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INTEGRAL TRANSFORMATIONS ASSOCIATED WITH DOUBLE FIBERINGS

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Introduction

For a double fibering $X \xleftarrow{\pi_1} B \xrightarrow{\pi_2} Y$ of differential manifolds, an integral transformation L_B: $C_0^\infty(Y) \to C^\infty(X)$ is defined by $L_B=(\pi_1)_!(\pi_2)^*$ (see Section 1 for the precise definition). This class of integral transformations, introduced in [2], includes many classical ones such as the Radon transformation, the spherical mean operators ([7]), the mean value operators on symmetric spaces ([4]), etc.. In each particular case deep investigations have been carried out (see [2], [5] for example).

In this paper, using the fact that L_B is a Fourier integral operator in the sense of [6], we study the question how much L_B improves the regularity of functions. An answer is described by an integer k_B geometrically associated with the double fibering $X \leftarrow B \to Y$ (Theorem 2.4).

Section 1 gives the precise definition and examples of L_B. Section 2 defines k_B, states the main theorem (Theorem 2.4) and applies it to the examples. Section 3 proves Theorem 2.4. Section 4 gives a sufficient condition for L_B to have a parametrix. Section 5 and 6 study the mean value operators on symmetric spaces of non-compact and compact type respectively.

This paper includes the results of [9], although the proof is somewhat different.

The author would like to express his hearty thanks to T. Sunada, whose suggestion that the theory of Fourier integral operators might be used to study the spherical mean operator was the motivation of this paper, and who suggested also that k_B might be expressed in terms of the root systems in the case of the mean value operators on symmetric spaces. The author would like to thank also Professor H. Ozeki who suggested the possibility of describing the number $v_{x_0}(\phi)$ of Section 2 in terms of the Schubert varieties.

1. Definitions and examples

1.1. Let X and Y be manifolds*) and B a submanifold of $X \times Y$ such that

*) In this paper the word *manifold* will always be used for a connected, paracompact, Hausdorff smooth manifold of finite dimension.

$\pi_1=\pi_X|_B\colon B\to X$ and $\pi_2=\pi_Y|_B\colon B\to Y$ make B locally trivial fiber spaces over X and Y respectively. Here $\pi_X\colon X\times Y\to X$ and $\pi_Y\colon X\times Y\to Y$ are natural projections. The obtained object $X\leftarrow B\to Y$ will be called a *double fibering*. Throughout this paper we will use the following notations:

$$\begin{aligned}
&n_X = \dim X,\ n_Y = \dim Y\,,\\
&N_x = \pi_2\pi_1^{-1}x\subset Y \qquad (x\in X)\,,\\
&M_y = \pi_1\pi_2^{-1}y\subset X \qquad (y\in Y)\,,\\
&p = \dim N_x\,,\ q = \dim M_y\,,\\
&d = n_X-q = n_Y-p\,.
\end{aligned}$$

Obviously

$$\begin{aligned}
&d = \operatorname{codim}_Y N_x = \operatorname{codim}_X M_y = \operatorname{codim}_{X\times Y} B\,,\\
&x\in M_y \Leftrightarrow (x, y)\in B \Leftrightarrow y\in N_x\,, \qquad x\in X,\ y\in Y\,.
\end{aligned}$$

Suppose that a smooth positive density $d\mu_x$ is given on each $\pi_1^{-1}x\cong N_x$ $(x\in X)$ which depends on x smoothly. Define $L_B\colon C_0^\infty(Y)\to C^\infty(X)$ by

$$(L_Bf)(x) = \int_{N_x} f|_{N_x} d\mu_x\,, \qquad x\in X,\ f\in C_0^\infty(Y)\,.$$

Here $C^\infty(X)$ is the set of all smooth functions on X and

$$C_0^\infty(Y)=\{f\in C^\infty(Y);\ \operatorname{supp} f \text{ is compact}\}\ .$$

Note that L_B is well defined since it is obvious that $L_Bf\in C^\infty(X)$ for $f\in C_0^\infty(Y)$. L_B is called *the integral transformation associated with the double fibering* $X\leftarrow B\to Y$ *and the densities* $\{d\mu_x\}$.

1.2. Examples of L_B.

EXAMPLE 1.1 (Radon transformation). Let $X=(\mathbf{R}^n\backslash 0)\times\mathbf{R}$, $Y=\mathbf{R}^n$. Denote the coordinates of the points of X and Y by $(\eta, p)=(\eta_1, \cdots, \eta_n, p)$ and $(y)=(y_1, \cdots, y_n)$ respectively. Put

$$B = \{(\eta, p, y);\ (\eta, y)-p = 0\}\subset X\times Y\,,$$

where $(\eta, y)=\eta_1y_1+\cdots+\eta_ny_n$. In this case $n_X=n+1$, $n_Y=n$, $N_{(\eta,p)}=\{(y);\ (\eta, y)=p\}$, $M_{(y)}=\{(\eta, p);\ (\eta, y)=p\}$, $p=n-1$, $q=n$, $d=1$. Let $d\mu_{(\eta,p)}$ be the smooth positive density on $N_{(\eta,p)}$ defined by the condition that

$$d\mu_{(\eta,p)}\cdot d(\eta, y) = dy_1\cdots dy_n \qquad \text{on } N_{(\eta,p)}\,.$$

Then the associated L_B is the classical Radon transformation (cf. [3]).

