n}-gz)}=|G_4|\frac{1-z^2}{(1+z^2)^n},$$

where $|G_4|$ is the number of elements in G_4.
Now, Proposition 2.9 follows immediately from this formula. q.e.d.

3. Classification of 3-dimensional spherical space forms

The classification of 3-dimensional spherical space forms has been obtained by W. Threllfall and H. Seifert [9] (see also [3], [5] and [10]). In this section, we shall describe their results and study some properties of the finite subgroups appearing in the classification.

Let $SO(3)$ be the special orthogonal group acting on R^4. It is well known that the finite subgroups of $SO(3)$ (up to conjugations in $SO(3)$) are given as follows (for details, see [10]);

$\boldsymbol{Z}_m$: the cyclic group of order m ($m\geqq 1$),
$\boldsymbol{D}_{2n}$: the dihedral group of order $2n$ ($n\geqq 2$),
$\boldsymbol{T}$: the tetrahedral group of order 12,
$\boldsymbol{O}$: the octahedral group of order 24,
$\boldsymbol{I}$: the icosahedral group of order 60.

These groups are given in terms of generators and relations by the followings;

(3.1) $\boldsymbol{Z}_m$: $A^m=1$,

(3.2) $\boldsymbol{D}_{2n}$: $A^n=B^2=1$, $BAB^{-1}=A^{-1}$,

(3.3) $\boldsymbol{T}$: $A^3=P^2=Q^2=1$, $PQ=QP$, $APA^{-1}=Q$, $AQA^{-1}=PQ$,

(3.4) $$\boldsymbol{O}:\ \begin{cases} A^3=P^2=Q^2=R^2=1, & PQ=QP,\quad RAR^{-1}=A^{-1}, \\ APA^{-1}=Q, & AQA^{-1}=PQ,\quad RPR^{-1}=QP,\quad RQR^{-1}=Q^{-1}, \end{cases}$$

(3.5) $\boldsymbol{I}$: $A^3=B^2=C^5=ABC=1$.

Moreover, it is known that $\boldsymbol{T}$ is isomorphic to the alternating group $\boldsymbol{A}_4$ of 4-symbols, $\boldsymbol{O}$ is isomorphic to the symmetric group $\boldsymbol{S}_4$ of 4-symbols and $\boldsymbol{I}$ is isomorphic to the alternating group $\boldsymbol{A}_5$ of 5-symbols.

Let $\boldsymbol{H}$ be the algebra of real quoternion numbers, which is a 4-dimensional vector space over the real number field $\boldsymbol{R}$ with basis $\{1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ such that $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$.

$$\boldsymbol{H} = \{a+bi+cj+dk;\ a, b, c, d \in R\} .$$

As usual, we have the conjugation on $\boldsymbol{H}$ over R given by

$$\overline{a+bi+cj+dk} = a-bi-cj-dk .$$

The subset $\boldsymbol{H}' = \{q \in \boldsymbol{H} : q\bar{q} = 1\}$ of $\boldsymbol{H}$ forms a multiplicative group, called *the group of unit quoternions,* and it is isomorphic to the special unitary group $SU(2)$ acting on C^2.

REMARK. The element -1 is the only element of order 2 in $\boldsymbol{H}'$.

Let $\boldsymbol{H}_0$ be the 3-dimensional real vector subspace of $\boldsymbol{H}$ with basis $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$, which will be identified R^3 by this basis. We have the map π of $\boldsymbol{H}'$ onto $SO(3)$

$$\pi\colon \boldsymbol{H}' \to SO(3)$$

defined by

$$\pi(q)q' = qq'\bar{q} ,$$

for $q \in \boldsymbol{H}'$ and $q' \in \boldsymbol{H}_0$.

π gives a homomorphism of the group $\boldsymbol{H}'$ onto $SO(3)$ and it is two-fold covering map with $\pi(q) = \pi(-q)$.

