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Osaka J. Math.
16 (1979), 529–537

ON REAL *J*-HOMOMORPHISMS

Dedicated to Professor A. Komatu on his 70th birthday

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(Received June 7, 1978)

1. In the present work we consider a Real analogue of J-homomorphisms in the sense of [3]. We use here the notation in [4], §§1 and 9 and [9], §2 for the equivariant homotopy groups which are discussed by Bredon [5] and Levine [10]. Moreover we shall use notations and terminologies of [4], §1 without any references.

Let us denote by $GL(n,\boldsymbol{C})$ (resp. $GL(\infty,\boldsymbol{C})$) the general linear group of degree n (resp. the infinite general linear group) over the complex numbers with involutions induced by complex conjugation. Let X be a finite pointed τ-complex. Then, by following the construction of usual J-homomorphisms (cf. [13], p. 314, [2]) we can define homomorphisms

$$(1.1) \qquad J_{R,n}\colon [\Sigma^{p,q}X, GL(n,\boldsymbol{C})]^\tau \to [\Sigma^{p+n,q+n}X, \Sigma^{n,n}]^\tau$$

and

$$J_R\colon [\Sigma^{p,q}X, GL(\infty,\boldsymbol{C})]^\tau \to \pi_s^{0,0}(\Sigma^{p,q}X)$$

for $p\geqq 0$ and $q\geqq 1$ where let $\pi_s^{0,0}(\Sigma^{p,q}X)=\lim_{n\to\infty}[\Sigma^{p+n,q+n}X, \Sigma^{n,n}]^\tau$. We now give definitions of $J_{R,n}$ and J_R below. Let $\Omega_d^{n,n}\Sigma^{n,n}$ denote the subspace of $\Omega^{n,n}\Sigma^{n,n}$ consisting of maps of degree d in the usual sense. Let γ be the τ-map of $\Sigma^{n,n}$ induced by the correspondence of $R^{n,n}$ such that $(x_1,\cdots,x_{2n})\mapsto(x_1,\cdots,x_{2n-1}, -x_{2n})$. By adding γ to the elements of $\Omega_1^{n,n}\Sigma^{n,n}$ with respect to the loop addition along fixed coordinates of $\Sigma^{n,n}$ we have a τ-map $t\colon \Omega_1^{n,n}\Sigma^{n,n}\to\Omega_0^{n,n}\Sigma^{n,n}$. Then we obtain $J_{R,n}$ by assigning to a base-point-preserving τ-map $f\colon \Sigma^{p,q}X\to GL(n,\boldsymbol{C})$ the adjoint of the composite

$$\Sigma^{p,q}X \xrightarrow{f} GL(n,\boldsymbol{C}) \overset{i}{\subset} \Omega_1^{n,n}\Sigma^{n,n} \xrightarrow{t} \Omega_0^{n,n}\Sigma^{n,n}$$

where i is the canonical inclusion map.

As is easily seen the diagram

$$\begin{array}{ccc} [\Sigma^{p,q}X, GL(n+1,\boldsymbol{C})]^\tau & \xrightarrow{J_{R,n+1}} & [\Sigma^{p+n+1,q+n+1}X, \Sigma^{n+1,n+1}]^\tau \\ \uparrow j_* & & \uparrow \Sigma_*^{1,1} \\ [\Sigma^{p,q}X, GL(n,\boldsymbol{C})]^\tau & \xrightarrow{J_{R,n}} & [\Sigma^{p+n,q+n}X, \Sigma^{n,n}]^\tau \end{array}$$

is commutative under the identification $\Sigma^{r,s}\wedge\Sigma^{p,q}=\Sigma^{r+p,s+q}$ where j_* is the homomorphism induced by a canonical inclusion map j: $GL(n,\boldsymbol{C})\subset GL(n+1,\boldsymbol{C})$ and $\Sigma_*^{1,1}$ is the suspension homomorphism ([4], (7.2)). Therefore, by taking the direct limits we get a homomorphism

$$J_{R,\infty}\colon \lim_{n\to\infty}[\Sigma^{p,q}X,\ GL(n,\boldsymbol{C})]^{\tau}\to \pi_s^{0,0}(\Sigma^{p,q}X)\,.$$

Also, as X is compact we have a canonical isomorphism μ: $\lim_{n\to\infty}[\Sigma^{p,q}X, GL(n,\boldsymbol{C})]^{\tau}\to[\Sigma^{p,q}X, GL(\infty,\boldsymbol{C})]^{\tau}$. So we define J_R to be the composite $J_{R,\infty}\mu^{-1}$.