EXAMPLE 1.2. Let $X=Y=\mathbf{R}^n$. Let S be a compact submanifold of codimension d. Put $B_S=\{(x, y); x-y\in S\}$. Then $X\leftarrow B_S\rightarrow Y$ is obviously a double fibering. Since $N_x=x-S\cong S$, we can define $d\mu_x$ as the smooth positive density corresponding to the volume element ω_S of the induced Riemannian metric on S. Then $L_S=L_{B_S}$ is defined by

$$(L_S f)(x)=\int_{s\in S} f(x-s)\omega_S\,, \qquad f\in C_0^\infty(\mathbf{R}^n)\,.$$

EXAMPLE 1.3 (Spherical mean operator). Let $X=Y$ be a compact Riemannian manifold, $SX=\{(x, \xi)\in TX;\ \|\xi\|=1\}$ its unit sphere bundle, and π: $SX\rightarrow X$ the natural projection. Let G_t: $TX\rightarrow TX$ $(t\in\mathbf{R})$ be the geodesic flow. Since $G_t(SX)\subset SX$, we can define $g_t=G_t|_{SX}$: $SX\rightarrow SX$. Put

$$e_t(\xi)=(\pi\xi,\ \pi g_t\xi)\in X\times X \qquad (\xi\in SX)\,,$$
$$T=\{t\in\mathbf{R};\ e_t \text{ is an embedding}\}\ .$$

Note that $T\cup\{0\}$ contains a neighbourhood of 0. Put $B_t=\operatorname{Im} e_t\subset X\times X$ $(t\in T)$. For t with sufficiently small $|t|$,

$$B_t=\{(x, y);\ d(x, y)=|t|\}\ ,$$

d being the metric defined by the Riemannian structure. For this double fibering $X\leftarrow B_t\rightarrow X$, $n_X=n_Y=n$, $p=q=n-1$, $d=1$. $M_x=N_x$ $(x\in X)$ is the geodesic sphere of radius $|t|$ with center x.

Let $d\mu_{t,x}$ be the smooth positive density on N_x corresponding to the natural one ω_x on $\pi^{-1}x\subset T_xX$ under the diffeomorphism πg_t: $\pi^{-1}x\rightarrow N_x$. The associated operator $L_t=L_{B_t}$ is then defined by

$$(L_t f)(x)=\int_{\xi\in\pi^{-1}x} f(\exp(t\xi))\omega_x, \qquad x\in X,\ f\in C^\infty(X)\,.$$

L_t is *the spherical mean operator with radius* $|t|$ *on* X (*cf.* [9]).

EXAMPLE 1.4 (Mean value operator [4]). Let G be a connected Lie group, K a compact subgroup of G, and $X=Y=G/K$. Fix an $a\in G$, put $H_a=K\cap aKa^{-1}$ and $B_a=G/H_a$. Define π_1, π_2: $G/H_a\rightarrow X$ by $\pi_1(gH_a)=gK$, $\pi_2(gH_a)=gaK$ respectively. Since $H_a\cdot aK\subset aKK=aK$, π_2 is well-defined. It is easy to see that $\pi_1\times\pi_2$: $B_a\rightarrow X\times X$ is an embedding and that $X\xleftarrow{\pi_1}B_a\xrightarrow{\pi_2}X$ is a double fibering. Obviously $M_{gK}=gKa^{-1}K/K$, $N_{gK}=gKaK/K$.

Define $\{d\mu_x;\ x\in X\}$ as follows: There is a natural isomorphism between K/H_a and $\pi_1^{-1}x$ $(x\in X)$, which is unique up to the action of K on K/H_a. So the K-invariant density $d\mu_{K/H_a}$ on K/H_a, normalized as

$$\int_{K/H_a} d\mu_{K/H_a}=1\ ,$$

gives rise to a uniquely determined density $d\mu_x$ on each $\pi_1^{-1}x$ which is clearly smooth with respect to $x \in X$. The obtained operator L_{B_a} will be called *the mean value operator determined by* $a \in G$ and denoted simply by M^a. By definition

$$(M^a f)(gK) = \int_{K/H_a \ni kH_a} f(gkaK) d\mu_{K/H_a}$$
$$= \int_{K \ni k} f(gkaK) dk ,$$

where $f \in C^\infty(X)$ and dk is the invariant density on K such that $\int_K dk = 1$.