Lemma 3.1. *Let g be an element of order t in $SO(3)$. Then the orders of elements of $\pi^{-1}(g)$ are $2t$ and t for odd t, and $2t$ for even t.*

Proof. Let s be an integer prime to t. Then we have

$$\begin{aligned}\pi\left(\cos\frac{s\pi}{t} + \sin\frac{s\pi}{t}k\right) &= \pi\left(-\cos\frac{s\pi}{t} - \sin\frac{s\pi}{t}k\right)\\ &= \begin{pmatrix} R\left(\frac{s}{t}\right) & \begin{matrix}0\\0\end{matrix}\\ \begin{matrix}0 & 0\end{matrix} & 0\end{pmatrix}.\end{aligned}$$

Now, our element g is conjugate to $\begin{pmatrix} R\left(\frac{s'}{t}\right) & \begin{matrix}0\\0\end{matrix}\\ \begin{matrix}0 & 0\end{matrix} & 0\end{pmatrix}$, for some integer s' prime to t, in $SO(3)$. Then the elements in $\pi^{-1}(g)$ are conjugate in H' to $\pm\left(\cos\frac{s'\pi}{t} + \sin\frac{s'\pi}{t}k\right)$. Suppose t is odd. Then we see easily that the orders of elements $\left(\cos\frac{s'\pi}{t} + \sin\frac{s'\pi}{t}k\right)$ and $-\left(\cos\frac{s'\pi}{t} + \sin\frac{s'\pi}{t}k\right)$ are $\{t, 2t\}$. Next suppose t is even. Then we see easily that they have the same order $2t$. q.e.d.

The finite subgroups of $\boldsymbol{H}'$ (up to conjugations in $\boldsymbol{H}'$) are given as follows (see [10]);

(3.6) $\boldsymbol{Z}_m$: the cyclic group of order m,

(3.7) $\boldsymbol{D}^*_{4n}=\pi^{-1}(\boldsymbol{D}_{2n})$: the binary dihedral group of order $4n$,

$\boldsymbol{D}^*_{4n}$ is also given by

$$\boldsymbol{D}^*_{4n}:\ A^{2n}=B^4=1\,,\quad BAB^{-1}=A^{-1}\,,$$

in th terms of generators and relations,

(3.8) $\boldsymbol{T}^*=\pi^{-1}(\boldsymbol{T})$: the binary tetrahedral group of order 24,

(3.9) $\boldsymbol{O}^*=\pi^{-1}(\boldsymbol{O})$: the binary octahedral group of order 48,

and

(3.10) $\boldsymbol{I}^*=\pi^{-1}(\boldsymbol{I})$: the binary icosahedral group of order 120.

The groups $\boldsymbol{T}^*$, $\boldsymbol{O}^*$, $\boldsymbol{I}^*$ are called the *binary polyhedral groups*.

Before describing the classification theorem, we need other two types of finite groups.

The finite group $\boldsymbol{D}'_{2^kn}$ ($k\geqq 3$, n odd and $n\geqq 3$) of order 2^kn is defined as follows.

Let $\boldsymbol{Z}_{2^k}$ be a cyclic subgroup of order 2^k of $\boldsymbol{H}'$ and $\boldsymbol{D}^*_{4n}$ the binary dihedral group of order $4n$. Let $\boldsymbol{Z}_{2^{k-1}}$ be the cyclic subgroup of order 2^{k-1} of $\boldsymbol{Z}_{2^k}$ and let $\boldsymbol{Z}_{2n}$ be the cyclic normal subgroup of order $2n$ of $\boldsymbol{D}^*_{4n}$ generated by A in (3.7). Let ϕ be the isomorphism of the quotient group $\boldsymbol{Z}_{2^k}/\boldsymbol{Z}_{2^{k-1}}$ onto $\boldsymbol{D}^*_{4n}/\boldsymbol{Z}_{2n}$. Set

$$\tilde{\boldsymbol{D}}_{2^{k+1}n}=\{(x,y)\in\boldsymbol{Z}_{2^k}\times\boldsymbol{D}^*_{4n}\colon\ [y]=\phi([x])\}\,, \tag{3.11}$$

where $[x]$, $[y]$ mean the quotient classes in $\boldsymbol{Z}_{2^k}/\boldsymbol{Z}_{2^{k-1}}$ and $\boldsymbol{D}^*_{4n}/\boldsymbol{Z}_{2n}$. Then we define

$$\tilde{\boldsymbol{D}}'_{2^kn}=\tilde{\boldsymbol{D}}_{2^{k+1}n}/\{(-1,-1),(1,1)\}\,. \tag{3.12}$$

Next, the finite group $\boldsymbol{T}'_{3^kn}(k\geqq 1)$ of order 3^k8 is defined as follows.