Taking $X=S^{0,1}$ in (1.1) J_R becomes the homomorphism from $\pi_{p,q}(GL(\infty,\boldsymbol{C}))$ to $\pi_{p,q}^s$. The aim of this paper is to prove the following theorem for the homomorphism

$$J_R\colon \pi_{2p-2k,2p+2k-1}(GL(\infty,\boldsymbol{C}))\to \pi^s_{2p-2k,2p+2k-1} \tag{1.2}$$

for $p\geqq k\geqq 0$ and $p+k\geqq 1$.

Theorem. *The image $J_R(\pi_{2p-2k,2p+2k-1}(GL(\infty,\boldsymbol{C})))$ of the homomorphism* (1.2) *is a cyclic group of the following order:*

$m(2p)$ *if either p, k are even or odd*

$\frac{1}{2}m(2p)$ *if p is even and k is odd*

$m(2p)$ *or* $2m(2p)$ *if p is odd and k is even*

where $m(t)$ is the numerical function as in [1], II, p. 139.

2. Let X be a compact pointed τ-space throughout this section.

Let KR denote the Real K-functor [3]. Then a similar proof to the complex case gives rise to a canonical isomorphism

$$[X,\ GL(\infty,\boldsymbol{C})]^{\tau}\cong \widetilde{KR}(\Sigma^{0,1}X) \tag{2.1}$$

(cf. [8], Chap. I, Theorem 7.6) and so we may consider J_R of (1.1) the homomorphism from $\widetilde{KR}^{-1}(\Sigma^{p,q}X)$ to $\pi_s^{0,0}(\Sigma^{p,q}X)$ through this isomorphism. In particular, there exist isomorphisms

$$\pi_{2p-2k,2p+2k-1}(GL(\infty,\boldsymbol{C}))\cong\widetilde{KR}(\Sigma^{2p-2k,2p+2k})\cong\widetilde{KO}(S^{4k})\cong Z \tag{2.2}$$

by (2.1) and the Real Thom isomorphism theorem [3]. Similarly we have isomorphisms

$$\pi_{4p-1}(GL(\infty,\boldsymbol{C}))\cong\tilde{K}(S^{4p})\cong\tilde{K}(S^{4k})\cong Z \tag{2.3}$$

in the complex K-theory.

Let ψ: $\pi_{p,q}(X)\to\pi_{p+q}(X)$ and ψ: $\pi^s_{p,q}(X)\to\pi^s_{p+q}(X)$ denote the forgetful homomorphisms [4,5]. Then, from the above discussion we have the following commutative diagram:

$$(2.4)\qquad \begin{array}{ccc} \widetilde{KO}(S^{4k}) & \xrightarrow{\quad c \quad} & \tilde{K}(S^{4k}) \\ \downarrow \cong & & \downarrow \cong \\ \pi_{2p-2k\ 2p+2k-1}(GL(\infty, \boldsymbol{C})) & \xrightarrow{\psi} & \pi_{4p-1}(GL(\infty, \boldsymbol{C})) \\ \downarrow J_R & & \downarrow J_U \\ \pi^s_{2p-2k, 2p+2k-1} & \xrightarrow{\quad \psi \quad} & \pi^s_{4p-1} \end{array}$$

where c is the natural complexification homomorphism and J_U is the complex stable J-homomorphism.

In the following we identify $\Sigma^{r,s} \wedge \Sigma^{p,q}$ with $\Sigma^{r+p,s+q}$. Regarding $\Sigma^{1,0}$ as the one-point compactification of $R^{1,0}$ with ∞ as base-point, the quotient $\Sigma^{1,0}/\{0, \infty\}$ is homeomorphic to $S^1 \vee S^1$ as τ-spaces where $S^1 \vee S^1$ has the invloution T interchanging factors. For a base-point-preserving map $f\colon S^{p+q} \to X$, define a τ-map $\tilde{f}\colon \sum^{p,q} \to X$ by the composition