1.3. We introduce here a notation which will be used throughout this paper.

Let $\mathcal{D}'(X)$ be the space of all distributions on X and

$$\mathcal{E}'(X) = \{u \in \mathcal{D}'(X);\ \text{supp}\ u \text{ is compact}\} .$$

We put for $s \in \mathbf{R}$

$$H_{(s)}^{\text{comp}}(X) = \{u \in \mathcal{E}'(X);\ Pu \text{ is square integrable for all } P \in \Psi^s(X)\} ,$$
$$H_{(s)}^{\text{loc}}(X) = \{u \in \mathcal{D}'(X);\ \phi u \in H_{(s)}^{\text{comp}}(X) \text{ for all } \phi \in C_0^\infty(X)\} .$$

Here $\Psi^s(X)$ is the set of all properly supported pseudo-differential operators of order s.

Let L: $C_0^\infty(Y) \to C^\infty(X)$ be a continuous linear mapping. If there is an $r \in \mathbf{R}$ such that, for every $s \in \mathbf{R}$, L extends to a continuous linear mapping

$$L:\ H_{(s)}^{\text{comp}}(Y) \to H_{(s+r)}^{\text{loc}}(X) ,$$

then we write

$$\text{reg}\ L \geqq r .$$

2. Regularity of L_B

2.1. Let V be an n-dimensional real vector space and X a manifold. Let $0 < l < n$ and suppose a smooth mapping ϕ: $X \to Gr(l, V)$ is given, where $Gr(l, V)$ denotes the Grassmann manifold of l-dimensional subspaces of V. Fix $x_0 \in X$ and put $W = \phi(x_0) \subset V$. The differential of ϕ at x_0 gives a linear mapping Φ: $T \to \text{Hom}(W, V/W)$, where $T = T_{x_0}X$ and $T_W Gr(l, V)$ is identified with $\text{Hom}(W, V/W)$ in a natural way (cf. Remark 2.1 below). For $u \in (V/W)^*$, define $u \circ \Phi$: $T \to W^*$ by

$$(u \circ \Phi)(t)(w) = u(\Phi(t)(w)) , \qquad t \in T,\ w \in W .$$

We put then

$$v_{x_0}(\phi) = \min_{u \in (V/W)^* \setminus 0} \text{rank}(u \circ \Phi) .$$

REMARK 2.1. Recall that Φ is obtained as follows. For a sufficiently small neighbourhood U of x_0, there is a smooth mapping $\psi: U \to \mathrm{Hom}(W, V)$ such that $\psi(x_0)=\mathrm{id}_W$ and $\phi(x)=\psi(x)W$ $(x\in U)$. Then

$$\Phi = d_{x_0}(\pi\psi),$$

where $\pi: V\to V/W$ is the natural projection.

EXAMPLE 2.2. Let $X=\mathbf{R}^n\backslash 0$, $V=\mathbf{R}^n$. Define $\phi: X\to Gr(n-1, V)$ by $\phi(x)=\{\xi\in V; (x, \xi)=0\}$, where $(x, \xi)=x_1\xi_1+\cdots+x_n\xi_n$, $(x=(x_1, \cdots, x_n)$, $\xi=(\xi_1, \cdots, \xi_n))$. Let $x_0=(1, 0, \cdots, 0)\in X$. Then $W=\phi(x_0)=\{0\}\times\mathbf{R}^{n-1}$. Define $\psi(x)\in\mathrm{Hom}(W, V)=\mathrm{Hom}(\mathbf{R}^{n-1}, \mathbf{R}^n)$ by the $(n, n-1)$-matrix

$$\begin{pmatrix} -x_2\cdots\cdots -x_n \\ x_1\cdots\cdots\cdots 0 \\ \vdots \ \ \ddots \ \ \ \ \vdots \\ \vdots \ \ \ \ \ddots \ \ \vdots \\ 0\cdots\cdots\cdots x_1 \end{pmatrix}.$$

It is obvious that $\psi(x_0)=id_W$ and $\phi(x)=\psi(x)W$ $(x_1\neq 0)$. We can identify V/W with $\mathbf{R}$ so that $\pi: V\to V/W$ is given by $(\xi_1, \cdots, \xi_n)\mapsto\xi_1$. Then $\pi\phi(x)\in\mathrm{Hom}(W, V/W)=\mathrm{Hom}(\mathbf{R}^{n-1}, \mathbf{R})$ is given by the $(1, n-1)$-matrix $(-x_2, \cdots, -x_n)$. Hence, $T_{x_0}X$ being identified with $X=\mathbf{R}^n$, $\Phi(x)\in\mathrm{Hom}(W, V/W)$ is given by $(-x_2, \cdots, -x_n)$. Thus

$$\begin{aligned} v_{x_0}(\phi) &= \min_{t\in\mathbf{R}^*\backslash 0}\mathrm{rank}(t\circ\Phi) \\ &= \mathrm{rank}(1\circ\Phi) \\ &= \mathrm{rank}(\mathbf{R}^n\ni x\mapsto(-x_2, \cdots, -x_n)\in\mathbf{R}^{n-1}) \\ &= n-1, \end{aligned}$$

where $1\in\mathbf{R}^*$ is the identity mapping $\mathbf{R}\to\mathbf{R}$.

