Sumioka, T.
Osaka J. Math.
16 (1979), 395-403

ON QF-3 AND 1-GORENSTEIN RINGS

TAKESHI SUMIOKA

(Received May 6, 1978)

Let R be a ring with identity, E the injective hull of R_R and Q the maximal right quotient ring of R. Consider the following conditions.

A_1: E is projective.

A_2: R has a minimal faithful right R-module.

A_3: R has a faithful injective right ideal.

A_4: E is torsionless.

A_5: Any finitely generated submodule of E is torsionless.

A_6: Any finitely generated submodule of Q is torsionless.

For arbitrary ring R, $A_2 \Rightarrow A_3 \Rightarrow A_4 \Rightarrow A_5 \Rightarrow A_6$ hold (see [14, p, 47]). On the other hand if R is a left perfect ring, A_2, A_3 and A_4 are equivalent (see [4, Theorem 3.2], [1, Theorem 2] and [13, Proposition 3.1]). Any commutative integral domain which is not a field satisfies A_5, but does not satisfy A_4. However, Rutter [8, Corollary 3] proved that for any right or left artinian ring R A_5 implies A_2. In § 2 we shall define a new ring, called a right DWA ring, which is a generalization of a right semi-artinian ring, and we shall show that for any right DWA and right perfect ring R A_5 implies A_3. As a consequence, for any perfect ring R A_5 implies A_2. We owe the method of the proof essentially to Rutter [8, Lemma 2] and the proofs of Colby and Rutter [1, Theorem 2] and Jans [4, Theorem 3.2]. We call a ring R right QF-3 if R satisfies A_5 and R right QF-3′ if R satisfies A_6.

By Wu, Mochizuki and Jans [16] (or Colby and Rutter [1, Theorem 1] and Kato [6, Remark, p. 236]), a characterization of right artinian rings satisfying A_1 (or rings satisfying A_4) was given. In § 3 we shall give an analogous result for right QF-3 rings. In § 4 we shall show that any right QF-3′ and right 1-Gorenstein ring is QF-3 (see § 4 for the definition of a 1-Gorenstein ring). Moreover using this result, we shall show that any noetherian right QF-3′ ring such that $E \oplus E/R$ is an injective cogenerator is right and left 1-Gorenstein.

1. Definitions and notations

Throughout this note we assume that R is a ring with identity and all R-modules are unitary. We denote by E the injective hull of R_R and by Q the maximal right quotient ring of R. For an R-module M we denote by M^* the dual module $\mathrm{Hom}_R(M, R)$. We assume that "torsion theory" means the Lambek torsion theory. We call R *right QF-3* if any finitely generated submodule of E is torsionless. (Note that we call such a ring right QF-3′ in [11] and [12]). On the other hand we call R *right QF-3′* if any finitely generated submodule of Q is torsionless.

Let M be a right R-module. For a submodule N of M, we say that N is saturated in M if M/N is torsion-free. (In [12] we call such a submodule closed). See [10] for saturated submodules. We say that M is *D-noetherian* (resp. *D-artinian*) if M satisfies the ascending (resp. descending) chain condition on saturated submodules of M. (The letter D comes from the topology consisting of dense right ideals of R.) A ring R is called *right D-noetherian* (resp. *right D-artinian*) if a right R-module R_R is D-noetherian (resp. D-artinian). There exists a maximal chain

$$M = M_0 \supset M_1 \supset \cdots \supset M_n$$

of saturated submodules M_i of M if and only if M is D-noetherian and D-artinian. If M has a maximal chain as above, we say that M is *finite Lambek-dimensional* (or simply *finite dimensional*) and denote it by $\dim M = n$. Otherwise we define it by $\dim M = \infty$. We call a module M *strongly uniform* (simply *s-uniform*) if M is torsion-free and $\dim M = 1$. We say that M is *D-weakly artinian* (or simply *DWA*) if any submodule N of M with $\dim N \geq 1$ has a submodule L with $\dim L = 1$ and R is *right DWA* if R_R is DWA. Note that any right semi-artinian (in particular left perfect) ring and any right D-artinian ring is right DWA

2. Perfect QF-3 rings

We can obtain the following theorem and corollaries (in particular Corollary 1) by Rutter [8, Lemma 2] and a slight modification of the proofs of Colby and Rutter [1, Theorem 1.2] and Jans [4, Theorem 3.2].