$$\begin{aligned} \Sigma^{p,q} = \Sigma^{p-1,0} \wedge \Sigma^{1,0} \wedge \Sigma^{0,q} &\xrightarrow{1 \wedge \pi \wedge 1} \Sigma^{p-1,0} \wedge (\Sigma^{1,0}/\{0, \infty\}) \wedge \Sigma^{0,q} \\ &\approx (\Sigma^{p-1,0} \wedge S^1 \wedge \Sigma^{0,q}) \vee (\Sigma^{p-1,0} \wedge S^1 \wedge \Sigma^{0,q}) \xrightarrow{f \vee \tau f \tau'} X \end{aligned}$$

for $p, q \geqq 1$ where π is the natural projection, τ is the involution of X and τ' is the involution of $(\Sigma^{p-1,0} \wedge S^1 \wedge \Sigma^{0,q}) \vee (\Sigma^{p-1,0} \wedge S^1 \wedge \Sigma^{0,q})$ induced by that of $\Sigma^{p-1,q+1} = \Sigma^{p-1,0} \wedge S^1 \wedge \Sigma^{0,q}$ and T. Then the correspondence $f \mapsto \tilde{f}$ determines a homomorphism

$$(2.5)\qquad \alpha\colon \pi_{p+q}(X) \to \pi_{p,q}(X)$$

for $p, q \geqq 1$ (cf. [5], p. 286, [4], (10.5)).

Let $J_{U,n}\colon \pi_{4p-1}(GL(n, \boldsymbol{C})) \to \pi_{4p-1+2n}(S^{2n})$ be the complex J-homomorphism. Let $\alpha_n\colon \pi_{4p-1}(GL(n, \boldsymbol{C})) \to \pi_{2p-2k, 2p+2k-1}(GL(n, \boldsymbol{C}))$ and $\alpha_n\colon \pi_{4p-1+2n}(S^{2n}) \to \pi_{2p-2k+n, 2p+2k-1+n}(\Sigma^{n,n})$ denote the homomorphisms of (2.5) for $X = GL(n, \boldsymbol{C})$ and $X = \Sigma^{n,n}$ respectively. Then we have the commutative diagram:

$$\begin{array}{ccc} \pi_{4p-1}(GL(n, \boldsymbol{C})) & \xrightarrow{\alpha_n} & \pi_{2p-2k, 2p+2k-1}(GL(n, \boldsymbol{C})) \\ \downarrow J_{U,n} & & \downarrow J_{R,n} \\ \pi_{4p-1+2n}(S^{2n}) & \xrightarrow{\alpha_n} & \pi_{2p-2k+n, 2p+2k-1+n}(\Sigma^{n,n}) . \end{array}$$

The commutativity is proved as follows. For a τ-map $g\colon \sum^{2p-2k, 2p+2k-1} \to GL(n, \boldsymbol{C})$ we denote by ad$g$ the adjoint of the composition: $\Sigma^{2p-2k, 2p+2k-1} \xrightarrow{g} GL(n, \boldsymbol{C}) \subset \Omega_1^{n,n} \Sigma^{n,n} \xrightarrow{t} \Omega_0^{n,n} \Sigma^{n,n}$. Then $J_{R,n}$ is given by the assignment $g \mapsto \mathrm{ad}g$ as in §1. In the above, forgetting the Z_2-action we get the homomorphism $J_{U,n}$. Hence we also use the same notation for maps in the complex case. Let us define a map $\lambda\colon S^n \wedge S^{2p-2k} \wedge S^n \wedge S^{2p+2k-1} \to S^n \wedge S^n \wedge S^{2p-2k} \wedge S^{2p+2k-1}$ by $\lambda(u_1 \wedge v_1 \wedge u_2 \wedge v_2) = u_1 \wedge u_2 \wedge v_1 \wedge v_2$ ($u_1, u_2 \in S^n$, $v_1 \in S^{2p-2k}$, $v_2 \in S^{2p+2k-1}$). And we define a map

$f'\colon S^{4p-1+2n}\to S^{2n}$ by $f'=(\mathrm{ad}f)\lambda$ for a map $f\colon S^{4p-1}\to GL(n,\boldsymbol{C})$. Then $f'\simeq\mathrm{ad}f$ since the degree of λ is 1, and so $\tilde{f}'\simeq_\tau\widetilde{\mathrm{ad}f}$. Besides we see easily that $\tilde{f}'=\mathrm{ad}\tilde{f}$. Therefore $\widetilde{\mathrm{ad}f}\simeq_\tau\mathrm{ad}\tilde{f}$ which implies $\alpha_n J_{U,n}([f])=J_{R,n}\alpha_n([f])$ where $[f]$ denotes the homotopy class of f.