Let $\boldsymbol{Z}_{3^k}$ and $\boldsymbol{T}^*$ be as before, considered as subgroup of $\boldsymbol{H}'$. Let $\boldsymbol{Z}_{3^{k-1}}$ be the cyclic subgroup of order 3^{k-1} of $\boldsymbol{Z}_{3^k}$. Let $\boldsymbol{H}_8$ be the inverse image of π of the subgroup of $\boldsymbol{T}$ generated by P and Q in (3.3). The group $\boldsymbol{H}_8$ is a normal subgroup of $\boldsymbol{T}^*$ and is isomorphic to $\boldsymbol{D}^*_8$. Then the quotient groups $\boldsymbol{Z}_{3^k}/\boldsymbol{Z}_{3^{k-1}}$ and $\boldsymbol{T}^*/\boldsymbol{H}_8$ are isomorphic to $\boldsymbol{Z}_3$. Hence, there exist only two isomorphisms ψ_1, ψ_2 of $\boldsymbol{Z}_{3^k}/\boldsymbol{Z}_{3^{k-1}}$ onto $\boldsymbol{T}^*/\boldsymbol{H}_8$.

Let ϕ be an isomorphism of $Z_{3^k}/Z_{3^{k-1}}$ onto T^*/H_8. We define

$$T'_{3^k 8}(\phi) = \{(x, y) \in Z_{3^k} \times T^*:\ [y] = \phi([x])\}\,, \tag{3.13}$$

where $[x]$, $[y]$ denote the quotient classes in each factor group. It is easy to see $T'_{3^k 8}(\psi_1)$ is isomorphic to $T'_{3^k 8}(\psi_2)$. In the followings, we denote by $T'_{3^k 8}$ the finite group isomorphic to one of the above groups. Note that $T'_{3^1 1\cdot 8} = T^*$.

For two integers a and b, we denote by (a,b) the greatest common divisor of a and b.

Theorem 3.2 (W. Seifert and H, Threlfall [9], see also [5] and [10]).
Let S^3/G be a 3-dimensional spherical space form. Then G is isomorphic to one of the following groups of type (I), (II), (III), (IV), (V) and (VI).
(I) the group Z_m with $m \geqq 1$,
(II) the group $Z_q \times D'_{2^k n}$ with $(q, 2n)=1$, $k \geqq 3$, n is odd and $n \geqq 3$,
(III) the group $Z_q \times D^*_{4n}$ with $(q, 2n)=1$ and $n \geqq 2$,
(IV) the group $Z_q \times T'_{3^k 8}$ with $(q, 6)=1$ and $k \geqq 1$,
(V) the group $Z_q \times O^*$ with $(q, 6)=1$,
(VI) the group $Z_q \times I^*$ with $(q, 30)=1$.
Moreover, let S^3/G' be an another spherical space form. Assume G is isomorphic to G' and is not of type (I). Then G is conjugate to G' in $O(4)$, so that S^3/G is isometric to S^3/G'.

REMARK. If G is of type (I), then S^3/G is a lens space as stated in examples of **1**. For the isometric conditions for lens spaces, see [4].

In the latter half of this section, we shall study some properties of the finite groups appearing in the above theorem.

Lemma 3.3. *Let Z_{2n} be the cyclic subgroup of D^*_{4n} generated by the element A in* (3.7). *Then all the elements of $(D^*_{4n} - Z_{2n})$ are of order* 4.

Proof. Any element in the set $(D^*_{4n} - Z_{2n})$ is uniquely writen as BA^t, where $1 \leqq t \leqq 2n$. Using the elementary relations in (3.7), we have

$$\begin{aligned}(BA^t)^2 &= BA^tBA^t \\ &= BA^{t-1}(AB)A^t \\ &= BA^{t-1}BA^{t-1} \\ &= (BA^{t-1})^2 \\ &\ \ \vdots \\ &= B^2\,.\end{aligned}$$

Hence, the element BA^t is of order 4. q.e.d.

From this lemma, we have directly the following two corollaries.

