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OVERRINGS OF KRULL ORDERS

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Introduction. Recently, one of the authors introduced a Krull order R in a simple artinian ring Q[6], that is, R is called Krull if the following conditions hold:

(K1) $R=\cap_{i\in I}R_i\cap S(R)$, where R_i and $S(R)$ are essential overrings of R(see [6] for the definition), and $S(R)$ is the Asano overring of R;

(K2) each R_i is a noetherian, local, Asano order in Q, and $S(R)$ is a noetherian, simple ring;

(K3) if c is any regular element of R, then $cR_i\neq R_i$ for only finitely many i in I and $R_kc\neq R_k$ for only finitely many k in I.

The fundamental properties of Krull orders were studied in [6]. Let $\boldsymbol{P}$ be the set of all prime v-ideals of R and $\boldsymbol{P}_0$ any subset of $\boldsymbol{P}$. Then, in §1, we shall show that an order $T=\cap_{P\in\boldsymbol{P}_0}R_P\cap S(R)$ is also Krull and, in particular, T is an RI-order in the sense of Cozzens and Sandomierski [1], if we take $\boldsymbol{P}_0$ to be the set of all invertible prime ideals of R. In §2 we apply the results of §1 to the case where R is a D-order in a central simple algebra, where D is a unique factorization domain. §3 is devoted to state an example of a maximal order which has the noninvertible prime v-ideals.

1. Overrings of Krull orders.

Let R be an order in a simple artinian ring Q. A right R-submodule X of Q is called a *right R-ideal,* if $aR\supset X\supset bR$ for units a,b in Q. A *left R-ideal* and a *two-sided R-ideal* are defined by the similar way. An R-ideal in R is simply called an ideal. For a one-sided R-ideal X of R, put $O_r(X)=\{x\in Q;\ Xx\subset X\}$, $O_l(X)=\{x\in Q;\ xX\subset X\}$, $X^{-1}=\{x\in Q;\ XxX\subset X\}=\{x\in Q;\ Xx\subset O_l(X)\}=\{x\in Q;\ xX\subset O_r(X)\}$, and $X^*=X^{-1^{-1}}$. X is called a *v-ideal* (*invertible ideal*), if $X=X^*(R=XX^{-1}=X^{-1}X)$.

We state some results in [6] concerning Krull orders. Let R be a Krull order in a simple artinian ring Q. R is a maximal order [6, Proposition 2.1]. Let P_i' be a unique maximal ideal of R_i. Then $P_i=P_i'\cap R$ is a prime v-ideal of R(cf. [4, Lemma 1.5]), $P_i'=R_iP_i$ [3, Proposition 1.1], and $R_i=R_{P_i}$, where R_{P_i} is the localization of R at P_i, that is, $R_{P_i}=\{xy^{-1}\in Q;\ x\in R, y\in C(P_i)\}$ with $C(P_i)=\{y\in R; y+P_i$ is a regular element of $R/P_i\}$.

Let $\boldsymbol{P}=\{P_i; i\in I\}$ be the set of all prime v-ideals of R(*cf.* [4, Proposition

1.7]), $\boldsymbol{P}_0$ any subset of $\boldsymbol{P}$, and $T=\cap_{P\in P_0}R_P\cap S(R)$. In order to show that T is Krull, we prepare some definitions and lemmas. We sometimes drop the index i of $P_i\in\boldsymbol{P}$ for abbreviation.

Throughout this section, R is a Krull order in a simple artinian ring Q. Let M be a subset of Q. Then M is called a *right R-set,* if M is a right R-module and contains a regular element of Q. We put $\bar{M}_r=\cup I^*$, where I ranges over the set of all right R-ideals which are contained in M. M is called *closed* if $\bar{M}_r=M$. By the similar way we define a *left R-set* and a *closed left R-set*. Let A be a right v-ideal and B a left v-ideal. Then we define $A\circ B=(AB)^*$(cf. [5]).

Lemma 1.1. *The following statements hold for right R-sets M,N.*

(i) *$\bar{M}_r$ is a right R-set which contains M.*

(ii) *If $N\subset M$, then $\bar{N}_r\subset\bar{M}_r$.*

(iii) $\overline{(\bar{M}_r)}_r=\bar{M}_r$.

(iv) *If I is a right R-ideal, then* $\bar{I}_r=I^*$.

(v) $\overline{(\bar{M}_r\cap\bar{N}_r)}_r=\bar{M}_r\cap\bar{N}_r$.

(vi) *If M is closed, then* $M=\cap_{P\in\boldsymbol{P}}MR_P\cap MS(R)$.

In the following, we assume M,N to be closed and A,B v-ideals. Let $M\circ A=\cup X\circ A$, *where X ranges over the set of all right v-ideals which are contained in M.*

(vii) $M\circ A=\overline{(MA)}_r$.