Here, by taking the direct limits we get the commutative diagram

$$\begin{array}{ccc} \pi_{4p-1}(GL(\infty,\boldsymbol{C})) & \xrightarrow{\alpha} & \pi_{2p-2k,2p+2k-1}(GL(\infty,\boldsymbol{C})) \\ \downarrow J_U & & \downarrow J_R \\ \pi^s_{4p-1} & \xrightarrow{\alpha} & \pi^s_{2p-2k,2p+2k-1} \end{array} \tag{2.6}$$

where each α is defined as the direct limit of α_n. As in proof of the commutativity of the above diagram, we can show that the lower homomorphism α is well-defined.

By the definition of α it follows that the realification homomorphism $r\colon\tilde{K}^{-1}(S^{4p-1})\to\widetilde{KR}^{-1}(\Sigma^{2p-2k,2p+2k-1})$ [12] coincides with $\alpha\colon\pi_{4p-1}(GL(\infty,\boldsymbol{C}))\to\pi_{2p-2k,2p+2k-1}(GL(\infty,\boldsymbol{C}))$ through the natural isomorphisms. Because, $\psi\alpha=1+*$, $\psi=c$, $cr=1+*$ and c is injective where $*$ is the operation on $K(X)$ defined in [12], §2. Thus, by (2.2), (2.3) and (2.6) we get the commutative diagram

$$\begin{array}{ccc} \tilde{K}(S^{4k}) & \xrightarrow{r} & \widetilde{KO}(S^{4k}) \\ \downarrow\cong & & \downarrow\cong \\ \pi_{4p-1}(GL(\infty,\boldsymbol{C})) & \xrightarrow{\alpha} & \pi_{2p-2k,2p+2k-1}(GL(\infty,\boldsymbol{C})) \\ \downarrow J_U & & \downarrow J_R \\ \pi^s_{4p-1} & \xrightarrow{\alpha} & \pi^s_{2p-2k,2p+2k-1} \end{array} \tag{2.7}$$

where r is the realification homomorphism.

Let $GL(\infty,\boldsymbol{R})$ denote the infinite general linear group over the real numbers and J_O denote the real stable J-homomorphism in stable dimensions $4p$-1. Let us put

$$g_\Lambda=J_\Lambda(1),\quad \Lambda=O,\ U\text{ or }R\,,$$

identifying $\pi_{4p-1}(GL(\infty,\boldsymbol{R}))$, $\pi_{4p-1}(GL(\infty,\boldsymbol{C}))$ and $\pi_{2p-2k,2p+2k-1}(GL(\infty,\boldsymbol{C}))$ with Z. Then, from (2.4), (2.7) and [12], (2.2) we see that

$$\alpha(g_U)=\begin{cases}2g_R & \text{if } k \text{ is even}\\ g_R & \text{if } k \text{ is odd}\end{cases} \tag{2.8}$$

and

$$\psi(g_R)=\begin{cases}g_U & \text{if } k \text{ is even}\\ 2g_U & \text{if } k \text{ is odd}\end{cases}$$

Furthermore it is known that

$$g_U=\begin{cases}2g_O & \text{if } p \text{ is even}\\ g_O & \text{if } p \text{ is odd}\end{cases} \tag{2.9}$$

and the order of g_O is equal to the number $m(2p)$ ([1], II, Theorem (2.7) and [11]) which is divisible by 8 ([1], II, p.139).

Let $o(p,k)$ denote the order of the image of (1.2). Then, by (2.8) and (2.9), we obtain

Lemma. *For* $p>k$,

$$o(p,k)=\begin{cases} dm(2p) & \textit{if either } k, p \textit{ are even or odd} \\ 2dm(2p) & \textit{if } k \textit{ is even and } p \textit{ is odd} \\ \frac{1}{2}dm(2p) & \textit{if } k \textit{ is odd and } p \textit{ is even} \end{cases}$$

where $d=\frac{1}{2}$ *or* 1.

We shall give a proof of Theorem in §§3–5.