Corollary 3.4. *Let $G=\boldsymbol{Z}_q\times\boldsymbol{D}^*_{2n}$ be of type (III) in Theorem 3.1. Then the number of the elements of order 4 in G is 2n for odd n and 2n+2 for even n, respectively.*

Corollary 3.5. *Let $G=\boldsymbol{Z}_q\times\boldsymbol{D}^*_{4n}$ be as in Corollary 3.4. Then we have*

$$\max\sigma(G)=2nq=\frac{|G|}{2}.$$

Lemma 3.6. *Let $G=\boldsymbol{D}'_{2^kn}$ be a finite group as in (3.11). Then we have*

$$\sigma(\boldsymbol{D}'_{2^kn})=\{2^k, 2^{k'}n' : 0\leqq k'<k,\ n' \text{ divides } n\}.$$

Proof. The group $\boldsymbol{D}_{2^{k+1}n}$ is divided into two subsets, $X=\boldsymbol{Z}_{2^{k-1}}\times\boldsymbol{Z}_{2n}$ and $Y=(\boldsymbol{Z}_{2^k}-\boldsymbol{Z}_{2^{k-1}})\times(\boldsymbol{D}^*_{4n}-\boldsymbol{Z}_{2n})$. Since $k\geqq 3$, the sets of orders of elements in X and Y are $\{2^{k'}n' : 0\leqq k'<k,\ n' \text{ divides } n\}$ and $\{2^k\}$, respectively. It is easy to see that the union of these sets coincides with the set $\sigma(\boldsymbol{D}'_{2^kn})$, since $k\geqq 3$.

q.e.d.

From this lemma, we see easily

Corollary 3.7. *Let $G=\boldsymbol{Z}_q\times\boldsymbol{D}'_{2^kn}$ be a finite group of type (II). Then*

i) $\max\sigma(G)=2^{k-1}nq=\dfrac{|G|}{2}$.

ii) *$2^kq'\in\sigma(G)$ if and only if q' divides q.*

Lemma 3.8. *Let $G=\boldsymbol{Z}_q\times\boldsymbol{D}'_{2^kn}$ be as in Corollary 3.7. Then the number of elements of order 4 in G is 2.*

Proof. It is sufficient to show that the number of elements of order 4 in $\boldsymbol{D}'_{2^kn}$ is 2. Let X and Y be the sets as in the proof of Lemma 3.6. Since $k\geqq 3$, any element in Y does not yield an element of order 4 in $\boldsymbol{D}'_{2^kn}$. Let $(x, y)\in X$. By $[(x, y)]$, we mean its quotient class in $\boldsymbol{D}'_{2^kn}$. If $[(x, y)]$ is of order 4, then the order of y must be one or two, i.e., $y=\pm 1$, (as elements in $\boldsymbol{H}'$), since n is odd. Thus, x must be of order 4. But the elements of order 4 in $\boldsymbol{Z}_{2^{k-1}}$ consists of two elements which differ in sign (as elements in $\boldsymbol{H}'$). Therefore the number of the elements of order 4 in $\boldsymbol{D}'_{2^kn}$ is 2. q.e.d.

Lemma 3.9. *Let $\boldsymbol{T}^*$ be the binary tetrahedral group and $\boldsymbol{H}_8$ the subgroup of $\boldsymbol{T}^*$ defined before. Then $\sigma(\boldsymbol{T}^*-\boldsymbol{H}_8)=\{3, 6\}$.*

Proof. Let $\boldsymbol{T}$ be the group as in (3.3). Then T is isomorphic to the alternating group $\boldsymbol{A}_4$ of 4 symbols 1, 2, 3 and 4. We use the usual notations for permutations (see [2]). Then the subgroup $\pi(\boldsymbol{H}_8)$ corresponds to the subgroup $\boldsymbol{D}_4=\{1, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$. Since $\boldsymbol{A}_4-\boldsymbol{D}_4$ $(=\{(1,2,3), (1,3,2), (1,2,4), \cdots\})$ consists of all 3-cycles in $\boldsymbol{A}_4$, we have $\sigma(\boldsymbol{A}_4-\boldsymbol{D}_4)=\{3\}$. Now, Lemma 3.9 follows from Lemma 3.1. q.e.d.

Lemma 3.10. *Let* $G=\boldsymbol{T}'_{3^k8}$ *be the finite group defined in* (3.13). *Then* $\sigma(G)=\{3^k, 2\cdot 3^k, 4\cdot 3^{k'}, 2\cdot 3^{k'}, 3^{k'}: 1\leqq k'<k\}$.

