Ōtsubo, S.
Osaka J. Math.
16 (1979), 667-673

ORDER OF LIOUVILLIAN ELEMENTS SATISFYING AN ALGEBRAIC DIFFERENTIAL EQUATION OF THE FIRST ORDER

SHŪJI ŌTSUBO

(Received September 25, 1978)

0. Introduction. Let k be an algebraically closed ordinary differential field of characteristic 0, and Ω be a universal extension of k. A finite chain of extending differential subfields $k=L_0\subset L_1\subset\cdots\subset L_n$ in Ω is called a Liouville chain over k if the following two conditions are satisfied:

(i) The field of constants of L_n is k_0, where k_0 is the field of constants of k;

(ii) For each $i(1\leqq i\leqq n)$ there exists a finite system of elements $w_1, w_2, \cdots, w_r$ of L_i which satisfies the following two conditions; either $w'_j\in L_{i-1}$ or w'_j/w_j is the derivative of an element of L_{i-1} for each j $(1\leqq j\leqq r)$, L_i is an algebraic extension of $L_{i-1}(w_1, w_2, \cdots, w_r)$ of finite degree.

Let z be an elemen of Ω. Then, z is called a liouvillian element over k if there exists a Liouville chain over k such that its end contains z. The following definition is due to Liouville [2] (cf. [8, p. 111]):

DEFINITION. A liouvillian element z over k is said to be of order m if m is the minimum of those n such that the end of a Liouville chain $L_0\subset\cdots\subset L_n$ over k contains z.

Let F be an algebraically irreducible element of the first order of the differential polynomial algebra $k\{u\}$ in a single indeterminate u over k. Suppose that z is a solution of $F=0$. Then, z is a generic point of the general solution of $F=0$ over k if and only if z is transcendental over k. Suppose that two liouvillian elements over k satisfy $F=0$ and that they are transcendental over k. Then, their orders are the same.

Theorem. *The order of a liouvillian element over k satisfying $F=0$ is at most three.*

For example, suppose that k is the algebraic closure of $k_0(x)$ with $x'=1$ and that $F=u'-\alpha u/x$, where $\alpha\in k_0$. Then, any non-trivial solution of $F=0$ is of the second order if α is not a rational number (cf. Liouville [2, pp. 94–98]).

REMARK 1. If we replace "liouvillian" by "generalized elementary" and modify the definition of "order" to fit the replacement, then a similar result to our theorem can be derived from a theorem of Singer (cf. [7], [6, Theorem 1]).

In order to prove our theorem we shall prepare several lemmas: Suppose that y is a generic point of the general solution of $F=0$ over k. Then, $k(y, y')$ is a one-dimensional algebraic function field over k with $F(y, y')=0$. The following lemma is due to the author [3]:

Lemma 1. *Suppose that $\nu_P(\tau')\leqq 0$ for every prime divisor P of $k(y, y')$, where ν_P is the normalized valuation belonging to P and τ is a prime element in P. Then, the order of any liouvillian element over k satisfying $F=0$ is 0.*

Let k^* be a differential subfield of Ω containing k such that k^* is finitely generated over k and the field of constants of k^* is k_0, and η be a generic point of the general solution of $F=0$ over k^*:

Lemma 2. *Suppose that there exists a liouvillian element z over k satisfying $F=0$ whose order is not 0. Then, we have such k^* that z is algebraic over k^* and that*

(1) $k^*(\eta, \eta')$ *contains a transcendental constant over* k^*.

Lemma 3. *Suppose that the condition* (1) *is satisfied by some k^* and that $\nu_P(\tau')>0$ for some prime divisor P of $k(y, y')$. Then, there exists in $k(\eta, \eta')$ a transcendental element ϕ over k such that $\phi'=a\phi+b$, where $a, b\in k$.*

Lemmas 2, 3 and Theorem will be proved in the sections 1, 3 and 4 respectively. In the section 2 we shall show the following:

Proposition. *Suppose that some k^* has the property* (1). *Then, in the algebraic closure of k^* there exists a liouvillian extension $k^\sharp$ of k such that $k^\sharp(\eta, \eta')$ has a transcendental constant over $k^\sharp$, if and only if $\nu_P(\tau')>0$ for some prime divisor P of $k(y, y')$.*

Remark 2. Suppose that k is the algebraic closure of $k_0(x)$ with $x'=1$ and that

$$F = u'-\alpha u/x-1/(1+x)\,, \qquad \alpha\in k_0\,.$$

Then, any solution of $F=0$ is of the third order if α is not a rational number. This remark is due to M. Matsuda.