(viii) $(M\circ A)\circ B=M\circ(A\circ B)$.

(ix) $M\circ(A\cap B)=M\circ A\cap M\circ B$.

Proof. (i) and (ii) are easily proved. (iii); It holds that $x\in\overline{(\bar{M}_r)}_r$ if and only if there is a right R-ideal $I\subset\bar{M}_r$ such that $x\in I^*$. Since we can take I to be finitely generated by [6, proposition 2.1], $I=a_1R+\cdots+a_kR$ with $a_i\in\bar{M}_r$. Thus, there are right R-ideals $I_i\subset M$ such that $a_i\in I_i^*(i=1,\cdots,k)$. We have $(I_1+\cdots+I_k)^*=(I_1^*+\cdots+I_k^*)^*\subset I^*$ by [5, Lemma 2] and $I_1+\cdots+I_k\subset M$. Hence $x\in\bar{M}_r$, that is, $\overline{(\bar{M}_r)}_r=\bar{M}_r$. (iv) is easily proved. (v); It holds that $\overline{(\bar{M}_r\cap\bar{N}_r)}_r\ni x$ if and only if there is a right R-ideal I such that $I\subset\bar{M}_r\cap\bar{N}_r$ and $x\in I^*$. We have $I^*=\bar{I}_r\subset\overline{(\bar{M}_r)}_r=\bar{M}_r$ and, similarly, $I^*\subset\bar{N}_r$. Hence $x\in I^*\subset\bar{M}_r\cap\bar{N}_r$. (vi); Let $x\in\cap_{P\in\boldsymbol{P}}MR_P\cap MS(R)$ be a regular element, Then we have $xc_P\in M$ for $c_p\in C(P)$ and $xB\subset M$ for a nonzero ideal B. If we put $X=\Sigma c_PR+B$, then $R=\cap XR_P\cap XS(R)\subset X^*$ by [6, Proposition 2.1 and 3, Corollary 4.2]. Thus $R=X^*$ and $x\in xX^*=(xX)^*=\overline{xX}_r\subset\bar{M}_r=M$. Hence $\cap MR_P\cap MS(R)=M$ by [3, Lemma 2.2]. (vii); Let X be a right v-ideal in M. Then $X\circ A=(XA)^*=\overline{(XA)}_r\subset\overline{(MA)}_r$. Conversely, if $x\in\overline{(MA)}_r$, then $x\in I^*$ for a finitely generated right R-ideal $I\subset MA$. By the same way as (iii) we have $I^*\subset X\circ A$ for a right v-ideal X in M. Hence $x\in M\circ A$. (viii); It holds that $(X\circ A)\circ B=((XA)B)*=(X(AB))^*=X\circ(AB)^*=X\circ(A\circ B)$ for a right v-ideal X. Hence $(M\circ A)\circ B$

$=M\circ(A\circ B)$. (ix); By [5, Lemma 3] the lattice anti-isomorphism between the set of right v-ideals and one of left v-ideals yields the proof of (ix).

Let $\boldsymbol{P}_0$ be an arbitrary subset of $\boldsymbol{P}$, $T=\cap_{P\in\boldsymbol{P}_0}R_P\cap S(R)$, $F=\{X\subset R;\ X$ is a right R-ideal such that $XR_P=R_P$ for every $P\in\boldsymbol{P}_0$ and $XS(R)=S(R)\}$, and $F_l=\{Y\subset R;\ Y$ is a left R-ideal such that $R_PY=R_P$ for every $P\in\boldsymbol{P}_0$ and $S(R)Y=S(R)\}$. In the following, the notation provided above will be preserved.

Lemma 1.2. *F is a right additive topology on R and $R_F=T$. Similarly, F_l is a left additive topology on R and $T=R_{F_l}$.*

Proof. To show that F is a right additive topology, we shall prove the followings [10]:

(i) If $X\in F$ and $r\in R$, then $r^{-1}X=\{x\in R;\ rx\in X\}\in F$.

(ii) If Y is a right ideal and $X\in F$ such that $x^{-1}Y\in F$ for all $x\in X$, then $Y\in F$.

(i); Considering the fact $XR_P=R_P\Leftrightarrow C(P)\cap X\neq\phi$ and $XS(R)=S(R)\Leftrightarrow X$ contains a nonzero ideal we can show $r^{-1}X\in F$ by the right Ore condition of $C(P)$. (ii); We have $R_P\supset YR_P\supset\Sigma_{x\in X}x(x^{-1}Y)R_P=\Sigma_{x\in X}xR_P=XR_P=R_P$, and then $R_P=YR_P$. Similarly, $S(R)=YS(R)$ which implies $Y\in F$. If $X\in F$, then $X^{-1}\subset X^{-1}T\subset\cap X^{-1}R_P\cap X^{-1}S(R)=\cap R_P\cap S(R)=T$. Thus $R_F\subset T$. Conversely, if $t\in T$, then there are $c_P\in C(P)$ and a nonzero ideal B such that $tc_P\in R$ and $tB\subset R$. Putting $X=\Sigma c_PR+B$ we have that $X\in F$ and $tX\subset R$. Hence $t\in R_F$. This ccompletes the proof.