2.2. We give here a geometric description of the number $v_{x_0}(\phi)$. It will not be used later.

Put $E=\mathrm{Im}\,\Phi$. Let $\mathcal{S}$ be the set of all the Schubert varieties of type $(\underbrace{n-l-1, \cdots, n-l-1}_{l})$ which contain W as an interior point. Thus $S\in\mathcal{S}$ is given by

$$S = \{Q\subset V; \dim(Q\cap V_i)\geqq i, i=1, \cdots, l\},$$

where $V_1\subset\cdots\subset V_l$ is a flag in V such that $\dim V_i=n-l+i-1$ and $\dim(W\cap V_i)=i$ $(1\leqq i\leqq l)$.

Proposition 2.3. $v_{x_0}(\phi)=\dim E-\max_{S\in\mathcal{S}}\dim(E\cap T_W S)$.

Proof. Let Y be a complementary subspace of W in V: $V=W\oplus Y$. V/W will be identified with Y. Let $S\in\mathcal{S}$ be defined by a flag $V_1\subset\cdots\subset V_l$. Since $W\in S$, there are a uniquely determined subspace Z of Y of codimension one and a $\psi\in\mathrm{Hom}(Z, W)$ such that

$$V_i=(W\cap V_i)\oplus\Gamma_\psi \qquad (1\leqq i\leqq l)\,,$$

where $\Gamma_\psi=\{\psi(y)+y;\, y\in Z\}$. Extend ψ to a $\tilde{\psi}\in\mathrm{Hom}(Y, W)$. Then $V=W\oplus\Gamma_{\tilde{\psi}}$.

First we determine $T_W S$. Let A: $(-\varepsilon, \varepsilon)\to\mathrm{Hom}(W, Y)$ $(\varepsilon>0)$ be a smooth mapping such that $A(0)=0$ and $\Gamma_{A(t)}\in S$ $(t\in(-\varepsilon, \varepsilon))$, where $\Gamma_{A(t)}=\{w+A(t)w;\, w\in W\}$. We may assume $1-\tilde{\psi}A(t)$ $(t\in(-\varepsilon, \varepsilon))$ is invertible. Since

$$w+A(t)w=(1-\tilde{\psi}A(t))w+(A(t)+\tilde{\psi}A(t))w\,,$$

we have $\Gamma_{A(t)}=\{w+B(t)w;\, w\in W\}$, where

$$B(t)=(A(t)+\tilde{\psi}A(t))(1-\tilde{\psi}A(t))^{-1}\in\mathrm{Hom}(W, \Gamma_{\tilde{\psi}})\,.$$

$\Gamma_{A(t)}\in S$ implies $\dim\Gamma_{A(t)}\cap(W\oplus\Gamma_\psi)=l$, that is, $\Gamma_{A(t)}\subset W\cap\Gamma_\psi$, whence $B(t)W\subset\Gamma_\psi$. This in turn is equivalent to $A(t)(1-\tilde{\psi}A(t))^{-1}W\subset Z$. Taking the differential at $t=0$, we get $\dot{A}(0)\in\mathrm{Hom}(W, Z)$. Thus

$$T_W S\subset\mathrm{Hom}(W, Z)\,.$$

Since $\dim T_W S=(n-l-1)l=\dim\mathrm{Hom}(W, Z)$, we get

$$T_W S=\mathrm{Hom}(W, Z)$$

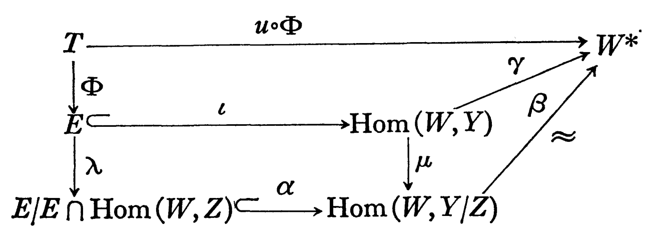
Put $\mathcal{Z}=\{Z\subset Y;\, \mathrm{codim}_Y Z=1\}$. It is easily verified that for each $Z\in\mathcal{Z}$ there is an $S\in\mathcal{S}$ such that $T_W S=\mathrm{Hom}(W, S)$. Thus

$$\{T_W S;\, S\in\mathcal{S}\}=\{\mathrm{Hom}(W, Z);\, Z\in\mathcal{Z}\}\,.$$

It remains to show

$$v_{x_0}(\phi)=\dim E-\max_{Z\in\mathcal{Z}}\dim(E\cap\mathrm{Hom}\,(W, Z))\,.$$

For $Z\in\mathcal{Z}$, take a $u\in Y^*\backslash 0$ such that $u|_Z=0$. Consider the following commutative diagram of linear mappings.