Theorem 1. *Let R be a right DWA and right perfect ring. Then the following conditions are equivalent:*

(1) *R is right QF-3.*

(2) *R has a faithful injective right ideal.*

Proof. It always holds that (2) implies (1). Assume (1). Let I be an s-uniform right ideal of R. $E(I)$ is clearly imbedded in E, where $E(I)$ is the injective hull of I. If M is a non-zero finitely generated submodule of $E(I)$, M is torsionless, i.e. $\bigcap_{f \in M^*} \mathrm{Ker} f = 0$. Hence there exists a map $f \in M^*$ such that $I \cap M \not\subset \mathrm{Ker} f$, since $I \cap M \neq 0$. Suppose $\mathrm{Ker} f \neq 0$. Then $I \cap \mathrm{Ker} f$ is a non-zero proper submodule of $I \cap M$. But $I \cap M/I \cap \operatorname{Ker} f$ is imbedded in a torsion-free R-module $M/\operatorname{Ker} f$. Since $\dim I \cap M=1$, we have $I \cap M/I \cap \operatorname{Ker} f=0$ or equivalently $I \cap M \subset \operatorname{Ker} f$. This is a contradiction. Thus we have $\operatorname{Ker} f=0$, so M is imbedded in R (see Proposition 1). Therefore by [8, Lemma 2] $E(I)$ is flat and so is projective. Since $E(I)$ is clearly indecomposable, $E(I)$ is isomorphic to a right ideal eR of R, where e is a primitive idempotent of R. Let $e_1R, \cdots, e_tR$ be a complete set of pairwise non-isomorphic injective indecomposable right ideals of R and put $P=e_1R+\cdots+e_tR$, where the sum is clearly direct. We show P is faithful. Let $0 \neq a \in R$. Then there exists an s-uniform right ideal I contained in aR. Since we have a monomorphism $I \rightarrow e_iR$ for some i and e_iR is injective, there exists $x \in e_iR$ such that xI $(\cong I) \neq 0$. Thus we have $xa \neq 0$ and this shows P is faithful.

Corollary 1. *Let R be a perfect ring. Then the following conditions are equivalent*:

(1) *R is right QF-3.*

(2) *R has a minimal faithful right R-module.*

Corollary 2. *Let R be a semi-perfect right DWA ring. Then the following conditions are equivalent*:

(1) *E is torsionless, where E is the injective hull of* R_R.

(2) *R has a faithful injective right ideal.*

We call a module M *essentially finite dimensional* if there exists a finite dimensional submodule N of M such that N is essential in M. Then the property that the torsionless module M in the proof of Theorem 1 is imbedded in R is extended as follows:

Proposition 1. *Let M be an essentially finite dimensional right R-module. If M is torsionless, then M can be imbedded in a finitely generated free right R-module.*

Proof. Let N be a finite dimensional and essential submodule of M. Since M is torsionless, we have a monomorphism $f\colon M \rightarrow \prod_{\alpha \in A} R_\alpha$ with R_α copies of R_R. Then by [12, Lemma 1′] there exists a finite subset B of A such that $pfg\colon N \rightarrow \prod_{\beta \in B} R_\beta$ is monomorphic, where $p\colon \prod_{\alpha \in A} R_\alpha \rightarrow \prod_{\beta \in B} R_\beta$ and $g\colon N \rightarrow M$ are the canoical projection and the canonical injection, respectively. This shows $N \cap \operatorname{Ker} pf=0$. Since N is essential in M, $\operatorname{Ker} pf=0$, and so M is imbedded in a finitely generated free right R-module.

