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CORRECTIONS TO "VANISHING THEOREMS FOR COHOMOLOGY GROUPS ASSOCIATED TO DISCRETE SUBGROUPS OF SEMISIMPLE LIE GROUPS"

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The paper referred to in the title appeared in this journal in 1966 (vol. 3, pp. 243–256). Unfortunately Theorem 1 of the paper as it stands is incorrect. There is a wrong assertion made on p. 249 in the proof of Assertion III (about representations of $SL(2)$). (I am indebted to Dr. Ruh and Dr. Im Hof for drawing my attention to the error). We give here a modified (weaker) result with the requisite additional arguments for its proof.

The notation is as in the 1966 paper: G will denote a connected real simple Lie group and ρ a representation of G on a complex vector space F; K will be a maximal compact subgroup and Γ a discrete uniform subgroup; $X=G/K$ and $H^p(\Gamma, X, \rho)$ will be the p^{th} cohomology of the complex of F- valued Γ-equivariant smooth forms on X (if Γ acts fixed point free on X, these are simply the Eilenberg-Maclane groups of Γ with coefficients in ρ). Let $\mathfrak{g}_0$ (resp $\mathfrak{k}_0$) denote the Lie algebra (resp. Lie subalgebra) of G (resp. K). Let $\mathfrak{g}$ (resp. $\mathfrak{k}$) denote the complexification of $\mathfrak{g}_0$ (resp. k_0). Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ *w.r.t.* the Killing form. Let $\mathfrak{h}_{\mathfrak{k}_0}$ be a Cartan-subalgebra of $\mathfrak{k}_0$ and $\mathfrak{h}_0 \supset \mathfrak{h}_{\mathfrak{k}_0}$ a Cartan-subalgebra of $\mathfrak{g}_0$. Let $\mathfrak{h}$ (resp. $\mathfrak{h}_{\mathfrak{k}}$) be the $\boldsymbol{C}$-span of $\mathfrak{h}_0$ (resp. $\mathfrak{h}_{\mathfrak{k}_0}$) in $\mathfrak{g}$. Let $\mathfrak{h}_{\mathfrak{p}} \subset \mathfrak{h} \cap \mathfrak{p}$. Let φ denote the Killing form on $\mathfrak{g}$ and Δ be the root system of $\mathfrak{g}$ *w.r.t.* $\mathfrak{h}$. For $\alpha \in \Delta$, let H_α be the unique element of $\mathfrak{h}$ such that $\varphi(H, H_\alpha)=\alpha(H)$ for all $H \in \mathfrak{h}$. Let $\mathfrak{h}^*=\sum_{\alpha\in\Delta} \boldsymbol{R}H_\alpha=i\mathfrak{h}_{\mathfrak{k}} \oplus \mathfrak{h}_{\mathfrak{p}}$. Let θ be the Cartan-involution of $\mathfrak{g}_0$ determined by $\mathfrak{k}_0$ as well as its extension to $\mathfrak{g}$. Let

$$
\begin{aligned}
\mathrm{A} &= \{\alpha \in \Delta \mid \theta(E_\alpha) = E_\alpha\} \\
\mathrm{B} &= \{\alpha \in \Delta \mid \theta(\alpha) \neq \alpha\} \\
\mathrm{C} &= \{\alpha \in \Delta \mid \theta(E_\alpha) = -E_\alpha\}
\end{aligned}
$$

(Here E_α is a root vector corresponding to α); then $\Lambda=A\cup B\cup C$. Also θ stabilises $\mathfrak{h}$ as well as Δ, $A\cup C$ and B; moreover $\theta(\alpha)=\alpha$ if $\alpha\in A\cup C$. We will say in the sequel that an order on the (real) dual of $\mathfrak{h}^*$ is *admissible* if it is obtained in the following manner: let $H_1, \cdots, H_1$ be an orthonormal basis of $\mathfrak{h}^*$

so chosen that $H_1, \cdots, H_p$ constitute a basis of $i\mathfrak{h}_{\mathfrak{k}_0}$; and $\alpha \in$ dual of $\mathfrak{h}^*$ is positive if the first nonvanishing $\alpha(H_i)$ is positive. If we denote by O one such an order for any subset E in the dual of $\mathfrak{h}$, $E^+(O)$ denotes the positive elements in E.