Proof. We divide the set G into two subsets, $X=\boldsymbol{Z}_{3^{k-1}}\times \boldsymbol{H}_8$ and $Y=G-X$. The orders of elements in X are $3^{k'}$, $2\cdot 3^{k'}$, $4\cdot 3^{k'}$ $(1\leqq k'<k)$. On the other hand, since $\sigma(\boldsymbol{T}^*-\boldsymbol{H}_8)=\{3, 6\}$ and $\sigma(\boldsymbol{Z}_{3^k}-\boldsymbol{Z}_{3^{k-1}})=\{3^k\}$, the orders of elements in the set Y are 3^k and $2\cdot 3^k$. q.e.d.

Lemma 3.11. *We have* $\sigma(\boldsymbol{O}^*)=\{1, 2, 3, 6, 8\}$ *and* $\sigma(\boldsymbol{I}^*)=\{1, 2, 3, 4, 5, 6, 10\}$.

Proof. This follows directly from the facts that $\boldsymbol{O}$ is isomorphic to $\boldsymbol{S}_4$ and $\boldsymbol{I}$ is isomorphic to $\boldsymbol{A}_5$. q.e.d.

By Lemma 3.10 and Lemma 3.11, we have

Corollary 3.12. *Let* $G_{\mathrm{IV}}=\boldsymbol{Z}_q\times \boldsymbol{T}'_{3^k8}$, $G_{\mathrm{V}}=\boldsymbol{Z}_q\times \boldsymbol{O}^*$ *and* $G_{\mathrm{VI}}=\boldsymbol{Z}_q\times \boldsymbol{I}^*$ *be finite groups of type (IV), (V) and (VI), respectively. Then* $\max\sigma(G_{\mathrm{IV}})=\dfrac{|G_{\mathrm{IV}}|}{4}$, $\max\sigma(G_{\mathrm{V}})=\dfrac{|G_{\mathrm{V}}|}{6}$ and $\max\sigma(G_{\mathrm{VI}})=\dfrac{|G_{\mathrm{VI}}|}{12}$, *respectively.*

4. Proof of Theorem I

Let S^3/G and S^3/G' be spherical space forms and $F_G(z)$, $F_{G'}(z)$ their generating functions. Suppose $F_G(z)=F_{G'}(z)$. Then we obtained in **2** the followings;

(4.1) $|G|=|G'|$,

(4.2) $\sigma(G)=\sigma(G')$, particularly $\max\sigma(G)=\max\sigma(G')$.

(4.3) The numbers of elements of order 4 in G and G' are equal.

First, suppose G is of type (I), i.e., G is cyclic. Then $\max\sigma(G)=|G|$. By (4,1), (4.2), we have $\max\sigma(G')=|G'|$. This implies G' is also cyclic. Thus S^3/G, S^3/G' are lens spaces. Hence, S^3/G is isometric to S^3/G' by [4], [7] and [11]. Next, suppose G is of type (II). Then by Corollary 3.5, 3.7 and 3.12, G' is of type (II) or type (III). Comparing the numbers of elements of order 4 in G and G' (Corollary 3.4 and Lemma 3.8), we see G' is of type (II). Let $G=\boldsymbol{Z}_q\times \boldsymbol{D}'_{2^kn}$ and $G'=\boldsymbol{Z}_{q'}\times \boldsymbol{D}'_{2^{k'}n'}$. Then by Lemma 3.6, $k=k'$. From ii) in Corollary 3.7, $q=q'$. Hence, $n=n'$. These implies G is isomorphic to G'. Thus S^3/G is isometric to S^3/G'. Suppose G is of type (III). Then G' is also of type (III), by Corollary 3.5 and 3.12. By Corollary 3.4, we can see easily G is isomorphic to G'. Hence, by Theorem 3.2, S^3/G is isometric to S^3/G'. Finally, suppose G is of type (IV), (V) or (VI). Then by Corollary 3.12, we obtain G' is of the same type as G and isomorphic to G. By Theorem

3.1, this means S^3/G is isometric to S^3/G'. Thus the proof is completed. q.e.d.

5. The spectrum of a homogeneous spherical space form

In this section, we shall show a homogeneous spherical space form is completely characterized by its spectrum among all spherical space forms. First, we recall the classification of homogeneous spherical space forms due to J.A. Wolf.