3. Proof for $p>k$, k odd and p even. By [5], Fig. we have an exact sequence

$$\pi^s_{2p-2k-1,2p+2k} \xrightarrow{\psi} \pi^s_{4p-1} \xrightarrow{\alpha} \pi^s_{2p-2k,2p+2k-1}$$

(cf. [4], (10.5)). Therefore, if we suppose that $o(p,k)=\frac{1}{4}m(2p)$ then $\alpha(\frac{1}{2}m(2p)g_O)=\frac{1}{4}m(2p)g_R=0$ by (2.8), (2.9) and so there exists an equivariant map

$$f\colon \Sigma^{2p-2k-1+n,2p+2k+n} \to \Sigma^{n,n} \quad \text{for } n \text{ sufficiently large}$$

such that the image of the homotopy class of f by ψ is $\frac{1}{2}m(2p)g_O$.

Since k is odd,

$$\widetilde{KR}(\Sigma^{2p-2k-1+n,2p+2k+n}) \cong \widetilde{KO}(S^{4k+1})=0$$

and

$$\widetilde{KR}(\Sigma^{2p-2k-1+n,2p+2k+n+1}) \cong \widetilde{KO}(S^{4k+2})=0\,.$$

Therefore we have the commutative diagram

$$\begin{array}{ccccccc} 0\leftarrow \widetilde{KR}(\Sigma^{n,n}) & \leftarrow & \widetilde{KR}(\Sigma^{n,n}\bigcup_f C\Sigma^{2p-2k-1+n,2p+2k+n}) & \leftarrow & \widetilde{KR}(\Sigma^{2p-2k-1+n,2p+2k+n+1}) & \leftarrow 0 \\ c\downarrow\cong & & c\downarrow & & c\downarrow & =0 \\ 0\leftarrow \tilde{K}(S^{2n}) & \longleftarrow & \tilde{K}(S^{2n}\bigcup_{f'} CS^{4p-1+2n}) & \longleftarrow & \tilde{K}(S^{4p+2n}) & \leftarrow 0 \end{array}$$

where f' is a representative of $\frac{1}{2}m(2p)g_O$, CA is the cone of A and c is the natural complexification homomorphism ([12], §2). This diagram implies that $e_C(f')=0$, which contradicts to the fact that $e_C(f')=\frac{1}{2}$ ([1], IV, §7). Hence we see by Lemma that $o(p,k)=\frac{1}{2}m(2p)$.

4. Proof for $p>k$ and p, k even or odd. Using the notation of Landweber for the stable homotopy groups [9], by [5], Fig. and (12) we have the following commutative diagram in which the columns and the rows are exact sequences:

$$\begin{array}{ccccc} 0 & & & & 0 \\ \downarrow & & & & \downarrow \\ \lambda^s_{2p-2k-1,2p+2k} & \xrightarrow{\psi^*} & \pi^s_{4p-1} & \xrightarrow{\alpha^*} & \lambda^s_{2p-2k,2p+2k-1} \\ \downarrow & & \| & & \downarrow \\ \pi^s_{2p-2k-1,2p+2k} & \xrightarrow{\psi} & \pi^s_{4p-1} & \xrightarrow{\alpha} & \pi^s_{2p-2k,2p+2k-1} \end{array}$$

for $k\geqq 0$. ($\lambda^s_{p,q}$ and $\pi^s_{p,q}$ are Bredon's $\pi^*_{p+q,p}$ and $\pi_{p+q,p}$ respectively.) If we assume that $o(p,k)=\frac{1}{2}m(2p)$, then $\alpha(\frac{1}{2}m(2p)g_O)=\frac{1}{2}m(2p)g_R=0$ by (2.8), (2.9) and therefore there is an equivariant map

$$\tilde{f}\colon \Sigma^{2p-2k-1+n,2p+2k+n}/\Sigma^{0,2p+2k+n} \to \Sigma^{n,n} \text{ for } n \text{ sufficiently large}$$

such that the image of the homotopy class of $\tilde{f}$ by ψ^* is $\frac{1}{2}m(2p)g_O$.

Consider the diagram

$$\begin{array}{ccc} \Sigma^{2p-2k-1+n,2p+2k+n}/\Sigma^{0,2p+2k+n} & & \\ \uparrow \pi & & \searrow \tilde{f} \\ \Sigma^{2p-2k-1+n,2p+2k+n} & \xrightarrow{\quad f \quad} & \Sigma^{n,n} \end{array}$$

where $f=\tilde{f}\pi$ and π is the map collapsing $\Sigma^{0,2p+2k+n}$ to a point.