Remark 3. The following theorem due to Rosenlicht [5] can be derived from Lemma 3: Assume that $k=k_0$ and $F=u'-f(u)$, where $f\in k(u)$. Then, the condition (1) is satisfied by some k^* if and only if we are in one of the following three cases: $f=0$, $1/f=\partial g/\partial u$, $1/f=c(\partial g/\partial u)/g$ with $g\in k(u)$ and $c\in k$.

The author wishes to express his sincere gratitude to Dr. M. Matsuda for his kind advices and to Mr. K. Nishioka for fruitful discussions with him.

1. Proof of Lemma 2. There exist in Ω an element t and a differential subfield k_1 containing k which satisfy the following conditions: k_1 is finitely generated over k; either t' or $t'/t \in k_1$; the field of constants of $k_1(t)$ is k_0; z is transcendental over k_1 and algebraic over $k_1(t)$. Let us set $k^* = k_1(t)$. Then, η is a generic differential specialization of z over k_1. Hence, there exists an element u of Ω such that (η, u) is a generic differential specialization of (z, t) over k_1. We have either $u' \in k_1$ or $u'/u \in k_1$, and η is algebraic over $k^*(u)$. Since η is transcendental over k^*, u is transcendental over k^*. Either $u-t$ or u/t is a transcendental constant over k^*. Hence, $k^*(\eta, \eta')$ contains a transcendental constant over k^*, since u is algebraic over $k^*(\eta, \eta')$.

2. Proof of Proposition. Firstly we shall prove the "only if" part. By the assumption there exists in $k^{\sharp}(\eta, \eta')$ a transcendental constant c over $k^{\sharp}$. The solution η of $F=0$ is algebraic over $k^{\sharp}(c)$ Since $k^{\sharp}$ is a liouvillian extension of k, η is a weakly liouvillian element over k Hence, $\nu_P(\tau')>0$ for some prime divisor P of $k(y, y')$ (cf. [3]).

Secondly we shall prove the "if" part. By the assumption there exists such a prime divisor P of $k(\eta, \eta')$ that $\nu_P(\tau')>0$. As τ we can take an element of $k(\eta, \eta')$. In the completion of $k(\eta\ \eta')$ with respect to P we have

$$\tau' = \sum b_i \tau^i, \qquad 1 \leqq i < \infty,\ b_i \in k. \tag{2}$$

Let k_2 denote the algebraic closure of k^* in Ω. Then, there exists uniquely a prime divisor Q of $k_2(\eta, \eta')$ such that the restriction of ν_Q^* to $k(\eta, \eta')$ is ν_P, where ν_Q^* is the normalized valuation belonging to Q. In this Q, τ is a prime element. In the completion of $k_2(\eta, \eta')$ with respect to Q we have

$$\eta,\ \eta' \in k((\tau)), \tag{3}$$

because $\tau \in k(\eta, \eta')$. There exists in $k_2(\eta, \eta')$ a transcendental constant c over k_2 by the assumption (1). Since c^{-1} is a constant, we may assume that $\nu_Q^*(c) \geqq 0$;

$$c = \sum \gamma_i \tau^i, \qquad 0 \leqq i < \infty,\ \gamma_i \in k_2. \tag{4}$$

Differentiating both sides we have

$$0 = c' = \sum(\gamma_i' \tau^i + i\gamma_i \tau^{i-1}\tau'), \qquad 0 \leqq i < \infty.$$