The following holds on essentially finite dimensional R-modules.

Proposition 2. *Let R be a ring. Then we have*

(1) *Any torsion-free and essentially finite dimensional right R-module is DWA and finite Goldie-dimensional.*

(2) *Any DWA and finite Goldie-dimensional right R-module is essentially finite dimensional.*

(3) *Any D-artinian right R-module is essentially finite dimensional.*

Proof. (1) and (2) are immediate consequences.

(3) Let M be a D-artinian right R-module and $t(M)$ the torsion submodule of M. Then $M/t(M)$ is DWA and finite Goldie-dimensional by [12, Lemma 2]. Hence by (2) there exists an essential and finite dimensional submodule $N/t(M)$ of $M/t(M)$. Then N is clearly an essential and finite dimensional submodule of M.

3. QF-3 rings

Wu, Mochizuki and Jans [16] gave a characterization of right artinian rings satisfying the condition A_1 (in the introduction). Moreover this characterization is also that of rings satisfying A_4 (see Colby and Rutter [1, Theorem 1] and Kato [6, Remark, p. 236]). We shall show that an analogous characterization holds for right QF-3 rings. A right R-module M is called *finitely imbedded* (simply FI) if M is imbedded in some finitely generated right R-module. Note that any submodule and any factor module of FI modules is also FI, and any FI torsion-free right R-module is isomorphic to a submodule of some finitely generated torsion-free right R-module.

Theorem 2. *Let R be a ring. Then the following conditions* (1) *and* (2) *are equivalent*:

(1) *R is right QF-*3.

(2) (a) *If* $0\to L\to M\to N\to 0$ *is an exact sequence such that M is FI* (*or finitely generated*), *and L and N are torsionless, then M is torsionless.*

(b) *For every FI right R-module M,* $M^*=0$ *if and only if M is a torsion module.*

Proof. (1)$\Rightarrow$(2). (a) is clear since any torsionless module is torsion-free. (b) follows from the definition of FI-modules.

(2)$\Rightarrow$(1). The proof is due to Wu, Mochizuki and Jans [16]. Let M be an FI torsion-free right R-module and K the kernel of a canonical map $M\to M^{**}$. As is well-known, K is the smallest submodule of M such that M/K is torsionless. Moreover let K' be the kernel of a canonical map $K\to K^{**}$. Then we have an exact sequence $0\to K/K'\to M/K'\to M/K\to 0$. Since M/K and K/K' are torsionless, so is M/K' by (a). Hence $K=K'$, so $K^*=0$ and K is a torsion module by (b). Consequently $K=0$ since K is torsion-free. Thus M is torsionless.

Let $\mathfrak{M}$ be the category of right R-modules and let $\mathfrak{M}_f$ be the full subcategory of $\mathfrak{M}$ whose objects are FI R-modules. An R-module M is torsionless, i.e., M can be imbedded in a products of copies of R if and only if for any non-zero submodule N of M a canonical map $M^* \to N^*$ is non-zero. Now we adopt the latter condition as the definition of torsionless R-modules in $\mathfrak{M}_f$. We denote by $\mathfrak{L}$ (resp. $\mathfrak{L}_f$) the class of torsionless modules in $\mathfrak{M}$ (resp. $\mathfrak{M}_f$), and denote by $\mathfrak{T}$ (resp. $\mathfrak{T}_f$) the class of modules M in $\mathfrak{M}$ (resp. $\mathfrak{M}_f$) such that M^* ($=\mathrm{Hom}_R(M, R))=0$. Then it is clear that R is right QF-3 if and only if $\mathfrak{L}_f$ is closed under taking essential extensions in $\mathfrak{M}_f$. On the other hand since M is a torsion module if and only if $N^*=0$ for any submodule N of M, (2)-(b) in Theorem 2 is equivalent to the condition that $\mathfrak{T}_f$ is closed under taking submodules (see [15, Proposition 1]). Thus Theorem 2 shows that we can replace $\mathfrak{L}$, $\mathfrak{T}$ and $\mathfrak{M}$ with $\mathfrak{L}_f$, $\mathfrak{T}_f$ and $\mathfrak{M}_f$, respectively, in [1, Theorem 1] (if we consider right R-modules instead of left R-modules in it). That is,