Lemma 1.3. *We have $\bar{T}_r=T=\bar{T}_l$, $\overline{S(R)}_r=S(R)=\overline{S(R)}_l$, and $(\bar{R}_P)_r=R_P=(\bar{R}_P)_l$.*

Proof. In general, if G is a right additive topology which consists of a family of essential right ideals, then $(\bar{R}_G)_l=R_G$. For, it holds that $x\in(\bar{R}_G)_l$ if and only if $x\in I^*$ for a finitely generated left R-ideal $I\subset R_G$. Then we can take $X\in G$ such that $I\subset X^{-1}$ by the same way as the proof of Lemma 1.1 (iii). Hence $x\in I^*\subset(X^{-1})^*=X^{-1}$. This completes the proof.

Lemma 1.4. *If $X\in F$ and $Y\in F_l$, then $Y^{-1}\circ X^{-1}\subset T$.*

Proof. We can assume X to be a right v-ideal and Y a left v-ideal. Then $Y^{-1}\circ X^{-1}=(X\circ Y)^{-1}=(XY)^{-1}$ by [5, Lemma 3]. If $c\in(XY)^{-1}$, then $cX\subset Y^{-1}\subset T=R_F$. Thus $c\in\mathrm{Hom}_R(X,R_F)$ which implies $c\in R_F$ by [10, Proposition 7.8].

Let $I_F=\cup I\circ X^{-1}$, where X ranges over F, for a right v-ideal I.

Lemma 1.5. *The following statements hold for a right v-ideal I.*

(i) *I_F is a right T-ideal and a v-ideal as a right T-ideal.*

(ii) *If $I\subset R$, then $I_F=T$ if and only if $I\in F$.*

(iii) *If* $X \in F$, *then* $(X^{-1})_{F_l} = T$.

Proof. (i); If $X, Y \in F$, then $I \circ X^{-1} + I \circ Y^{-1} \subset (I \circ X^{-1} + I \circ Y^{-1})^* = I \circ (X^{-1} + Y^{-1})^* = I \circ (X \cap Y)^{-1}$ by [5, Lemma 3]. Thus $I \circ X^{-1} + I \circ Y^{-1} \subset I_F$. Let $t \in T$, $Y \in F$ such that $t \in Y^{-1}$, $x \in I \circ X^{-1}$ for $X \in F$, and $Z = \{r \in R; tr \in X\} \in F$. Then we have $xX \subset I$ and $xtZ \subset xX \subset I$, which implies $(xtS + I \circ Z^{-1})Z \subset I$ and $xtS + I \circ Z^{-1}$ to be a right S-ideal with $S = O_l(Z)$. Since we can take X to be a right v-ideal, Z is also a right v-ideal by [5, Lemma 3]. Thus $xt \in (xtS + I \circ Z^{-1}) \subset (xtS + I \circ Z^{-1})S \subset (xtS + I \circ Z^{-1})^* \circ Z \circ Z^{-1} = ((xtS + I \circ Z^{-1})Z)^* \circ Z^{-1} \subset I \circ Z^{-1} \subset I_F$. Hence I_F is a right T-ideal. To prove that I_F is a v-ideal as a right T-ideal, it suffices to show $(I_F)^{-1} = (I^{-1})_{F_l}$. Then we have $(I_F)^{-1-1} = (I_{F_l}^{-1})^{-1} = (I^{-1-1})_F = I_F$. Now, let x be a regular element in $(I_F)^{-1}$ and $I = (a_1R + \cdots + a_nR)^*$. Then there exists $Y \in F_l$ with $xa_i \in Y^{-1}$, since $xI \subset T$. We have $xI = x(a_1R + \cdots + a_nR)^* = (xa_1R + \cdots + xa_nR)^* \subset (Y^{-1})^* = Y^{-1}$, and then $x \in x(I \circ I^{-1}) = (xI) \circ I^{-1} \subset Y^{-1} \circ I^{-1} \subset (I^{-1})_{F_l}$. Hence $(I_F)^{-1} \subset (I^{-1})_{F_l}$ by [3, Lemma 2.2]. Conversely, let y be a regular element in $(I^{-1})_{F_l}$. Then there exists $Y \in F_l$ with $y \in Y^{-1} \circ I^{-1}$. For any $X \in F$, we have $y(I \circ X^{-1}) = yI \circ X^{-1} \subset Y^{-1} \circ X^{-1} \subset T$ by Lemma 1.4. Thus $yI_F \subset T$, that is, $y \in (I_F)^{-1}$. Hence again by [3, Lemma 2.2] $(I^{-1})_{F_l} \subset (I_F)^{-1}$. This completes the proof. (ii); If $I \subset R$, then $I \circ X^{-1} \subset R \circ X^{-1} = X^{-1} \subset T$ for any $X \in F$, that is, I_F is an integral right T-ideal. It holds that $I_F = T \Leftrightarrow 1 \in I_F \Leftrightarrow 1 \in I \circ X^{-1}$ for some $X \in F \Leftrightarrow X \subset I \Leftrightarrow I \in F$. (iii); We have $(X^{-1})_{F_l} = (X^{-1})_{F_l}^{-1-1} = (X^{-1-1})_F^{-1} = T$ by (i), (ii), and $X^{-1-1} \in F$.

