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A REMARK ON SIMPLE SYMMETRIC SETS

Dedicated to Professor Yoshikazu Nakai on his 60th birthday

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Nobusawa [1] has shown that if A is a simple symmetric set then the group of displacements $H(A)$ is almost simple. The purpose of this note is to prove the converse. For the completeness we shall restate the result of Nobusawa in a slightly extended way.

A symmetric set is a set A carrying a binary operation $a\circ b$ which satisfies the following identical relations:

(1) $a\circ a=a$.
(2) $(x\circ a)\circ a=x$.
(3) $(x\circ y)\circ a=(x\circ a)\circ(y\circ a)$.

The mapping $\iota(a)$: $A\to A$ defined by $x^{\iota(a)}=x\circ a$ is an automorphism of A by (3) and we have the following:

(4) $a^{\iota(a)}=a$.
(5) $\iota(a)^2=1$.
(6) For any automorphism σ of A

$$\sigma^{-1}\iota(a)\sigma=\iota(a^\sigma)\,.$$

Particularly we have

(7) $\iota(b)^{-1}\iota(a)\iota(b)=\iota(a^{\iota(b)})=\iota(a\circ b)$.

The group $G(A)$ is the subgroup of Aut A (the automorphism group of A) generated by $\iota(A)=\{\iota(a)\,|\,a\in A\}$. The group $H(A)$ is the subgroup of $G(A)$ generated by $\{\iota(a)\iota(b)\,|\,a, b\in A\}$ and is called the group of displacements. Then $|G(A)\colon H(A)|\leqq 2$ and $H(A)=\langle\iota(e)\iota(a)\,|\,a\in A\rangle$ for a fixed element e of A.

The set $\iota(A)$ is a collection of conjugate classes of involutions in $G(A)$, and is a symmetric set with the binary operation $\iota(a)\circ\iota(b)=\iota(b)^{-1}\iota(a)\iota(b)$. The mapping ι: $A\to\iota(A)$ is an epimorphism, and if ι is an isomorphism then A is called *effective*.

If $G(A)$ acts transitively (or primitively) on A, then we call A *transitive* (or *primitive*). Note that A is transitive if and only if $H(A)$ is transitive on A, and if A is transitive then $\iota(A)$ is a conjugate class of involutions in $G(A)$.

Let $f\colon A\to B$ be a homomorphism of a symmetric set A to another symmetric set B. An inverse image $f^{-1}(b)$ for $b\in f(A)$ is called a *coset* of f. We call f *proper* if $|f(A)|>1$ and f is not a monomorphism. Thus f is proper if and only if there is a coset C such that $A\supsetneqq C$ and $|C|>1$.

A symmetric set A is called *simple* if $|A|>2$ and there is no proper homomorphism of A to another symmetric set. If A is simple then A is not trivial, where A is called *trivial* if $G(A)=1$.

Proposition 1. *Let A be a symmetric set and K a normal subgroup of $G(A)$. Denote the K-orbit containing $a\in A$ by $\bar{a}$ and let $\bar{A}=\{\bar{a}\,|\,a\in A\}$. Define a binary operation on $\bar{A}$ by $\bar{a}\circ\bar{b}=\overline{a\circ b}$. Then this is well defined and $\bar{A}$ is a symmetric set. Further the mapping $f\colon A\to\bar{A}$ $(a\mapsto\bar{a})$ is an epimorphism.*

Proof. Let $a'=a^{\sigma}$ and $b'=b^{\rho}$ for σ, $\rho\in K$. Then

$$\begin{aligned}a'\circ b' &= a^{\sigma}\circ b^{\rho}=(a\circ b^{\rho\sigma^{-1}})^{\sigma}\\ &= (((a\circ b)\circ b)\circ b^{\rho\sigma^{-1}})^{\sigma}\\ &= (a\circ b)^{\iota(b)\iota(b^{\rho\sigma^{-1}})\sigma},\end{aligned}$$

where $\iota(b)\iota(b^{\rho\sigma^{-1}})\sigma=\iota(b)^{-1}(\rho\sigma^{-1})^{-1}\iota(b)(\rho\sigma^{-1})\sigma\in K$. Hence $\overline{a'\circ b'}=\overline{a\circ b}$ and the binary operation on $\bar{A}$ is well defined. The other parts are evident.