Hence, for each i $(0 \leqq i < \infty)$

$$\gamma_i' + i\gamma_i b_1 + \sum j\gamma_j b_{i-j+1} = 0 \qquad (0<j<i) \tag{5}$$

by (2). For $i=0$ we have $\gamma_0'=0$ and $\gamma_0 \in k$. There is a positive integer m such that $\gamma_i=0$ for each $i\ (1 \leqq i < m)$ and $\gamma_m \neq 0$, because $c \notin k$. We have

$$\gamma_m' + m\gamma_m b_1 = 0 .$$

Let δ be a root of $\delta^m=\gamma_m$. Then, $\delta'+b_1\delta=0$. For each $i(m<i<\infty)$ let us define an element u_i of k_2 by $\gamma_i=u_i\delta^i$. Then,

$$\gamma_i' + i\gamma_i b_1 = u_i'\delta^i , \qquad m<i<\infty,$$

and

$$u_i' \in k(\delta, u_{m+1}, \cdots, u_{i-1}) , \qquad m<i<\infty$$

by (5). Since $c \in k_2(\eta, \eta')$, we have

(6) $$c = S(\eta, \eta')/T(\eta) , \qquad (S, T)=1,\ \deg_{\eta'} S < \deg_{y'} F ,$$

where $S(Y, Z)$ and $T(Y)$ are polynomials over k_2:

$$S = \sum \alpha_{ij} Y^i Z^j \qquad (0 \leqq i \leqq p,\ 0 \leqq j \leqq q),\ \alpha_{ij} \in k_2 ,$$
$$T = \sum \beta_i Y^i \qquad (0 \leqq i \leqq r),\ \beta_r = 1,\ \beta_i \in k_2 .$$

Let L and M denote

$$k(\gamma_0, \gamma_1, \cdots, \gamma_n, \cdots) , \quad 0 \leqq n < \infty$$

and

$$k(\alpha_{00}, \cdots, \alpha_{ij}, \cdots, \alpha_{pq};\ \beta_0, \cdots, \beta_r), \qquad 0 \leqq i \leqq p,\ 0 \leqq j \leqq q$$

respectively. We shall prove that

(7) $$L=M .$$

For each n $(0 \leqq n < \infty)$ we have

$$\gamma_n = \phi_n(\alpha_{00}, \cdots, \alpha_{ij}, \cdots, \alpha_{pq};\ \beta_0, \cdots, \beta_r)$$

by (3), (4) and (6), where ϕ_n is a rational function of Y_{ij} $(0 \leqq i \leqq p,\ 0 \leqq j \leqq q)$ and Z_i $(0 \leqq i \leqq r)$ over k. Hence, $L \subset M$. Take an algebraic automorphism σ of k_2 over L. Let S^σ and T^σ be the polynomials obtained from S and T respectively by operating σ on each of their coefficients. Then, we have

$$S^\sigma(\eta,\eta')/T^\sigma(\eta) = \sum \gamma_n \tau^n = S(\eta, \eta')/T(\eta), \qquad 0 \leqq n < \infty,$$

since each of $\gamma_n (0 \leqq n < \infty)$ is left invariant by σ. Hence, each of $\alpha_{ij} (0 \leqq i \leqq p,\ 0 \leqq j \leqq q)$ and $\beta_i (0 \leqq i \leqq r)$ is left invariant by σ, and it is an element of L. Thus, we have (7). There exists a positive integer e such that $L=k(\gamma_0, \cdots, \gamma_e)$. As k^* we can take $L(\delta)$.

3. Proof of Lemma 3. We may assume that the field of constants of $k(\eta, \eta')$ is k_0: For, in the contrary case a transcendental constant of $k(\eta, \eta')$ over k can be taken as ϕ. By the discussions of the previous section, there exists such an extending chain $k=N_{-1}\subset N_0\subset N_1\subset\cdots\subset N_n$ of differential subfields of the algebraic closure k_2 of k^* in Ω that satisfies the following three conditions:

(iii) Each of the fields of constants of $N_{n-1}(\eta, \eta')$ and N_n is k_0;

(iv) the field of constants of $N_n(\eta, \eta')$ is not k_0;

(v) there exist elements $t_0, \cdots, t_n$ of k_2 which satisfy the following conditions; for each $i(0<i\leqq n)$, $N_i=N_{i-1}(t_i)$ and $t_i'\in N_{i-1}$; $N_0=k(t_0)$ and $t_0'=b_1t_0$, $b_1\in k$. We may assume that t_i is transcendental over N_{i-1} for each i $(1\leqq i\leqq n)$: For, $t_i\in N_{i-1}$ if t_i is algebraic over N_{i-1}.