Lemma 1.6. *If* $P \in \boldsymbol{P}$ *and* A *is a nonzero ideal of* R, *then* $AR_P = (\overline{AR_P})_l = (\overline{R_PA})_r = R_PA$.

Proof. If $AR_P = R_P$, then $A \cap C(P) \neq \phi$ which implies $R_PA = R_P = AR_P$. Thus we assume $R_PA \neq R_P$, that is, $A \subset P$. If $AP^{-m}R_P \neq R_P$ for every integer $m \geq 1$, then $AP^{-m} \subset P$. For $AP^{-k} \subset P$ implies, $AP^{-(k+1)} \subset PP^{-1} \subset R$. If $AP^{-(k+1)} \not\subset P$, then $AP^{-(k+1)} \cap C(P) \neq \phi$, that is, $AP^{-(k+1)}R_P = R_P$ which is a contradiction. Thus $AP^{-(k+1)} \subset P$ and the assertion holds by induction. There is an increasing chain of proper right ideals of R_P;

$$AR_P \subset AP^{-1}R_P \subset AP^{-2}R_P \subset \cdots \subset AP^{-m}R_P \subset \cdots$$

which must stabilize since R_P is noetherian. Therefore there is an integer n such that $AR_PP^{-n} = AR_PP^{-(n+1)}$. We have $R_PAR_P = R_PAR_PP^{-1}$ and $R_PP^{-1} \subset O_r(R_PAR_P) = R_P$ which is a contradiction. Thus $AP^{-m}R_P = R_P$ for some integer $m \geq 1$. Let m be the smallest such integer. We conclude that $AP^{-(m-1)} \subset P$ and then $AP^{-m} \subset R$ which implies $AP^{-m} \cap C(P) \neq \phi$. Thus $AP^{-m}R_P = R_P = R_PAP^{-m}$, and then $R_PA \supset R_PAP^{-m}P^m = AP^{-m}R_PP^m = AR_P$. By the similar way we have $AR_P \supset R_PA$. Hence $AR_P = R_PA$. Now, it suffices to show $AR_P = (\overline{AR_P})_l$. Let a be a regular element in A such that $AR_P = aR_P = R_Pa$ and $x \in (\overline{AR_P})_l$. Then there exists $J=Ra_1+\cdots+Ra_n\subset AR_P$ with $x\in J^*$, Put $a_i=r_ic^{-1}a$ for $c\in C(P)$ $(i=1,\cdots,n)$. We have $x\in J^*=(\Sigma_{i=1}^n Rr_ic^{-1}a)^*=(\Sigma Rr_i)^*c^{-1}a\subset Rc^{-1}a\subset R_Pa=AR_P$. Hence $(\overline{AR_P})_l\subset AR_P$. This completes the proof.

Lemma 1.7. *Let A be an R-ideal in R. Then $A^*=\cap_{P\in\boldsymbol{P}}AR_P\cap S(R)$.*

Proof. Since $(A^{-1}A)^*=R$, we have $A^*A^{-1}\not\subset P$ and $A^{-1}A\not\subset P$, that is, $R_P=R_PA^*A^{-1}=A^{-1}AR_P$ for all $P\in\boldsymbol{P}$. Thus $A^*R_P=A^*(A^{-1}A)R_P=R_P(A^*A^{-1})A=R_PA=AR_P$ by Lemma 1.6 and $A^*S(R)=AS(R)=S(R)$. Hence we have $A^*=\cap_{P\in\boldsymbol{P}}A^*R_P\cap A^*S(R)=\cap_{P\in\boldsymbol{P}}AR_P\cap AS(R)$ by [3, Corollary 4.2].