Theorem 5.1 (See [10]). *Let G be a fixed point free finite subgroup of $SO(2n)$ $(n\geqq 2)$. Then the following conditions are equivalent.*

1. *S^{2n-1}/G is a riemannian homogeneous space.*
2. *For any $g\in G$, either $g=\pm 1_{2n}$ or there is a unimodular complex number λ such that half the eigenvalues of g are λ and the other half are $\bar{\lambda}$.*
3. *Either* (i) *G is cyclic of order $q\geqq 1$, and G is conjugate to the image of $\boldsymbol{Z}_q=\{g^t\}_{t=0}^{q-1}$ under the representation $\tau\oplus\cdots\oplus\tau$ (n-times) of $\boldsymbol{Z}_q$, where $\tau(g^t)=R(t/q)\subset SO(2)$, or* (ii) *$G$ is isomorphic to a binary dihedral or binary polyhedral group $\boldsymbol{P}^*$, n is even and G is conjugate to the image of $\boldsymbol{P}^*$ under the representation $\rho\oplus\cdots\oplus\rho$ $\left(\frac{n}{2}-times\right)$ of $\boldsymbol{P}^*$, where $\rho\colon \boldsymbol{P}^*\subset$(the group of unit quoternions)$=SU(2)\subset SO(4)$.*

Let S^{2n-1}/G be a homogeneous spherical space form. Then by 2 in Theorem 5.1, for any element g in G with $g\neq\pm 1_{2n}$, g has only two eigenvalues λ, $\bar{\lambda}$ with the same multiplicities n.

Lemma 5.2. *Let S^{2n-1}/G be a spherical space form and λ a unimodular complex number with $\lambda\neq\pm 1$. Then we have*

$$\lim_{z\to\lambda}(z-\lambda)^n F_G(z)=\frac{1}{|G|}\frac{1-\lambda^2}{(\lambda-\bar{\lambda})^n}N_\lambda(G)\,,$$

where $N_\lambda(G)$ denotes the number of elements in G of which eigenvalues are λ and $\bar{\lambda}$ with the same multiplicities n.

Proof. We have

$$\lim_{z\to\lambda}(z-\lambda)^n\frac{1-z^2}{(z-\lambda)^n(z-\bar{\lambda})^n}=\frac{1-\lambda^2}{(\lambda-\bar{\lambda})^n}\,.$$

If $g\in G$ with $g\neq\pm 1_{2n}$, then the multiplicities of eigenvalues of g are at most n. These imply the lemma. q.e.d.

Now, we give a proof of Theorem III.

Let S^{2n-1}/G, S^{2n-1}/G' be spherical space forms. Assume S^{2n-1}/G is homogeneous riemannian manifold and isospectral to S^{2n-1}/G'. By Theorem 5.1, we have

$$\sum_{\lambda} N_\lambda(G) = 2(|G|-2) \quad \text{if} \quad |G| \text{ is even,}$$
$$= 2(|G|-1) \quad \text{if} \quad |G| \text{ is odd},$$

where the summation runs through all unimodular complex numbers which are eigenvalues of some $g \in G$ with $g \neq \pm 1_{2n}$. By Lemma 5.2, we have

$$\frac{N_\lambda(G)}{|G|} = \frac{N_\lambda(G')}{|G'|}.$$

Since $|G| = |G'|$ by Corollary 2.4, we see

$$N_\lambda(G) = N_\lambda(G').$$

Hence, we have

$$\sum_{\lambda} N_\lambda(G') = 2(|G'|-2) \quad \text{if} \quad |G'| \text{ is even,}$$
$$= 2(|G'|-1) \quad \text{if} \quad |G'| \text{ is odd}.$$

This implies that any $g \in G$ with $g \neq \pm 1_{2n}$ has only two eigenvalues with the same multiplicities n. Therefore S^{2n-1}/G' is a riemannian homogeneous manifold by Theorem 5.1. By 3 in Theorem 5.1, G, G' are isomorphic to one of the groups $\boldsymbol{Z}_q$, $\boldsymbol{D}^*_{4n}$, $\boldsymbol{T}^*$, $\boldsymbol{O}^*$ or $\boldsymbol{I}^*$. On the other hand, we have seen before (Corollary 3.5 and 3.12), $\max \sigma(\boldsymbol{Z}_q) = |\boldsymbol{Z}_q| = q$, $\max \sigma(\boldsymbol{D}^*_{4n}) = \frac{|\boldsymbol{D}^*_{4n}|}{2} = 2n$, $\max \sigma(\boldsymbol{T}^*) = \frac{|\boldsymbol{T}^*|}{4} = 6$, $\max \sigma(\boldsymbol{O}^*) = \frac{|\boldsymbol{O}^*|}{6} = 8$ and $\max \sigma(\boldsymbol{I}^*) = \frac{|\boldsymbol{I}^*|}{12} = 10$. By the assumption, we have $\sigma(G) = \sigma(G')$, particularly $\max \sigma(G) = \max \sigma(G')$. From these facts, we see that G is isomorphic to G'. This completes the proof by 3 in Theorem 5.1. q.e.d.