For an irreducible representation ρ of G, let $\Lambda_\rho(O)$ be the highest weight of ρ *w.r.t.* $\mathfrak{h}$ and the order O. Let $\Sigma_2(O)=C^+(O) \cup \{\alpha \in B^+(O) \mid \alpha > \theta(\alpha)\}$ and $\sum_o (O)=\{\alpha \in \sum_2 (O) \mid \varphi(\Lambda_\rho(O), \alpha \neq 0)\}$. With this notation we have

Theorem *Let ρ be a finite dimensional irreducible representation of G. Then if $\sum_o(O)$ contains (strictly) more than q elements for every admissible O, then $H^p(\Gamma, X, \rho)=0$ for $0 \leq p \leq q$.*

As is deduced in [1] from the work of Matsushima and Murakami, the vanishing of $H^p(\Gamma, X, \rho)$, $0 \leq p \leq q$, will follow from the following:

Let $X_1, \cdots, X_n$ be a basis of $\mathfrak{g}$ with $\varphi(X_i, X_j)=\pm\delta_{ij}$ and $X_i \in k$ for $1 \leq i \leq N$ forming a basis of K. Let $c'=-\sum_{1\leq i\leq N} X_i^2$ and $c=-\sum_{1\leq i\leq N} X_i^2+\sum_{N<i\leq n} X_i^2$ in the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. We denote by ρ the extension of ρ to $U(\mathfrak{g})$ as well. Let E be the p^{th} exterior power of $\mathfrak{p}$ and T_ρ^p: $F \otimes E \to F \otimes E$ be the endomorphism

$$T_\rho^p = 2(\rho\otimes 1)(c)+(1\otimes\sigma)(c')-(\rho\otimes 1)(c')-(\rho\otimes\sigma)(c') .$$

Let $\langle\, , \rangle$ denote the natural hermitian scalar product on $F \otimes E$ deduced from the involution θ and φ on $\mathfrak{g}_0$. Then the hermitian quadratic form,

$$\eta \mapsto \langle T_\rho^p \eta, \eta\rangle$$

is positive definite for $p \leq q$ provided the hypothesis of the theorem holds.

As in [1], let $E=\sum_{\mu \in M(O)} E_\mu$ be a decomposition of E into its ($\mathfrak{k}$-) irreducible components and similarly $F=\sum_{\lambda\in L(O)} F_\lambda$ a decomposition of F into irreducible $\mathfrak{k}$-modules. The indexing sets are the dominant weights *w.r.t.* an *admissible* order O on the dual of $\mathfrak{h}^*$. We denote by $V_{\lambda\mu}^\nu$ the $\mathfrak{k}$-irreducible components of $V_{\lambda\mu}=F_\lambda \otimes E_\mu$; then each $V_{\lambda\mu}^\nu$ is contained in a eigen-space of T^p and let $a(\lambda, \mu, \nu)$ be the corresponding eigen-value. Let $a(\lambda, \mu)=a(\lambda, \mu, \lambda+\mu)$.

We assume in the sequel that G is simple (over $\boldsymbol{R}$)

Assertion I. *$a(\lambda, \mu, \nu) \geq a(\lambda, \mu)$ for all $\lambda \in L(O)$, $\mu \in M(O)$; equality occurs only if $\nu=\lambda+\mu$.*

This is proved in [1].

Assertion II. *Let f_λ be a dominant weight vector of F_λ, $\lambda \in L(O)$. Suppose that there exists $\alpha \in B^+(O)$ such that $E_{\alpha_0} f_\lambda \neq 0 \in F_{\lambda_1}$, $\lambda_1 \in L(O)$; then $a(\lambda, \mu) > a(\lambda_1, \mu)$.*

For a proof see [1].

Assertion III. *Suppose that* f_λ *is a (non-zero) dominant weight vector in* F_λ *and that* $E_\alpha f_\lambda = 0$ *for all* $\alpha \in A^+(O) \cup B^+(O)$. *Then if* $a(\lambda, \mu) = 0$ *for some* $\mu \in M(0)$, *there exists an admissible order* O' *on the dual of* $\mathfrak{h}^*$ *such that* f_λ *is dominant for* O' *(i.e.* $E_\alpha f = 0$ *for all* $\alpha \in \Delta^+(O')$*).*

We need the following Lemma.

Lemma. *Let* $E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ *and* $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *be the standard basis of* $SL(2)$. *Let* τ *be a finite dimensional irreducible representation of* $SL(2)$ *and* v *a weight vector for* H *of weight* λ. *Then*

$$\tau(E_+E_- + E_-E_+)(v) = \lambda' \cdot v$$

with $\lambda' \geq |\lambda|$; *also equality occurs if and only if* λ *is dominant (positive or negative).*

Proof. Let $b = E_+E_- + E_-E_+ + H^2/2$ be the Casimir element. Then $\tau(b)$ is a scalar equal to $\lambda_0(H)^2//2 + \lambda_0(H)$ where λ_0 is the positive dominant weight. Thus

$$\tau(E_+E_- + E_-E_+)v = (\tau(b) - \tau(H^2/2))v = \lambda_0(H)^2/2 + \lambda_0(H) - \lambda(H)^2/2.$$

Now for any weight ν of τ, $\nu(H)^2 \leq \lambda_0(H)^2$, equality occuring if and only if ν is extremal. The lemma is now immediate.