Let $P\in\boldsymbol{P}_0$, P' the corresponding unique maximal ideal of R_P, and $P''=P'\cap T$ the minimal prime v-ideal of T. Then $P''R_P=P'=PR_P$, since T is Krull in the sense of [4]. It is noted that the same proof as Lemma 1.7 yields $B^*=\cap_{P\in\boldsymbol{P}_0}BR_P\cap S(R)$ for every ideal B of T. We shall write $I_F=I\circ R_F=I\circ T$ for a right v-ideal I.

Lemma 1.8. *We have $(P^n)^*\circ T=T\circ(P^n)^*=(P''^n)^*=P^nR_P\cap T$ for every $P\in\boldsymbol{P}_0$ and every natural number n.*

Proof. Since $(P^n)^*\circ T$ is a right v-ideal by Lemma 1.5, we have $(P^n)^*\circ T=\cap_{P_i\in\boldsymbol{P}_0}((P^n)^*\circ T)R_{P_i}\cap S(R)\supset\cap_{P_i\in\boldsymbol{P}_0}P^nR_{P_i}\cap S(R)=P^nR_P\cap T=P''^nR_P\cap T=(P''^n)^*$. On the other hand, $(P^n)^*\circ T=(\overline{(P^n)^*T})_l\subset(P^n)^*R_P\cap T=P^nR_P\cap T$ by Lemmas 1.1(ii), 1.6, and 1.7. Hence $(P^n)^*\circ T=P^nR_P\cap T=(P''^n)^*$. By the similar way $T\cap P^nR_P=T\circ(P^n)^*$. This completes the proof.

Lemma 1.9. (i) *If A is a v-ideal of R, then $A\circ T=T\circ A$ is a v-ideal of T.* (ii) *If A'' is a v-ideal of T, then $A=A''\cap R$ is a v-ideal of R and $A''=A\circ T$.*

Proof. (i); Let $A=(P_1^{n_1}\cdots P_k^{n_k})^*$ with $P_i\in\boldsymbol{P}_0(i=1,\cdots,l)$ and $P_j\in\boldsymbol{P}-\boldsymbol{P}_0$ $(j=l+1,\cdots,k)$. Since $A\circ T$ is a right v-ideal, $A\circ T=\cap_{P_j\in\boldsymbol{P}_0}(A\circ T)R_{P_j}\cap S(R)\supset\cap_{P_j\in\boldsymbol{P}_0}AR_{P_j}\cap S(R)=\cap_{i=1}^{l}P_i^{n_i}R_{P_i}\cap T=(P_1''^{n_1})^*\cap\cdots\cap(P_l''^{n_l})^*$. On the other hand, $A\circ T=(\overline{AT})_l\subset\cap_{P\in\boldsymbol{P}_0}AR_P\cap S(R)=\cap_{i=1}^{l}P_i^{n_i}R_{P_i}\cap T=(P_1''^{n_1})^*\cap\cap\cdots(P_l''^{n_l})^*$. Thus $A\circ T=(P_1''^{n_1})^*\cap\cdots\cap(P_l''^{n_l})^*=T\circ A$ by the similar way. (ii); It holds that $\{P''; P\in\boldsymbol{P}_0\}$ is the set of all prime v-ideals of T(cf. [4, Proposition 1.7]). Thus we have $A''=(P_1''^{n_1})^*\cap\cdots\cap(P_k''^{n_k})^*=P_1^{n_1}R_{P_1}\cap T\cap\cdots\cap P_k^{n_k}R_{P_k}\cap T$ by Lemma 1.8. Hence $A=(P_1^{n_1}R_{P_1}\cap R)\cap\cdots\cap(P_k^{n_k}R_{P_k}\cap R)=(P_1^{n_1})^*\cap\cdots\cap(P_k^{n_k})^*$ is a v-ideal of R. Moreover, we have $A\circ T=((P_1^{n_1})^*\cap\cdots\cap(P_k^{n_k})^*)\circ T=(P_1^{n_1})^*\circ T\cap\cdots\cap(P_k^{n_k})^*\circ T=(P_1''^{n_1})^*\cap\cdots\cap(P_k''^{n_k})^*=A''$ by Lemmas 1.1 (ix) and 1.8. This completes the proof.

Theorem 1.10. *T is a Krull order in Q.*

Proof. In order to prove T to be Krull, it suffices to show $S(R)=S(T)$, because T is Krull in the sense of [4] (cf. [4, Proposition 1.2]). Let A be a

v-ideal of R. Then $A''=A\circ T$ is also a v-ideal of T and $A^{-1}\subset T\circ A^{-1}=(A\circ T)^{-1}\subset S(T)$ by the proof of Lemma 1.5 (i). Thus $S(R)\subset S(T)$. Conversely, let A'' be a v-ideal of T and $A=A''\cap R$ a v-ideal of R. Then $(A'')^{-1}=T\circ A^{-1}=(\overline{TA^{-1}})_r\subset(\overline{S(R)})_r=S(R)$. Hence $S(R)=S(T)$ and thus T is Krull.