Putting

$$\begin{aligned} A &= \widetilde{KO}_{Z_2}(\Sigma^{2p-2k-1+n,2p+2k+n}/\Sigma^{0,2p+2k+n}), \\ B &= \widetilde{KO}_{Z_2}(\Sigma^{2p-2k-1+n,2p+2k+n}), \\ C &= \widetilde{KO}_{Z_2}(\Sigma^{2p-2k-1+n,2p+2k+n+1}) \end{aligned}$$

and taking

$$n \equiv 0 \mod 8,$$

we have by [9], Lemma 4.1

$$A \cong KO^{-2p-2k-n-1}(P^{2p-2k-2+n})$$

where P^m is the real projective m-space and we have by [6] and [9], Theorem 3.1

$$A \cong \begin{cases} 0 & \text{if } p=2q,\ k=2l \text{ and } q+l \text{ is odd} \\ & \text{or } p=2q+1,\ k=2l+1 \text{ and } q+l \text{ is even} \\ Z_2\oplus Z_2 & \text{if } p=2q,\ k=2l \text{ and } q+l \text{ is even} \\ & \text{or } p=2q+1,\ k=2l+1 \text{ and } q+l \text{ is odd}, \end{cases}$$

$$B \cong Z,\ C \cong Z_2 \quad \text{if } p, k \text{ are even}$$

and

$$B \cong Z,\ C = 0 \quad \text{if } p, k \text{ are odd}.$$

In any case A, C are torsion groups and B is a free abelian group. Hence $f^* = \pi^* \tilde{f}^* \colon \widetilde{KO}_{Z_2}(\Sigma^{n,n}) \to B$ is a zero map since $\pi^* \colon A \to B$ is so. And therefore we have the commutative diagram

$$(4.1)\qquad \begin{array}{ccccccc} 0 \leftarrow & \widetilde{KO}_{Z_2}(\Sigma^{n,n}) & \leftarrow & \widetilde{KO}_{Z_2}(\Sigma^{n,n} \bigcup_f C\Sigma^{2p-2k-1+n,2p+2k+n}) & \longleftarrow & C & \\ & \rho \downarrow & & \rho \downarrow & & \rho \downarrow & \\ 0 \leftarrow & \widetilde{KO}(S^{2n}) & \longleftarrow & \widetilde{KO}(S^{2n} \bigcup_{f'} CS^{4p-1+2n}) & \longleftarrow & \widetilde{KO}(S^{4p+2n}) & \leftarrow 0 \\ & & & & & \cong Z & \end{array}$$

where f' is a representative of $\frac{1}{2} m(2p) g_O$ and ρ is the forgetful homomorphism.

From [9], Theorem 3.1 and Proposition 3.4 we see that $\widetilde{KO}_{Z_2}(\Sigma^{8m,8m})$ is a free $RO(Z_2)$-module with a single generator u for which the Adams operation ψ^k satisfy

$$(4.2)\qquad \psi^k(u) = \begin{cases} k^{8m}u + \frac{1}{2} k^{8m}(H-1)u & \text{if } k \text{ is even} \\ k^{8m}u + \frac{1}{2}(k^{8m} - k^{4m})(H-1)u & \text{if } k \text{ is odd} \end{cases}$$

for $m > 0$ where H is a canonical, non-trivial, 1-dimensional representation of Z_2. Since $\rho(u)$ becomes a generator of $\widetilde{KO}(S^{16m})$, (4.1) and (4.2) imply that $e'_R(f') = 0$. On the other hand $e'_R(f') = \frac{1}{2}$ ([1], IV, §7). This contradiction and Lemma show that $o(p,k) = m(2p)$.

5. Proof for $p = k$. Considering the following diagram

$$\begin{array}{ccc} \pi_{0,4p-1}(GL(\infty, \boldsymbol{C})) & \underset{\cong}{\overset{\varphi}{\rightarrow}} & \pi_{4p-1}(GL(\infty, \boldsymbol{R})) \\ \downarrow J_R & & \downarrow J_O \\ \pi^s_{0,4p-1} & \xrightarrow{\quad\varphi\quad} & \pi^s_{4p-1} \end{array}$$

where φ is the fixed-point homomorphism [4, 5] we see that this diagram is commutative and therefore $o(p,p)$ is divisible by $m(2p)$.