Let f_λ be a unit dominant weight vector in F_λ ($\lambda \in L(O)$). Then we have

$$\begin{aligned} \rho(c)f_\lambda &= \{\textstyle\sum_{\alpha\in\Delta^+(O)} \rho(E_\alpha E_{-\alpha} + E_{-\alpha}E_\alpha) + \sum_{1\leq i\leq l} \rho(H_i)^2\} f_\lambda \\ &= (\textstyle\sum_{\alpha\in A^+(O)\cup B^+(O)} \lambda(H_\alpha + H_{\theta(\alpha)})/2) \cdot f_\lambda \\ &\quad + (\textstyle\sum_{1\leq i\leq p'} \lambda(H_i))^2 f_\lambda + (\sum_{p'<i\leq 1} \rho(H_i)^2) f_\lambda \\ &\quad + \textstyle\sum_{\alpha\in C^+(O)} \rho(E_\alpha E_{-\alpha} + E_{-\alpha}E_\alpha) f_\lambda \end{aligned}$$

We have used here the fact $E_\alpha f_\lambda = 0$ for $\alpha \in A^+(O) \cup B^+(O)$. Also, we have

$$\rho(c')f_\lambda = \textstyle\sum_{1\leq i<p'} \lambda(H_i)^2 + \sum_{\alpha\in\Sigma_1(O)} \lambda(H_\alpha + H_{\theta(\alpha)})/2 \cdot f_\lambda$$

where $\sum_1(O) = A^+(O) \cup \{\alpha \in B^+(O) \mid \alpha > \theta(\alpha)\}$. Also for dominant weight vector $e_\mu \in E_\mu (\mu \in M(O))$ of unit length

$$\begin{aligned} \sigma(c')e_\mu &= \{\textstyle\sum_{1\leq i\leq p'} \mu(H_i)^2 + \sum_{\alpha\in\Sigma_1(O)} \mu(H_\alpha + H_{\theta(\alpha)})/2\} \cdot e_\mu \\ (\rho\otimes\sigma)(c')(f\otimes e_\mu) &= \{\textstyle\sum_{1\leq i\leq p'} (\lambda+\mu)(H_i)^2 \\ &\quad + \textstyle\sum_{\alpha\in\Sigma_1(O)} (\lambda+\mu)(H_\alpha + H_{\theta(\alpha)})/2\} f_\lambda \otimes e_\mu \end{aligned}$$

leading to:

$$\begin{aligned} \langle T^p_\rho f_\lambda \otimes e_\mu, f_\lambda \otimes e_\mu \rangle &= 2\textstyle\sum_{\alpha\in C^+(O)} \langle \rho(E_\alpha E_{-\alpha} + E_{-\alpha}E_\alpha) f_\lambda \otimes e_\mu, f_\lambda \otimes e_\mu \rangle \\ &\quad + 2\textstyle\sum_{p<i\leq l} \langle \rho(H_i)^2 f_\lambda \otimes e_\mu, f_\lambda \otimes e_\mu \rangle \\ &\quad + \textstyle\sum_{\alpha\in B^+(O)} \lambda(H_\alpha + H_{\theta(\alpha)}) \\ &\quad - 2\textstyle\sum_{1\leq i\leq p'} \lambda(H_i)\mu(H_i) . \end{aligned}$$

Consider now $\langle\rho(E_\alpha E_{-\alpha}+E_{-\alpha}/E_\alpha)f_\lambda\otimes e_\mu, f_\lambda\otimes e_\mu\rangle$ for $\alpha\in C^+(O)$. We claim that this can be zero if and only if f_λ is an extremal vector for $\mathfrak{g}(\alpha)$. To see this we decompose F_λ into $\mathfrak{g}(\alpha)$-irreducible components

$$F_\lambda=\sum_q V(r)$$

We assume as we may that $V(r)$ are mutually orthogonal with respect to the scalar product on F. Let $f_\lambda=\sum f(r)$ with $f(r)\in V(r)$. We then have

$$\langle\rho(E_\alpha E_{-\alpha}+E_{-\alpha}E_\alpha)f_\lambda\otimes e_\mu, f_\lambda\otimes e_\mu\rangle=\sum\langle\rho(E_\alpha E_{-\alpha}+E_{-\alpha}E_\alpha)f(r)\otimes e_\mu, f(r)\otimes e_\mu\rangle$$