The epimorphism $f\colon A\to\bar{A}$ in the proposition above is called the *canonical epimorphism.*

Proposition 2. *If a symmetric set A is simple then it is transitive.*

Proof. Suppose that A is simple and intransitive. Let $\bar{a}$ be the $G(A)$-orbit containing $a\in A$ and $\bar{A}=\{\bar{a}\,|\,a\in A\}$. Then there is the canonical epimorphism $f\colon A\to\bar{A}$. Here $|\bar{A}|>1$ by the intransitivity. Hence f must be an isomorphism by simplicity and we have $a^{G(A)}=a$ for any $a\in A$. Thus $G(A)=1$, which is a contradiction.

Proposition 3. *If a symmetric set A is primitive, then A is simple.*

Proof. Suppose A is not simple. Then there is a proper epimorphism $f\colon A\to B$. Thus there exists a coset $C=f^{-1}(b)$ such that $A\supsetneqq C$ and $|C|>1$. Clearly C is a non-trivial set of imprimitivity of $G(A)$, and hence A is imprimitive.

Now we have the following

Theorem 1 (Nobusawa). *If A is a simple symmetric set, then $H(A)$ is a unique minimal normal subgroup of $G(A)$. Hence $H(A)$ is either a simple group or a direct product of two simple groups which are isomorphic.*

Proof. Let $K\neq 1$ be a normal subgroup of $G(A)$, $\bar{A}$ the symmetric set consisting of all K-orbits on A and let f: $A\rightarrow\bar{A}$ be the canonical epimorphism. Since $K\neq 1$ there is an element $a\in A$ such that $|a^K|>1$, and hence f is not an isomorphism. Then by the simplicity of A we have $|\bar{A}|=1$, that is, K is transitive on A. Thus for any a, $b\in A$ there is an element $\rho\in K$ such that $b=a^{\rho}$, and then $\iota(a)\iota(b)=\iota(a)^{-1}\rho^{-1}\iota(a)\rho\in K$. Therefore $K\supseteq H(A)$ and this shows that $H(A)$ is a unique minimal normal subgroup of $G(A)$. The second half of the theorem follows from the fact $|G(A)$: $H(A)|\leqq 2$.

Remark. Nobusawa [1] states the theorem under the assumption that A is primitive.

We also remark the following

Proposition 4. *If A is simple then it is effective.*

Proof. Consider the epimorphism ι: $A\rightarrow\iota(A)$. If $|\iota(A)|=1$, then for any x, $a\in A$ $x^{\iota(a)}=x^{\iota(x)}=x$ and hence $G(A)=1$, which contradicts the simplicity of A. Thus $|\iota(A)|>1$ and hence ι is an isomorphism.

From Proposition 2 and 4, a simple symmetric set A may be regarded as a conjugate class of involutions in $G(A)$, where the group $G(A)$ has the property property as in Theorem 1.

We have also the following

Theorem 2. *Let A be a simple symmetric set. If A is imprimitive then $H(A)$ is a simple group.*

Proof. Let $G=G(A)$ and $H=H(A)$. we may assume that A is a set of involutions in G. Suppose H is not simple. Then $H=K\times K^a$ with $a\in A$ and K is simple. Let $L=\{kk^a|k\in K\}$. Then L is a subgroup of H, $C_G(a)\cap H=L$ and $C_G(a)=L+La$. We claim that $C_G(a)$ is a maximal subgroup of G and hence A is primitive. Suppose that there is a subgroup M of G which contains $C_G(a)$ properly. Then $M^*=M\cap H\gneqq L$ and hence there is an element kk_1^a in M^* such that $k\neq k_1$. Then $(kk^a)(kk_1^a)^{-1}=(kk_1^{-1})^a\in M^*\cap K^a$ and hence $M^*\cap K^a\neq 1$. Since the projection of M^* on K^a covers the whole of K^a, $M^*\cap K^a$ is a normal subgroup of K^a and hence we have $K^a\leqq M^*$. In the same way we have $K\leqq M^*$. Thus M contains H and we have $M=G$, which shows the maximality of $C_G(a)$.