Firstly suppose that n is positive. By the induction on i we shall prove that for each i $(0<i\leqq n)$ there exists in $N_{n-1}(\eta, \eta')$ a transcendental element Φ_{n-i} over N_{n-i} such that the derivative of Φ_{n-i} is an element of N_{n-i}. By (iii) and (iv) our statement is true for $i=1$, because $N_n=N_{n-1}(t_n)$ and $t_n'\in N_{n-1}$. Suppose that our statement is true for i $(1\leqq i<n)$. For convenience let us represent Φ_{n-i} by Φ, t_{n-i} by t, $N_{n-i-1}(\eta, \eta')$ by H and N_{n-i-1} by M respectively. Then, t is transcendental over H: For, in the contrary case Φ is algebraic over $M(t)$; this contradicts our assumption that Φ is transcendental over N_{n-i}. Since $\Phi\in H(t)$, we have

$$\Phi = S/R\,,\ (S, R) = 1\,, \qquad S,\, R\in H[t];$$

here the coefficient of the highest degree in R is assumed to be 1. We shall prove that $R\in M[t]$. Let P_j run over all irreducible factors of R in which the coefficient of the highest degree is 1. Then,

$$\Phi = U+\sum Q_j/P_j^{\lambda_j} \qquad (1\leqq j\leqq \mu),\ U,\ Q_j,\ P_j\in H[t];$$

here,

$$\deg Q_j<\lambda_j \deg P_j\,, \qquad 1\leqq j\leqq \mu\,. \tag{8}$$

Since $t'\in M$, we have

$$\deg\,(Q_j'P_j^{\lambda_j}-\lambda_j Q_j P_j^{\lambda_j-1}P_j')<2\lambda_j \deg P_j\,, \qquad 1\leqq j\leqq \mu$$

by (8). Suppose that some P_j is not an element of $M[t]$. Then,

$$(Q_j/P_j^{\lambda_j})' = 0\,,$$

because $\Phi'\in M(t)$. This contradicts our assumption that the field of constants of $H(t)$ is k_0. Hence, $P_j\in M[t]$ for each j. Thus, we have $R\in M[t]$ and

$$S'R-SR'\in M[t]\,. \tag{9}$$

Since $\Phi\notin M(t)$, we have $S\notin M[t]$. Set

$$S = s_0+s_1t+\cdots+s_mt^m\,, \qquad s_i\in H,\ s_m\neq 0\,.$$

Then, there exists an integer j ($0\leqq j\leqq m$) such that $s_i\in M$ if $i>j$ and $s_j\notin M$. We have $s_j'\in M$ by (9). Since the field of constants of H is k_0, s_j is transcendental over M. Hence, s_j can be taken as Φ_{n-i-1}. Thus, the induction is completed. In particular, for $i=n$ there exists in $N_0(\eta, \eta')$ a transcendental element Φ_0 over N_0 such that $\Phi_0'\in N_0$. We are in one of the following two cases: In the first case $t_0\in k$; we have $\Phi_0\in k(\eta, \eta')$, $\Phi_0'\in k$ and $\Phi_0\notin k$. In this case Φ_0 can be taken as ϕ. In the second case $t_0\notin k$; let us set $i=n$ in the above induction on i. Then, we have an element s_j of $k(\eta, \eta')$ such that $s_j\notin k$ and

$$s_j'+(j-r)b_1s_j\in k\,, \quad r = \deg R\,,$$

because $t'=b_1/t$, $b_1\in k$. Hence, s_j can be taken as ϕ in this case.

Secondly suppose that $n=0$. Then, t_0 is transcendental over $k(\eta, \eta')$. By our assumption there exists in $k(t_0, \eta, \eta')$ a transcendental constant over $k(\eta, \eta')$. Hence, in $k(\eta, \eta')$ we have a nontrivial solution ϕ of $\phi'=hb_1\phi$ for some positive integer h, because $t_0'=b_1t_0$ with $b_1\in k$. Since the field of constants of $k(t_0)$ is k_0, ϕ is transcendental over k.