Here λ is the quotient mapping, μ is induced from the quotient mapping $Y \to Y/Z$, and α, β, γ are defined respectively by the following conditions:

$$\begin{aligned}
&\alpha\lambda = \mu\iota\,,\\
&\beta\mu = \gamma\,,\\
&\gamma(\Lambda)(w) = u(\Lambda(w)),\ \Lambda \in \mathrm{Hom}(W,\ Y),\ w \in W.
\end{aligned}$$

Note that Φ, λ are surjective and α, β are injective, whence

$$\begin{aligned}
\mathrm{rank}\, u\circ\Phi &= \dim E/E \cap \mathrm{Hom}(W,\ Z)\\
&= \dim E - \dim E \cap \mathrm{Hom}(W,\ Z)\,.
\end{aligned}$$

In view of the definition of $v_{x_0}(\phi)$, this completes the proof. Q.E.D.

2.3. Now we go back to the situation of Section 1. Let $X \xleftarrow{\pi_1} B \xrightarrow{\pi_2} Y$ be a double fibering and smooth positive densities $\{d\mu_x\}$ are given. Let L_B: $C_0^\infty(Y) \to C^\infty(X)$ be the associated integral transformation. For each $y \in Y$, define a smooth mapping ϕ_y: $M_y \to Gr(p;\ T_y Y)$ by

$$\phi_y(x) = T_y N_x \subset T_y Y\,.$$

Put

$$k_B = \min_{(x,y)\in B} v_x(\phi_y)\,.$$

We can state now the main theorem of this paper.

Theorem 2.4. $\mathrm{reg}\, L_B \geqq \frac{1}{2} k_B$.

The proof will be given in the next section.

2.4. Applications of Theorem 2.4 to the examples of Section 1.

EXAMPLE 1.1 (continued). By virtue of the homogenity of the situation, $k_B = v_{x_0}(\phi_{y_0})$, where $y_0 = 0$, $x_0 = (1,\ 0,\ \cdots,\ 0) \times (0)$. We have

$$\begin{aligned}
M_{y_0} &= \{(\xi,\ 0);\ \xi \in \mathbf{R}^n \backslash 0\} \cong \mathbf{R}^n \backslash 0\,,\\
T_0 N_{(\xi,0)} &= \{(x);\ (x,\ \xi) = 0\} \subset \mathbf{R}^n\,,
\end{aligned}$$

$T_0 Y$ and $\mathbf{R}^n$ being identified. By Example 2.2,

$$v_{x_0}(\phi_{y_0}) = n-1\,.$$

Thus

$$\mathrm{reg}\, L_B \geqq \frac{1}{2}(n-1)\,.$$

EXAMPLE 1.2 (continued). Suppose $d=1$. Then

$$v_x(\phi_y) = \operatorname{rank} II_{x-y} \qquad (x, y) \in B_S ,$$

where II_s ($s \in S$) stands for the second fundamental form of S at s. In fact ϕ_y is essentially the Guass map of S and, in the case $d=1$, $v_x(\phi_y)$ is just the rank of its Jacobian at $x-y$, which coincides with the rank of II_{x-y}. Hence

$$k_{B_S} = \min_{s \in S} \operatorname{rank} II_s .$$

EXAMPLE 1.3 (continued). We can verify easily

$$k_{B_t} = v_{(1,0,\cdots,0)}(\phi) = n-1 ,$$

where ϕ: $S^{n-1} \to Gr(n-1, \mathbf{R}^n)$ is defined by

$$\phi(x) = \{(\xi) \in \mathbf{R}^n; (x, \xi) = 0\} .$$

Hence

Proposition 2.5. $\operatorname{reg} L_t \geqq \frac{1}{2}(n-1)$, $t \in T$.

REMARK. Another proof of this fact is given in [9].

Note that it can be easily verified without the use of Proposition 2.5 that L_t can be extended to an operator $\bar{L}_t$: $H_{(0)}(X) \to H_{(0)}(X)$, where $H_{(0)}(X) = H_{(0)}^{\text{loc}}(X) = H_{(0)}^{\text{comp}}(X)$. By virtue of the Sobolev's lemma, Proposition 2.5 implies

Corollary 2.6. *If $n \geqq 2$ and $t \in T$, then the eigenfunctions of $\bar{L}_t$ with non-zero eigenvalues are smooth.*

Corollary 2.7. *If $n \geqq 2$ and $t \in T$, then the elements of $H_{(0)}(X)$ that are fixed by $\bar{L}_t$ are constant functions.*

REMARK (i). This fact was conjectured by T. Sunada in [8]. There it is shown that the above fact is equivalent to the mixing property of an abstract dynamical system on the space of certain random walks over X.