Appendix. In the proof of the Main theorem in [11], Y. Yamamoto assumed that our isospectral problem for 3-dimensional lens spaces is equivalent to some number theoretical problem (Lemma in the below). For the completeness, we give a proof of it in this appendix.

Let $a(u: p, q)$ denote the number of lattice points $(x, y) \in Z^2$ such that (i) $|x| + |y| \leqq u$ (ii) $x + py \equiv 0 \pmod{q}$, where u, p, q are given integers with p prime to q. Let p' be another integer prime to q. Let $L(q: p)$ and $L(q: p')$ be 3-dimensional lens spaces defined in [4]. In [4] and [11], the author and Y. Yamamoto proved.

Theorem. *If the lens space $L(q: p)$ is isospectral to the lens space $L(q: p')$ then they are isometric.*

In [11], Y. Yamamoto assumed the following lemma to show the above theorem for any composite number q.

Lemma. *The spaces* $L(q:p)$, $L(q:p')$ *are isospectral if and only if* $a(u:p,q)=a(u:p,q')$ *for every positive integer* u.

Proof. We retain the notations in [4]. Let $G=\{g^k\}_{k=0}^{q-1}$ be the cyclic subgroup of $SO(4)$ defining the lens space $L(q:p)=S^3/G$, where $g=\begin{pmatrix}\gamma & 0\\ 0 & \gamma^p\end{pmatrix}\in U(2)\subset SO(4)$ and $\gamma=\exp 2\pi\sqrt{-1}/q$. Let $E_{k(k+2)}$ be the eigenspace of the Laplacian on $L(q:p)$ with the eigenvalue $k(k+2)$. Then the dimension of $E_{k(k+2)}$, $\dim E_{k(k+2)}$, is the dimension of the space consisting of g-invariant elements in H^k, because g is a generator of G. Let P_G^k be the space of G-invariant homogeneous polynomials of degree k on R^4. By (2.2) in [4], we have

$$\dim E_{k(k+2)}=\dim P_G^k-\dim P_G^{k-2}\,.$$

From (3.3) in [4], $\dim P_G^k$ is equal to the number of the set

$$I(k:p,q)=\left\{(i_1,i_2,j_1,j_2)\in Z^4:\begin{array}{l}\text{(i) } i_1+i_2p-j_1-j_2p\equiv 0 \pmod q\,,\\ \text{(ii) } i_1+i_2+j_1+j_2=k\end{array}\right\}.$$

We can define the injection map ψ of $I(k-2:p,q)$ into $I(k:p,q)$ by

$$\psi(i_1,i_2,j_1,j_2)=(i_1,i_2+1,j_1,j_2+1)\,.$$

By this map, we can see easily $\dim E_{k(k+2)}$ is equal to the number of the set

$$\left\{(i,j,y)\in Z^3:\begin{array}{l}\text{(i) } i-j+py\equiv 0 \pmod q,\\ \text{(ii) } i+j+|y|=k,\\ \text{(iii) } i\geqq 0,\ j\geqq 0\end{array}\right\}.$$

Put $x=i-j$ in the above. Then the number of the above set is equal to the number of the set

$$\left\{(j,x,y)\in Z^3:\begin{array}{l}\text{(i) } x+py\equiv 0 \pmod q,\\ \text{(ii) } 2j+x+|y|=k,\\ \text{(iii) } j+x\geqq 0,\ j\geqq 0\end{array}\right\}.$$

From this, we see easily the number of the above set is equal to the number of the set

$$J(k:p,q)=\left\{(x,y)\in Z^2:\begin{array}{l}\text{(i) } x+py\equiv 0 \pmod q,\\ \text{(ii) } x+y\equiv 2 \pmod k,\\ \text{(iii) } |x|+|y|\leqq k\end{array}\right\}.$$

It is clear that $J(k: p, q)=J(k: p', q)$ for every positive integer k if and only if $a(u: p, q)=a(u: p', q)$ for every positive integer u. This completes the proof of the lemma. q.e.d.

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