4. Proof of Theorem. By Lemmas 1, 2 and 3 it is sufficient to prove the following: Suppose that $k(y, y')$ contains a transcendental element ϕ over k such that $\phi'=a\phi+b$, $a, b\in k$. Then, any liouvillian element over k satisfying $F=0$ is at most of the third order. We may set

$$\phi = Q(y, y')/P(y)\,, \qquad P, Q\in k\{u\}\,.$$

Let Γ be the set of all solutions of $F=0$ contained in k. Firstly assume that Γ is infinite. In this case we shall prove that $k(y, y')$ contains a transcendental constant over k and hence any liouvillian element over k satisfying $F=0$ is of order 0. There exists an element J of $k\{u\}$ satisfying $J(y, y', \cdots)\neq 0$ such that any differential specialization w of y over k with $J(w, w', \cdots)\neq 0$ can be extended to a differential specialization (w, ϕ_0) of (y, ϕ) over k(cf. Ritt [4], Koichin [1, p. 928]). Since Γ is infinite, there exists an element w of Γ such that $J(w, w', \cdots)\neq 0$ and $P(w)\neq 0$. Let (w, ϕ_0) be a differential specialization of (y, ϕ) over k. Then, $\phi_0'=a\phi_0+b$, and $\phi_0\in k$. Set $\psi=\phi-\phi_0$. Then, $\psi'=a\psi$. In a similar way to the above we have an element ψ_0 of k satisfying $\psi_0'=a\psi_0$ and $\psi_0\neq 0$. The element ψ/ψ_0 of $k(y, y')$ is a transcendental constant over k. Secondly assume that Γ is finite. Take elements A, t of Ω such that $A'=a$, $t'=at$ and $t\neq 0$. Let Λ be the prime differential ideal in $k\{z_1, z_2, z_3, z_4\}$ whose generic zero over k is (A, t, ϕ, y). We define an element T of $k[z_2, z_4]$ by

$$T = z_2 \prod (z_4 - w), \qquad w \in \Gamma .$$

Then, $T \notin \Lambda$. There exists a zero (A_0, t_0, ϕ_0, y_0) of Λ such that $T(t_0, y_0) \neq 0$ and the field of constants of $k\langle A_0, t_0, \phi_0, y_0\rangle$ is k_0 (cf. Kolchin [1]). We have $A_0' = a$, $t_0' = at_0$, $t_0 \neq 0$, $(\phi_0/t_0)' = b/t_0$, $F(y_0, y_0') = 0$ and $y_0 \notin \Gamma$. The element y_0 is transcendental over k and algebraic over $k(\phi_0)$. Hence, y_0 is a liouvillian element over k whose order is not 0 and at most 3.

OSAKA UNIVERSITY

Bibliography

[1] E.R. Kolchin: *Existence theorems connected with the Picard-Vessiot theory of homogeneous linear ordinary differential equations,* Bull. Amer. Math. Soc. **54** (1948) 927–932.

[2] J. Liouville: *Mémoire sur la classification des transcendantes, et sur l'impossibilité d'éxprimer les racine des certains equations en fonction finie explicite des coefficients,* J. Math. Pures Appl. **2** (1837), 56–104.

[3] S. Ōtsubo: *Solutions of an algebraic differential equation of the first order in a liouvillian extension of the coefficient field,* Osaka J. Math. **16** (1979), 289–293.

[4] J.F. Ritt: *On a type of algebraic differential manifold,* Trans. Amer. Math. Soc. **48** (1940), 542–552.

[5] M. Rosenlicht: *The nonminimality of the differential closure,* Pacific J. Math. **52** (1974), 529–537.

[6] ——— and M.F. Singer: *On elementary, generalized elementary, and liouvillian extension fields,* "Contributions to Algebra," Academic Press, New York, 1977, 329–342.

[7] M.F. Singer: *Elementary solutions of differential equations,* Pacific J. Math. **59** (1975), 535–547.

[8] G.N. Watson: A treaties on the theory of Bessel functions, Cambridge, Univ. Press, London, 1922.