To prove the converse of Theorem 1, we need the following

Proposition 5. *Let $f: A\rightarrow\bar{A}$ be an epimorphism of a symmetric set A to another symmetric set $\bar{A}$ and denote $f(a)$ by $\bar{a}$. Then there is an epimorphism f^*: $G(A)\rightarrow G(\bar{A})$ such that $\overline{\iota(a)}=\iota(\bar{a})$ and $\overline{a^{\sigma}}=\bar{a}^{\bar{\sigma}}$ for $a\in A$ and $\sigma\in G(A)$, where $\overline{\iota(a)}$ and $\bar{\sigma}$ denote $f^*(\iota(a))$ and $f^*(\sigma)$ respectively.*

Proof. For x, $a\in A$, $\overline{x\circ a}=\bar{x}\circ\bar{a}$, *i.e.* $\overline{x^{\iota(a)}}=\bar{x}^{\iota(\bar{a})}$. Hence for $\sigma=\iota(a_1)\cdots\iota(a_r)$ $\in G(A)$ $\overline{x^{\sigma}}=\bar{x}^{\iota(\bar{a}_1)\cdots\iota(\bar{a}_r)}$. Particularly if $\sigma=1$ then $\iota(\bar{a}_1)\cdots\iota(\bar{a}_r)=1$, and hence the mapping $f^*: G(A) \to G(\bar{A})$ defined by $f^*(\iota(a_1)\cdots\iota(a_r)) = \iota(\bar{a}_1)\cdots\iota(\bar{a}_r)$ is well defined. It is now easy to see that f^* is an epimorphism satisfying the conditions in the proposition.

The epimorphism f^* in the proposition is called the *extension* of f.

Now we have the following

Theorem 3. *Let G be a group, A a conjugate class of involutions in G and suppose $G = \langle A \rangle$. If $H = \langle ab \mid a, b \in A \rangle$ is a minimal normal subgroup of G and $|A| > 2$, then the symmetric set A with binary operation $a \circ b = b^{-1}ab$ is simple.*

Proof. Suppose A is not simple and $|A| > 2$. Then there is a proper epimorphism $f: A \to \bar{A}$. Denote $f(a)$ by $\bar{a}$. Then f can be extended to the group epimorphism $f^*: G(A) \to G(\bar{A})$. There is also a natural epimorphism $g: G \to G(A)$ such that $g(a_1 \cdots a_r) = \iota(a_1)\cdots\iota(a_r)$ for $a_i \in A$ $(i = 1, \cdots, r)$. Thus we have an epimorphism $f^{**} = f^* \circ g: G \to G(\bar{A})$, and we denote $f^{**}(\sigma)$ by $\bar{\sigma}$. Then we have

$$\overline{x^\sigma} = \bar{x}^{\bar{\sigma}}$$

for $x \in A$ and $\sigma \in G$.

Since $f^{**}(H) = H(\bar{A})$, f^{**} induces an epimorphism $f^{**}|_H: H \to H(\bar{A})$, and $\mathrm{Ker}\,(f^{**}|_H)$ is a normal subgroup of G which is contained in H.

Now since f is not an isomorphism there are different elements a and b of A such that $\bar{a} = \bar{b}$. Then $f^{**}(ab^{-1}) = \iota(\bar{a})\iota(\bar{b})^{-1} = 1$ and hence we have $1 \neq ab^{-1} \in \mathrm{Ker}\,(f^{**}|_H)$. Thus $\mathrm{Ker}(f^{**}|_H) \neq 1$. Next we show that $H \neq \mathrm{Ker}(f^{**}|_H)$. Since $|\bar{A}| > 1$ there are elements $a, b \in A$ such that $\bar{a} \neq \bar{b}$. Let $b = a^\sigma$ with $\sigma = c_1 \cdots c_r$, $c_i \in A$ $(i = 1, \cdots, r)$. Then there is an i such that $\overline{a^{c_1 \cdots c_i}} = \bar{a}$ and $\overline{a^{c_1 \cdots c_{i+1}}} \neq \bar{a}$. Let $d = a^{c_1 \cdots c_i}$, $c = c_{i+1}$. Then $d, c \in A$ and $\bar{d}^{\bar{c}} \neq \bar{d}$. Since $f^{**}(dc) = \iota(\bar{d})\iota(\bar{c})$ takes $\bar{d}$ to $\bar{d}^{\iota(\bar{c})} = \overline{d^c}$, $dc \in H - \mathrm{Ker}(f^{**}|_H)$.

Thus $\mathrm{Ker}(f^{**}|_H)$ is a non-trivial normal subgroup of G which is properly contained in H, and H is not a minimal normal subgroup of G.

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Reference

[1] N. Nobusawa: *Simple symmetric sets and simple groups*, Osaka J. Math. **14** (1977), 411–415.