Corollary 1.11. *If we choose $\boldsymbol{P}_0$ such that $\boldsymbol{P}-\boldsymbol{P}_0$ is a finite set, then $R=T\cap T'$, where $T=\cap_{P\in\boldsymbol{P}_0}R_P\cap S(R)$ is a Krull order and $T'=\cap_{P\in\boldsymbol{P}-\boldsymbol{P}_0}R_P$ is a bounded Dedekind prime ring, and is a right and left principal ideal ring.*

Proof. This follows from Theorem 1.10 and [3, Lemma 3.3].

Now, we specially choose $\boldsymbol{P}_0$ to be the set of all invertible prime ideals. Then T is an RI-order in the sense of Cozzens and Sandomierski [1], here an order in a simple artinian ring is said to be an *RI-order,* its two-sided v-ideals form a group under the ordinary multiplication.

Theorem 1.12. *If $\boldsymbol{P}_0$ is the set of all invertible prime ideals of R, then $T=\cap_{P\in\boldsymbol{P}_0}R_P\cap S(R)$ is an RI-order.*

Proof. It holds that $\{P''; P''=P'\cap T$ and P' is a unique maximal ideal of R_P, $P\in\boldsymbol{P}_0\}$ is the set of all minimal prime ideals of T. Thus by [1, Proposition 2.4] we only show that each P'' is invertible. Let $x\in P''$ and $X\in F$ with $x\in X^{-1}$. Then $xX\subset R\cap P'=P$ and $xR_P=xXR_P\subset PR_P$. We have $x\in PR_P\cap_{P_i\in\boldsymbol{P}_0}R_{P_i}\cap S(R)=\cap_{P_i\in\boldsymbol{P}_0}PR_{P_i}\cap S(R)=PT$ by the invertibility of P and [6, Proposition 2.1]. Thus $P''\subset PT$. The converse inclusion is clear, so that $P''=PT$. By the same way we have $P''=TP$. Hence P'' is invertible.

We shall give the structure of the integral v-ideals of R.

Theorem 1.13. *Let R be a Krull order in Q and A a v-ideal of R. Then $A=P_1^{n_1}\cdots P_k^{n_k}B=BP_1^{n_1}\cdots P_k^{n_k}$, where each P_i is invertible $(i=1,\cdots,k)$ and B is a v-ideal such that $B\not\subset P$ for every invertible prime ideal P.*

Proof. Let $A=(P_1^{n_1}\cdots P_k^{n_k}P_{k+1}^{n_{k+1}}\cdots P_l^{n_l})^*$, where each P_i is an invertible prime ideal $(i=1,\cdots,k)$ and each P_j is a noninvertible prime v-ideal $(j=k+1,\cdots,l)$. Then by [1, Lemma 2.1] we have that $(P_1^{n_1}\cdots P_k^{n_k})^{-1}A=((P_1^{n_1}\cdots P_k^{n_k})^{-1}A)^*=((P_1^{n_1}\cdots P_k^{n_k})^{-1}(P_1^{n_1}\cdots P_l^{n_l}))^*=(P_{k+1}^{n_{k+1}}\cdots P_l^{n_l})^*=B$ is a v-ideal. It is clear that B satisfies the condition of the theorem. Hence $A=P_1^{n_1}\cdots P_k^{n_k}B$. Starting with $A=(P_{k+1}^{n_{k+1}}\cdots P_l^{n_l}P_1^{n_1}\cdots P_k^{n_k})^*$ we get $A=BP_1^{n_1}\cdots P_k^{n_k}$. This completes the proof.

2. Maximal orders over unique factorization domains

Throughout this section, let R be a unique factorization domain and Λ a maximal R-order in the sense of Fossum [2], that is, Λ satisfies the following conditions;

(i) $R \subset \Lambda$.

(ii) $K\Lambda = \Sigma$, where K is the quotient field of R and Σ is a central simple K-algebra.

(iii) Each element of Λ is integral over R.

Let $\mathcal{P}$ be the set of all minimal prime ideals of R, $\mathcal{P}_1 = \{p \in \mathcal{P}$; the minimal prime ideal P of Λ with $P \cap R = p$ is invertible in $\Lambda\}$, $\mathcal{P}_2 = \mathcal{P} - \mathcal{P}_1$, $\Lambda_1 = \cap_{p \in \mathcal{P}_1} \Lambda_p$, and $\Lambda_2 = \cap_{q \in \mathcal{P}_2} \Lambda_q$. It is well-known that there is a unique minimal prime ideal P of Λ with $p = P \cap R$ for every $p \in \mathcal{P}$. Let δ be the different of Λ, that is, $\delta = C(\Lambda)^{-1}$, where $C(\Lambda)$ is the complementary ideal, $C(\Lambda) = \{x \in \Sigma; tr(x\Lambda) \subset R\}$, in which tr denotes the usual trace function in a simple algebra [8]. Let $P_1, \cdots, P_n$ be all minimal primes of Λ which contain δ. We shall classify the minimal primes of Λ with respect to the invertibility.