Let us denote by $\Omega^n_d S^n$ the space of base-point-preserving maps of S^n into itself of degree d, by $GL(n, \boldsymbol{R})$ the general linear group of degree n over the real numbers and by $GL(n, \boldsymbol{R})_0$ its identity component. Then the real J-homomorphism $J_{O,n} \colon \pi_{4p-1}(GL(n, \boldsymbol{R})) \to \pi_{4p-1+n}(S^n)$ is induced by the composition

$$GL(n, \boldsymbol{R})_0 \overset{i'}{\subset} \Omega^n_1 S^n \overset{t'}{\rightarrow} \Omega^n_0 S^n$$

where i' is the inclusion map and t' is a similar one to t in §1 ([2], §1). Particularly, if $n\geqq 4p+1$ then we may consider $J_{O,n}$ the stable real J-homomorphism J_O: $\pi_{4p-1}(GL(\infty,\boldsymbol{R}))\to\pi^s_{4p-1}$.

For a map f: $S^{4p-1}\to\Omega_1^n S^n$, define a map f': $S^{4p-1}\to\Omega_1^{n,n}\Sigma^{n,n}$ by $f'(x)=f(x)\wedge f(x)$ ($x\in S^{4p-1}$). Here we regard $S^n\wedge S^n$ as a space with involution switching factors and then $S^n\wedge S^n\approx\Sigma^{n,n}$ as τ-spaces. The assignment $f\mapsto f'$ determines a homomorphism ω': $\pi_{4p-1}(\Omega_1^n S^n)\to\pi_{0,p-1}(\Omega_1^{n,n}\Sigma^{n,n})$. And so we define a homomorphism

$$\omega\colon \pi_{4p-1}(\Omega_1^n S^n)\to\pi^s_{0,4p-1}$$

by the composition

$$\pi_{4p-1}(\Omega_1^n S^n)\xrightarrow{\omega'}\pi_{0,4p-1}(\Omega_1^{n,n}\Sigma^{n,n})\xrightarrow{t_*}\pi_{0,4p-1}(\Omega_0^{n,n}\Sigma^{n,n})\to\pi^s_{0,4p-1}$$

where the unlabelled arrow is the obvious homomorphism. Then we can easily check that the diagram with the natural isomorphism $\pi_{4p-1}(GL(n,\boldsymbol{R}))\cong\pi_{4p-1}(GL(\infty,\boldsymbol{R}))$

$$\begin{array}{ccc}\pi_{0,4p-1}(GL(\infty,\boldsymbol{C}))\overset{\varphi}{\underset{\cong}{\to}}\pi_{4p-1}(GL(\infty,\boldsymbol{R}))\cong & & \pi_{4p-1}(GL(n,\boldsymbol{R}))\\ \downarrow J_R & & \downarrow i'_*\\ \pi^s_{0,4p-1} & \xleftarrow{\quad\omega\quad} & \pi_{4p-1}(\Omega_1^n S^n)\end{array}$$

is commutative for $n\geqq 4p+1$. From the commutativity of this diagram and the fact that J_O factors into the following three homomorphism:

$$\pi_{4p-1}(GL(n,\boldsymbol{R}))\xrightarrow{i'_*}\pi_{4p-1}(\Omega_1^n S^n)\overset{t'_*}{\underset{\cong}{\to}}\pi_{4p-1}(\Omega_0^n S^n)\cong\pi_{4p-1+n}(S^n)$$

for $n\geqq 4p+1$ ([12], §1), it follows that $m(2p)$ is divisible by $o(p,p)$. This completes the proof of Theorem.

6. Finally we observe examples for the case k even and p odd.

By [5], (8) and [7], Table 1 we obtain

$$\lambda^s_{2,1}\cong Z_{12}\text{ and }\lambda^s_{6,5}\cong Z_{504}$$

using the Landweber's notation and so, making use of the exact sequence of [9], p.129, we have

$$\pi^s_{2,1}\cong Z_{24}\text{ and }\pi^s_{6,5}\cong Z_{504}\,.$$

Since $m(2p)=24$ and $m(2p)=504$ if $p=1$ and $p=3$ respectively, we get by Lemma and the above isomorphisms $o(p,k)=m(2p)$ for $(p,k)=(1,0)$, $(3,0)$. We therefore conjecture that $o(p,k)=m(2p)$ for k even and p odd generally.

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