Corollary 3. *Let R be a ring. Then the following conditions* (1) *and* (2) *are equivalent*:

(1) *$\mathfrak{L}_f$ is closed under taking essential extensions in $\mathfrak{M}_f$.*

(2) (*a*) *$\mathfrak{L}_f$ is closed under taking extensions in $\mathfrak{M}_f$ by elements of $\mathfrak{L}_f$.*

(*b*) *$\mathfrak{T}_f$ is closed under taking submodules.*

4. QF-3 and 1-Gorenstein rings

We shall give a sufficient condition for a right QF-3′ ring to be right QF-3. We call a ring R a *right n-Gorenstein* ring if R is (right and left) noetherian and R has the right self-injective dimension $\leq n$.

Lemma 1. *Let M be a D-noetherian R-module which is not a torsion module. Then there exists a submodule N of M such that M/N is s-uniform.*

Proof. If N is a maximal element in the set consisting of saturated submodules of M, N clearly satisfies the assertion.

Lemma 2. *Let M be an s-uniform right R-module. Then M can be imbedded into Q, where Q is the maximal right quotient ring of R.*

Proof. Since M is torsion-free, we have a non-zero map $M \to E$, where E denotes the injective hull of R_R. Then by [11, Lemma 1] this map is monomorphic. Thus we may assume that M is contained in E. Consider an exact sequence $0 \to M \cap Q \to M \to M/M \cap Q \to 0$. Since $M \cap Q \neq 0$, $M/M \cap Q$ is a torsion module. On the other hand, since $M/M \cap Q \cong M+Q/Q \subset E/Q$, $M/M \cap Q$ is torsion-free, so $M/M \cap Q=0$. Therefore we have $M \subset Q$.

REMARK. The proof of the above lemma shows that any s-uniform submodule of E is contained in Q if we regard Q as a submodule of E.

Proposition 3. *Let R be a right D-noetherian and right QF-3′ ring. Then for every finitely generated right R-module M, $M^*=0$ if and only if M is a torsion module.*

Proof. Let M be a finitely generated right R-module. If M is a torsion module, $M^*=0$ is clear. Assume M is not a torsion module. Since M is D-noetherian by [12, Lemma 2], there exists a submodule N of M such that M/N is s-uniform by Lemma 1. M/N can be imbedded in Q by Lemma 2 and so M/N is torsionless. Now we have an exact sequence $0 \to (M/N)^* \to M^*$. Thus we have $M^* \neq 0$ since $(M/N)^* \neq 0$.

Lemma 3. *Let R be a ring with right self-injective dimension ≤ 1. If $0 \to L_R \to M_R \to N_R \to 0$ is an exact sequence such that L is torsionless and N is a submodule of a finitely generated free right R-module, then M is torsionless.*

Proof. Since R has the right self-injective dimension ≤ 1, we have an exact sequence $0 \to N^* \to M^* \to L^* \to 0$. Hence we have a following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L^{**} & \to & M^{**} & \to & N^{**} . & & \end{array}$$

It follows that M is torsionless.

From Theorem 2, Proposition 3, Lemma 3 and [9, Theorem 1.1], the following theorem is immediate.

Theorem 3. *Let R be a right QF-3′ and right 1-Gorenstein ring. Then R is a QF-3 ring with finite dimension on both sides.*

The following theorem is a slight extension of Theorem 3.