Each term on the right hand side is non-negative so that the left hand side is non negative and equals zero if and only if

$$\langle\rho(E_\alpha E_{-\alpha}+E_{-\alpha}E_\alpha)f(r)\otimes e_\mu, f(r)\otimes e_\mu\rangle=0$$

for *all* r. Now by the lemma above, this means that for every r, $f(r)$ is an extremal weight vector for $\mathfrak{g}(\alpha)$. Also E_α (resp. $E_{-\alpha}$) annihilates $f(r)$ if $\lambda(H_\alpha)\geq 0$ (resp. ≤ 0)—$f(r)$ is a weight vector whose weight is $H_\alpha\mapsto\lambda(H_\alpha)$ for *all* r. Thus we see that all the $f(r)$ are annihilated by the same root vector: E_α or $E_{-\alpha}$. Thus $E_\alpha f_\lambda=0$ or $E_{-\alpha}f_\lambda=0$ proving our contention. Also snice $E_\alpha f_\lambda=0$ for all $\alpha\in A^+(O)\cup B^+(O)$ one sees that $\lambda(H_\alpha+H_{\theta(\alpha)})\geq 0$. Since $H_i\in\mathfrak{h}_\mathfrak{p}$ for $i>p'$, $\langle\rho(H_i)^2 f_\lambda, f_\lambda\rangle\geq 0$. Finally

$$\sum_{1\leq i\leq p}\lambda(H_i)\mu(H_i)=\langle\lambda,\mu\rangle=\langle\lambda,\sum_{1\leq i\leq p}\pm(\alpha_i+\theta(\alpha_i))/2\rangle$$

for suitable elements α_i, $1\leq i\leq p$, in $\sum_2(O)$. We see then that if $\langle T_\mathfrak{p}^p(f\otimes e_\mu), f_\lambda\otimes e_\mu\rangle=0$, we must necessarily have the following

(i) $\rho(H_i)f_\lambda=0$, $p'<i\leq l$

(ii) $\lambda(H_\alpha+H_{\theta(\alpha)})\geq 0$ for all $\alpha\in B^+(O)$

(iii) $E_\alpha f_{-\lambda}=0$ or $E_{-\alpha}f_\lambda=0$ for any $\alpha\in C^+(O)$.

From the fact that $\rho(H_i)f_\lambda=0$ for $p'<i\leq l$ we see that f_λ is an eigen-vector for all of $\mathfrak{h}$ with the corresponding weight $\tilde{\lambda}$ being the unique extension of λ which is zero on $\mathfrak{h}_\mathfrak{p}$. Let $h_\lambda(=h_{\tilde{\lambda}})$ be the unique element of $i\mathfrak{h}_\mathfrak{k}$ such that $\varphi(h_\lambda, h)=\lambda(h)$ for all $h\in\mathfrak{h}_\mathfrak{k}$. Then we take an orthonormal basis $H_1,\cdots,H_l$ of $\mathfrak{h}^*$ yielding an order O' on the dual of $\mathfrak{h}^*$ with H_1 being a *positive* multiple of h_λ. Now since $\alpha(h_\lambda)\geq 0$ for all $\alpha\in A^+(O)\cup B^+(O)$ and $E_\alpha f_\lambda=0$ for $\pm\alpha\in C^+$ if $\pm\alpha(h_\lambda)\geq 0$, we see that with respect to the order O', f_λ is an extremal vector. We need only conclude that O' is admissible and this is obvious. This completes the proof of Assertion III.

Assertion IV. *Fix an admissible order O on the dual of $\mathfrak{h}^*$. Let Λ be the dominant weight of ρ w.r.t. O and let λ_0 denote the restriction of λ to $i\mathfrak{h}_\mathfrak{k}$; then $a(\lambda_0,\mu)>0$ for all $\mu\in M(O)$ provided that there are at least $(q+1)$ roots $\alpha\in\sum_2(O)$ such that $\Lambda(H_\alpha+H_{\theta(\alpha)})>0$.*

The proof is given in [1].

REMARK. We have proved the theorem for the case of simple $\mathfrak{g}$ but the general case is immediate from this.

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References

[1] M.S. Raghunathan: *Vanishing theorems for cohomology groups associated to discrete subgroups of semisimple Lie groups,* Osaka J. Math. **3** (1966), 243–256.