(ii) Define $\tilde{g}_t$: $TX \backslash 0 \to TX \backslash 0$ by $\tilde{g}_t(\xi) = ||\xi|| g_t(\xi/||\xi||)$, which is a diffeomorphism. Let h_t: $T^*X \backslash 0 \to T^*X \backslash 0$ correspond to $\tilde{g}_t$ under the natural diffeomorphism $TX \backslash 0 \cong T^*X \backslash 0$ defined by the Riemannian metric. Let $\Gamma_t \subset (T^*X \backslash 0) \times (T^*X \backslash 0)$ be the graph of h_t. Then it turns out $T^*_{B_t}(X \times X) \backslash 0 = \Gamma'_t \cup \Gamma'_{-t}$ where $T^*_{B_t}(X \times X)$ is the dual of the normal bundle of B_t in $X \times X$ and $\Gamma'_t = \{(x, \xi, x', \xi');$ $(x, \xi', x', -\xi') \in \Gamma_t\}$ (cf. [9]). Since $\Gamma_t \cap \Gamma_{-t} = \emptyset$ for t with small $|t|$, this shows that the Lagrangean manifold of the Fourier integral operator L_t is composed of graphs of two canonical transformations on $T^*X \backslash 0$ (cf. Proposition 3.1).

EXAMPLE 1.4. (continued). We will express k_{B_a} in terms of Lie algebras. By virtue of the homogenity of the situation, $k_{B_a} = v_{x_0}(\phi_{y_0})$ ($y_0 = K$, $x_0 = a^{-1}K$). Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K respectively. Let $\mathfrak{p}$ be a complementary subspace of $\mathfrak{k}$: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Identify $\mathfrak{g}$ and $\mathfrak{k}$ respectively with T_eG and

$T_K(G/K)$ so that the projection $p: \mathfrak{g} \to \mathfrak{p}$ corresponds to the differential at e of the projection $\pi: G \to G/K$, e being the unit element of G. For $x = ka^{-1}K \in M_{y_0}$, we have

$$N_x = ka^{-1}KaK/K = \pi(ka^{-1}Kak^{-1}K) \qquad (x = ka^{-1}K),$$

whence

$$\phi_{y_0}(x) = T_K N_x = p(\mathrm{Ad}(ka^{-1})\mathfrak{k}).$$

Identify $K/H_{a^{-1}}$ with M_{y_0} by the diffeomorphism: $K/H_{a^{-1}} \to M_{y_0}$ sending $kH_{a^{-1}}$ to $ka^{-1}K$. Then ϕ_{y_0} is given by

$$\phi_{y_0}(kH_{a^{-1}}) = p(\mathrm{Ad}(ka^{-1})\mathfrak{k}) \subset \mathfrak{p}.$$

Fix a complementary subspace $W^\perp$ of $W = p(Ad(a^{-1}))\mathfrak{k}$ in $\mathfrak{p}$ and let $\Pi: \mathfrak{p} \to W^\perp$ be the projection. Since

$$p(Ad(ka^{-1})\mathfrak{k}) = \psi(k)W$$

where $\psi(k) = p(Ad(k)|_W) \in \mathrm{Hom}(W, \mathfrak{p})$, we have

$$d_{x_0}(\phi_{y_0}) = d_0(\Pi\psi): \mathfrak{k}/\mathfrak{k} \cap \mathrm{Ad}(a^{-1})\mathfrak{k} \to \mathrm{Hom}(W, W^\perp)$$

in view of Remark 2.1 ($0 = H_{a^{-1}} \in K/H_{a^{-1}}$). It is easy to verify

$$d_0(\Pi\psi)([X])(Y) = \Pi p([X, Y]), \qquad X \in \mathfrak{k},\ Y \in W.$$

Hence we have

Proposition 2.8. *Let $\Phi: \mathfrak{k} \times W \to W^\perp$ be the bilinear mapping defined by $\Phi(X, Y) = \Pi p([X, Y])$ $(X \in \mathfrak{k},\ Y \in W)$. Then*

$$k_{B_a} = \min_{u \in (W^\perp)^* \setminus 0} \mathrm{rank}\,(u\Phi).$$

In Sections 5, 6, we will compute k_{B_a} in the case G/K is a symmetric space.

3. Proof of Theorem 2.4

3.1. First we shall show that L_B is essentially a Fourier integral operator in the sense of [6].

Let $\Omega_{1/2}$ be the line bundle of the densities of order 1/2 over X(cf. [6]). Let $C^\infty(X, \Omega_{1/2})$ be the space of all the smooth cross-sections of $\Omega_{1/2}$ and

$$C_0^\infty(X, \Omega_{1/2}) = \{u \in C^\infty(X, \Omega_{1/2});\ \mathrm{supp}\ u \text{ is compact}\}.$$

Fix smooth positive densities ω_X, ω_Y of X, Y respectively. Define $\tilde{L}_B$ by the following commutative diagram:

$$\begin{array}{ccc} C_0^\infty(Y) & \xrightarrow{L_B} & C^\infty(X) \\ \alpha\downarrow\wr & & \beta\downarrow\wr \\ C_0^\infty(Y, \Omega_{1/2}) & \xrightarrow{\tilde{L}_B} & C^\infty(X, \Omega_{1/2}) \end{array},$$

where $\alpha(f)=f\sqrt{\omega_X}$, $\beta(g)=g\sqrt{\omega_Y}$ $(f\in C_0^\infty(Y),\ g\in C^\infty(X))$. Put $\Lambda_B=T_B^*(X\times Y)\backslash 0\subset T^*(X\times Y)$, where $T_B^*(X\times Y)$ is the dual of the normal bundle of B in $X\times Y$ and 0 is the zero section. The distribution kernel of $\tilde{L}_B$ will be denoted also by $\tilde{L}_B$.