Theorem 4. *Let R be a right noetherian right QF-3′ ring satisfying the descending chain condition on annihilator right ideals and assume R has the right self-injective dimension ≤ 1. Then R is right finite dimensional and right QF-3.*

Proof. Let M be a finitely generated torsion-free right R-module with finite dimension. Put dim $M=n$. First we show that such a module M is torsionless by induction on n. When $n=1$, the result holds by Lemma 2. Assume $n>1$. Then there exists a submodule N of M such that M/N is s-uniform. By inductional assumption N and M/N are torsionless. In particular M/N can be imbedded in R_R since M/N is s-uniform and $(M/N)^* \neq 0$. Hence by Lemma 3 M is torsionless.

Next we show that any finitely generated right R-module is finite dimensional. By using Lemma 1 inductively we have a following descending chain of right ideals I_i of R such that I_{i-1}/I_i is an s-uniform right R-module:

$$R = I_0 \supset I_1 \supset I_2 \supset \cdots .$$

Then R/I_i is a torsion-free module with dimension i for each i. Therefore R/I_i is torsionless and I_i is an annihilator right ideal. Hence the above chain terminates. Thus a right R-module R_R and consequently any finitely generated right R-module is finite dimensional.

Combining the above facts, we conclude that any finitely generated torsion-free right R-module is torsionless.

EXAMPLE. We give a noetherian QF-3′ ring which is not right (and left) finite dimensional. Let S be a commutative noetherian ring and put $R= S[X, Y]/(XY, Y^2)$, where $S[X, Y]$ is a polynomial ring with variables X and Y. Then R is a commutative (hence QF-3′) noetherian ring. Let x and y be the image of X and Y by the canonical map $S[X, Y] \to R$, respectively. Put $I_0=R$, $I_1=xR+yR$, $I_i=x^{i-1}R$ $(i\geq 2)$ and consider a following chain of ideals I_i of R:

$$R = I_0 \supset I_1 \supset I_2 \supset \cdots .$$

Since each factor I_i/I_{i+1} is isomorphic to an ideal yR, R is not finite dimensional. Hence in particular R is not QF-3 by [11. Proposition 1].

Recently Iwanaga [3, Theorem 1] showed that if R is a (right and left) 1-Gorenstein ring, $E\oplus E/R$ is an injective cogenerator. Here we show that the converse holds for noetherian right QF-3′ ring.

Lemma 4. *Let R be a right 1-Gorenstein ring. If $0\to M_R\to F_R\to L_R\to 0$ is an exact sequence where F is a finitely generated free module and L is torsionless, then M is reflexive.*

Proof. This is immediate from a following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M^{**} & \to & F^{**} & \to & L^{**} . & & \end{array}$$

Theorem 5. *Let R be a noetherain right QF-3′ ring and E the injective hull of R_R. If $E\oplus E/R$ is an injective cogenerator, then R is a 1-Gorenstein ring.*

Proof. Let Q be the maximal right quotient ring. If S is a simple torsion right R-module, S is imbedded in Q/R since Q/R is the torsion submodule of a cogenerator $E\oplus E/R$. On the other hand we have an isomorphism $\mathrm{Hom}_R(S, Q/R)\to \mathrm{Ext}^1_R(S, R)$. Therefore $\mathrm{Ext}^1_R(S, R)\neq 0$.