Lemma 2.1. *If P is a minimal prime of Λ with $P \neq P_i (i = 1, \cdots, n)$, then $P = p\Lambda (p = P \cap R)$ is invertible. If P is one of P_i's, then;*

(i) *There is an integer $t > 1$ such that $P^t = p\Lambda (p = P \cap R) \Leftrightarrow P$ is invertible.*

(ii) *$P^t \neq p\Lambda (p = P \cap R)$ for any integer $t \Leftrightarrow P$ is not invertible.*

Proof. By [9, §5] we only prove the second statement. (i); $\Rightarrow$: Since $p\Lambda$ is invertible by hypothesis, P is, too. $\Leftarrow$: There is an integer $t > 1$ such that $P_p^t = p\Lambda_p$. If $q \in \mathcal{P}$ with $q \neq p$, then $P_q^t = p\Lambda_q = \Lambda_q$. Thus $P^t = \cap_{q \in \mathcal{P}} P_q^t \cap_{q \in \mathcal{P}} p\Lambda_q = p\Lambda$, since P^t and $p\Lambda$ are v-ideals. Now, (ii) holds at once.

Theorem 2.2. *Let Λ, Λ_1, and Λ_2 are the same as in the first paragraph of this section. Then $\Lambda = \Lambda_1 \cap \Lambda_2$, where Λ_1 is an RI-order and Λ_2 is a bounded Dedekind prime ring, and is a right and left principal ideal ring.*

Proof. This follows from Lemma 2.1, Corollary 1.11, and Theorem 1.12.

Applying Theorem 1.13 to a v-ideal of Λ we get the following.

Proposition 2.3. *If A is a v-dieal of Λ, then $A\Lambda_1$ is also a v-ideal of Λ_1.*

Proof. Let $A = P_1^{n_1} \cdots P_t^{n_t} B$, where P_i's are minimal primes with $P_i \cap R \in \mathcal{P}_1$ and B is a v-ideal of Λ such that $B \not\subset P$ for every minimal prime P with $P \cap R \in \mathcal{P}_1$. Let Q be a minimal prime ideal of Λ with $q = Q \cap R \in \mathcal{P}_2$ and $R_1 = \cap_{p \in \mathcal{P}_1} R_p$. Then $q = zR$ for $z \in R$, since R is a unique factorization domain. We have $qR_1 = zR_1 = \cap_{p \in \mathcal{P}_1} zR_p = R_1$, and then $q\Lambda_1 = \Lambda_1$ which implies $Q\Lambda_1 = \Lambda_1$. Therefore, $B\Lambda_1 = \Lambda_1$. Hence $A\Lambda_1 = P_1^{n_1} \cdots P_t^{n_t} \Lambda_1$ is a v-ideal of Λ_1 (in fact, an invertible ideal).

3. Example. In this section, we study an example of a maximal R-order which has a noninvertible prime v-ideal.

REMARK. It was shown in [1, §2] that an arbitrary maximal R-order, where R is a noetherian integrally closed domain, is an RI-order. However, the following example turns out to be the counter example of this statement.

Now, our example is originated by Ramras [7]. Let k be a perfect field of characteristic $\neq 2$, $R=k[[X, Y]]$ with X and Y transcendental over k, and $K=k((X, Y))$. Let Σ be the quaternion algebra $K[1, \alpha, \beta, \alpha\beta]$ with $\alpha^2=X$, $\beta^2=Y(Y-X)(Y+X)$, and $\beta\alpha=-\alpha\beta$. Then the R-free order $R[1, \alpha, \beta, \alpha\beta]=\Lambda$ is a maximal R-order in Σ by [7]. We shall compute the different δ of Λ, and for the reader's convernience we state the process of it. Let $L=K(\beta)$ be the cyclic extension of K and $\Sigma=L\oplus L\alpha$. We put $\bar{a}=k-\beta l$ for any $a=k+\beta l\in L(k,l\in K)$. L is the splitting field of Σ and the ring isomorphism $L\otimes_K\Sigma\simeq M_2(L)$ is given by $1\otimes a\to\begin{pmatrix} a & 0 \\ 0 & \bar{a}\end{pmatrix}$ and $1\oplus\alpha b\to\begin{pmatrix} 0 & X\bar{b} \\ b & 0\end{pmatrix}$ for $a, b\in L$ (cf. [8, Example 9.4]). Thus $tr(x)=a+\bar{a}$ for $x=a+\alpha b\in\Sigma$. Let $x, y\in\Sigma$ and $x=k_1+\beta l_1+\alpha(k_2+\beta l_2)$, $y=m_1+\beta n_1+\alpha(m_2+\beta n_2)$ $(k_i, l_i, m_i, n_i\in K)$. Then:

$$\begin{aligned} xy = {} & k_1m_1+Xk_2m_2+\psi l_1n_1-X\psi l_2n_2+\beta(k_1n_1+Xk_2n_2+l_1m_1-Xl_2m_2) \\ & +\alpha(k_1m_2+k_2m_1-\psi l_1n_2+\psi l_2n_1+\beta(k_1n_2-l_1m_3+k_2n_1+l_2m_1)), \end{aligned}$$

where $\psi=Y(Y-X)(Y+X)$. Now, we get $tr(xy)=2(k_1m_1+Xk_2m_2+\psi l_1n_1-X\psi l_2n_2)$, $C(\Lambda)=\{x=k_1+\beta l_1+\alpha(k_2+\beta l_2)\in\Sigma;\ k_1\in R,\ k_2\in(1/X)R,\ l_1\in(1/\psi)R,\ l_2\in(1/X\psi)R\}$, and then $\delta=C(\Lambda)^{-1}=\{y=m_1+\beta n_1+\alpha(m_2+\beta n_2)\in\Lambda;\ m_1\in(\psi X),\ m_2\in(\psi),\ n_1\in(X),\ n_2\in R\}$. Let $P_0=\{y=m_1+\beta n_1+\alpha(m_2+\beta n_2)\in\Lambda;\ m_1, n_1\in(X),\ m_2, n_2\in R\}=\alpha\Lambda$, $P_i=\{y=m_1+\beta n_1+\alpha(m_2+\beta n_2)\in\Lambda;\ m_1, m_2\in(\psi_i),\ n_1, n_2\in R\}$, where $\psi_1=Y$, $\psi_2=Y-X$, $\psi_3=Y+X (i=1, 2, 3)$. Thus $P_i\supset\delta(i=0, 1, 2, 3)$, P_0 is invertible, each P_i is a v-ideal, since $P_i^{-1}=\{x=k_1+\beta l_1+\alpha(k_2+\beta l_2)\in\Sigma;\ l_1, l_2\in(1/\psi_i)R,\ k_1, k_2\in R\}$ $(i=1, 2, 3)$. Since $\alpha\Lambda_{(X)}=\mathrm{Rad}\,\Lambda_{(X)}$ and $\beta\Lambda_{(\psi_i)}=\mathrm{Rad}\,\Lambda_{(\psi_i)}$, each $P_i(i=0, 1, 2, 3)$ is a prime ideal by the equations $\alpha\Lambda_{(X)}\cap\Lambda=P_0$, $\beta\Lambda_{(\psi_i)}\cap\Lambda=P_i(i=1, 2, 3)$. If $P_i(i=1, 2, 3)$ is invertible, then there exist $k_{sj}, l_{sj}, m_{sj}, n_{sj}\in R(s=1, 2; j=1, \cdots, t)$ such that $\psi_i\Sigma_j k_{1j}m_{1j}+X\psi_i\Sigma_j k_{2j}n_{2j}+(\psi/\psi_i)\Sigma_j l_{1j}n_{1j}-X(\psi/\psi_i)\Sigma_j l_{2j}n_{2j}=1$. However, the left hand side of this equation is contained in (X, Y) which is a contradiction. Thus each P_i is not invertible $(i=1, 2, 3)$. It holds that $P_0^2=X\Lambda$, $P_i=\beta\Lambda+\psi_i\Lambda$, and no power of P_i equals $\psi_i\Lambda(i=1, 2, 3)$. If P is a minimal prime ideal which contains δ, then $P\cap R\supset(X\psi)$. Thus $P\cap R=(X)$ or (ψ_i) which implies that P equals one of P_i's$(i=0, 1, 2, 3)$.

Summarizing the above results we get the followings. Λ is a maximal R-order which has the prime v-ideals $P_i(i=0, 1, 2, 3)$. P_0 is invertible, $P_0\not\supseteq X\Lambda$, and $P_0^2=X\Lambda$ and P_i is noninvertible, $P_i^t\neq\psi_i\Lambda$ for any positive integer t, and $P_i=\beta\Lambda+\psi_i\Lambda(i=1, 2, 3)$. Every prime v-ideal $P\neq P_i(i=0, 1, 2, 3)$ is invertible and $P=p\Lambda(p=P\cap R)$.

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