Proposition 3.1. $\tilde{L}_B\in \mathrm{I}^{-1/4(p+q)}(X\times Y, \Lambda_B)$.

As to the notation $\mathrm{I}^m(X\times Y, \Lambda_B)=\mathrm{I}_1^m(X\times Y, \Lambda_B)$, see [6].

Proof. Let ω_B be the smooth positive density of B defined by

$$\omega_B(x, y) = d\mu_x(y)\cdot\omega_X(x)\,, \qquad (x, y)\in B\,.$$

Then the distribution $\tilde{L}_B$ is given by

$$\langle\tilde{L}_B, \phi\sqrt{\omega_X\omega_Y}\rangle = \int_B (\phi|_B)\omega_B\,, \qquad \phi\in C_0^\infty(X\times Y)\,.$$

In fact, for $f\in C_0^\infty(X)$, $g\in C_0^\infty(Y)$,

$$\begin{aligned}\langle\tilde{L}_B, fg\sqrt{\omega_X\omega_Y}\rangle &= \int_X f(x)\Big(\int_{N_x} g|_{N_x} d\mu_x\Big)\omega_x \\ &= \int_B (fg)|_B\omega_B\,.\end{aligned}$$

It suffices to show that for each $p\in X\times Y$, there is a neighbourhood U of p such that

$$\tilde{L}_B|_U\in \mathrm{I}^{-1/4(p+q)}(U, \Lambda_B|_U)\,,$$

where $\Lambda_B|_U=\pi^{-1}U\cap\Lambda_B$, π: $T^*(X\times Y)\to X\times Y$ being the natural projection. Since supp $\tilde{L}_B=B$, we may assume $p\in B$. Let $(U; z^1, \cdots, z^N, w^1, \cdots, w^d)$ $(N=n_X+n_Y-d)$ be a local chart of $X\times Y$ around p such that

$$U\cap B = \{(z, w);\ w^1 = \cdots = w^d = 0\}\,.$$

Let

$$\begin{aligned}&\omega_B|_{U\cap B} = f dz \\ &\omega_X\omega_Y|_U = g dz dw\,,\end{aligned}$$

where $dz=dz^1\cdots dz^N$, $dw=dw^1\cdots dw^d$, $f\in C^\infty(U\cap B)$, $g\in C^\infty(U)$, $f>0$, $g>0$. For $\phi\in C_0^\infty(U)$,

$$\begin{aligned}&\langle\tilde{L}_B, \phi\sqrt{dzdw}\rangle \\ &= \langle\tilde{L}_B, (\phi/\sqrt{g})\sqrt{\omega_X\omega_Y}\rangle\end{aligned}$$

$$= \int_{U \cap B} (\phi/\sqrt{g})|_B \omega_B$$
$$= \int_{\mathbf{R}^N} (\phi(z, 0)/\sqrt{g(z, 0)}) f(z) dz$$
$$= (2\pi\sqrt{-1})^{-d} \int_{\mathbf{R}^N \times \mathbf{R}^d \times \mathbf{R}^d} \phi(z, w) a(z, w, \theta) e^{i\phi(z,w,\theta)} dz dw d\theta ,$$

where $\theta=(\theta^1, \cdots, \theta^d)\in\mathbf{R}_d$, $a(z, w, \theta)=f(z)\sqrt{g(z, w)}\in S^0(U\times(\mathbf{R}^d\backslash 0))$, $\phi(z, w, \theta) = w^1\theta^1+\cdots+w^d\theta^d$ and the last expression is the oscillatory integral. (As to $S^0(U\times(\mathbf{R}^d\backslash 0))$, see [6].) Thus

$$\tilde{L}_B|_U \in I^m(U, \Lambda)$$

with

$$\Lambda = \{(z, 0; \Sigma\theta^i dw^i); z\in\mathbf{R}^N, \theta\in\mathbf{R}^d\} = \Lambda_B|_U ,$$
$$m+\frac{1}{4}(n_X+n_Y-2d) = 0 .$$

Hence

$$L_B|_U \in I^{-1/4(p+q)}(U, \Lambda_B|_U) . \qquad \text{Q.E.D.}$$

Since the symbol of L_B is homogeneous and non-zero at each point of Λ_B, we have

Corollary 3.2. *$\tilde{L}_B$ is non-characteristic everywhere on Λ_B.*

3.2. We quote a theorem from [6]. Let

$$H^{\mathrm{loc}}_{(s)}(X, \Omega_{1/2}) = H^{\mathrm{loc}}_{(s)}(X) \underset{C^\infty(X)}{\otimes} C^\infty(X, \Omega_{1/2}) ,$$
$$H^{\mathrm{comp}}_{(s)}(X, \Omega_{1/2}) = H^{\mathrm{comp}}_{(s)}(X) \underset{C^\infty_0(X)}{\otimes} C^\infty_0(X, \Omega_{1/2}) , \qquad s\in\mathbf{R} .$$