Now we show that every finitely generated torsionless right R-module is reflexive, which implies R is a left 1-Gorenstein ring by [5, Corollary 1.3]. Suppose that there exists a finitely generated torsionless right R-module M which is not reflexive. Then M_R can be imbedded in a finitely generated free module F_R. Since F is noetherian, there is a maximal element in the set consisting of submodules N of F such that N is not reflexive. We may assume that M is such a maximal element. By Theorem 3 R is QF-3. Hence, if F/M is torsion-free, F/M is torsionless and so M is reflexive by Lemma 4. This is a contradiction. Therefore the torsion submodule L/M of F/M is non-zero. If $L\supset L_1\supset L_2\supset\cdots\supsetneqq M$ is a descending chain of submodules L_i of L containing M, we have an ascending chain $L^*\subset L_1^*\subset L_2^*\subset\cdots\subset M^*$. Then there is an integer n such that $L_i^*=L_{i+1}^*$ for each $i\geq n$ since M^* is noetherian. But by the maximality of M, L_i is reflexive and hence we have $L_i=L_{i+1}$ for each $i\geq n$. This shows L/M is artinian. Let P/M is a simple submodule of L/M and consider an exact sequence $0\to M\to P\to P/M\to 0$. Then we have an exact sequence $0\to P^*\to M^*\to \mathrm{Ext}_R^1(P/M, R)\to 0$. Since M^* and P^* are reflexive and $\mathrm{Ext}_R^1(P/M, R)\neq 0$, a derived map $M^{**}\to P^{**}$ is not isomorphic. By [7, Proposition 1.2] and [12, Proposition 6] $\mathrm{Ext}_R^1(P/M, R)$ is a torsion module. Therefore we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & P & \longrightarrow & P/M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & M^{**} & \to & P^{**} . & & & & \end{array}$$

Since P is reflexive and P/M is simple, M is reflexive. This is a contradiction. Thus R is a 1-Gorenstein ring.

For any (right and left) noetherian ring R, the socle of R_R is zero if and only if so is that of ${}_RR$ (see [2, Theorem 5]). Therefore, by Theorem 5 and [3, Corollary 2], we have:

Corollary 4. *Let R be a noetherian right QF-3′ ring. If E/R is an injective cogenerator, then R is a 1-Gorenstein ring with zero socle (on both sides), where E is the injective hull of R_R.*

Osaka City University

References

[1] R.R. Colby and E.A. Rutter, Jr.: *Semi-primary QF-3 rings,* Nagoya Math. J. **32** (1968), 253–257.

[2] C.M. Ginn and P.B. Moss: *A decomposition theorem for noetherian orders in artinian rings,* Bull. London Math. Soc. **9** (1977), 177–181.

[3] Y. Iwanaga: *On rings with self-injective dimension* ≤ 1, Osaka J. Math. **15** (1978), 33–46.
[4] J.P. Jans: *Projective injective modules,* Pacific J. Math. **9** (1959), 1103–1108.
[5] J.P. Jans: *Duality in Notherian rings,* Proc. Amer. Math. Soc. **12** (1961), 829–835.
[6] T. Kato: *Torsionless modules,* Tohoku Math. J. **20** (1968), 233–242.
[7] K. Masaike: *On quotient rings and torsionless modules,* Sci. Rep. Tokyo Kyoiku Daigaku A **11** (1971), 26–30.
[8] E.A. Rutter, Jr.: *A characterization of QF-3 rings,* Pacific J. Math. **51** (1974), 533–536.
[9] H. Sato: *On localization of a 1-Gorenstein ring,* Sci. Rep. Tokyo Kyoiku Daigaku A **13** (1977), 188–193.
[10] B. Stenström: Rings of quotients, Grundlehren Math. Wiss. 217, Springer, 1975.
[11] T. Sumioka: *On non-singular QF-3′ rings with injective dimension* ≤ 1, Osaka J. Math. **15** (1978), 1–11.
[12] T. Sumioka: *On finite dimensional QF-3′ rings,* Proc. of the 10-th Symposium of Ring Theory, Okayama, Japan, 1978.
[13] H. Tachikawa: *On left QF-3 rings,* Pacific. J. Math. **32** (1970), 255–268.
[14] H. Tachikawa: Quasi-Frobenius rings and generalization, Lecture Notes in Math. 351, Springer, 1973.
[15] C. Vinsonhaler: *A note on two generalizations of QF-3,* Pacific J. Math. **40** (1972), 229–233.
[16] L.E.T. Wu, H.Y. Mochizuki and J.P. Jans, *A characterization of QF-3 rings,* Nagoya Math. J. **27** (1966), 7–13.