Theorem 3.3 (Hörmander [6]). *Let C be a homogeneous canonical relation from T^*Y to T^*X such that*

(i) *the restrictions to C of the natural projections $T^*(X\times Y)\to X$, $\to Y$ are submersions;*

(ii) *there is a non-negative integer k such that*

$$\mathrm{rank}_c (p_X) \geqq n_X+k ,$$
$$\mathrm{rank}_c (p_Y) \geqq n_Y+k ,$$

for all $c\in C$, $p_X\colon C\to T^*X$ and $p_Y\colon C\to T^*Y$ being the restrictions of the natural projections $T^*(X\times Y)\to T^*X$ and $T^*(X\times Y)\to T^*Y$ respectively.

Then, if $m\leqq\frac{1}{4}(2k-n_X-n_Y)$, every $A\in I^m(X\times Y, C')$ can be extended to a continuous mapping from $H^{\mathrm{comp}}_{(0)}(Y, \Omega_{1/2})$ to $H^{\mathrm{loc}}_{(0)}(X, \Omega_{1/2})$.

Recall that $C'=\{(x, \xi, y, \eta);\ (x, \xi, y, -\eta)\in C\}$.

Corollary 3.4. *Under the conditions* (i) *and* (ii) *of Theorem* 3.3,

$$\operatorname{reg} A\geqq -m-\frac{1}{4}(n_X+n_Y-2k)$$

for every $A\in I^m(X\times Y, C')$, *that is, A can be extended to a continuous linear mapping of* $H_{(s)}^{\mathrm{comp}}(Y, \Omega_{1/2})$ *into*

$$H^{\mathrm{loc}}_{(s-m-1/4(n_X+n_Y-2k))}(X, \Omega_{1/2}), \qquad \textit{for each } s, m\in\mathbf{R}.$$

REMARK 3.5. By Proposition 4.1.4 of [6],

$$\operatorname{rank}_c(p_X)-n_X=\operatorname{rank}_c(p_Y)-n_Y.$$

Hence the condition (ii) of Theorem 3.3 can be weakened:

(ii)′ $\operatorname{rank}_c(p_Y)\geqq n_Y+k$ $(c\in C)$.

3.3. We prove now the key lemma of the proof of Theorem 2.4. Fix $p_0=(x_0, y_0)\in B$ and put $\Lambda_{p_0}=\pi^{-1}p_0\setminus\{0\}$, where π: $T^*_B(X\times Y)\to B$ is the natural projection.

Lemma 3.6. $\min_{\lambda\in\Lambda_{p_0}} \operatorname{rank}_\lambda(p_Y)=n_Y+d+v_{x_0}(\phi_{y_0})$.

Proof. Consider the following commutative diagram of smooth mappings:

$$\begin{array}{ccccc}
 & & T^*_B(X\times Y)|_{M_{y_0}\times\{y_0\}} & \longrightarrow & M_{y_0}\times\{y_0\} \\
 & & \kappa\downarrow & & \downarrow\iota \\
T^*_B(X\times Y)_{p_0} & \hookrightarrow & T^*_B(X\times Y) & \xrightarrow{\ \pi\ } & B \\
\beta\downarrow & & \downarrow p_Y & & \downarrow \pi_2 \\
T^*_{y_0}Y & \hookrightarrow & T^*Y & \longrightarrow & Y \\
\gamma\downarrow & & & & \\
T^*_{y_0}N_{x_0} & & & &
\end{array}\tag{1}$$

γ being the restriction of linear forms on $T_{y_0}Y$ to $T_{y_0}N_{x_0}$. We note that β *is an isomorphism onto the subspace* $(T^*_{N_{x_0}}Y)_{y_0}$ *of* $T^*_{y_0}Y$:

(2) β: $T^*_B(X\times Y)_{p_0}\xrightarrow{\approx}(T^*_{N_{x_0}}Y)_{y_0}$.

In fact, let $(\xi, \eta)\in T^*_B(X\times Y)_{p_0}\subset T^*_{x_0}X\times T^*_{y_0}Y$. From $\{x_0\}\times N_{x_0}\subset B$, it follows $\eta|_{T_{y_0}N_{x_0}}=0$, that is, $p_Y((\xi, \eta))\in(T^*_{N_{x_0}}Y)_{y_0}$. Thus $\operatorname{Im}\beta\subset(T^*_{N_{x_0}}Y)_{y_0}$. Since

$$\dim(T^*_{N_{x_0}}Y)_{y_0} = d = \dim T^*_B(X\times Y)_{p_0},$$

it suffices to show that β is injective. Let $(\xi, 0)\in T^*_B(X\times Y)_{p_0}$. This means that $\pi_1^*\xi=0$, where $\pi_1^*\colon T^*_{x_0}X\to T^*_{p_0}B$ is induced by $\pi_1\colon B\to X$. Since π_1 is a submersion, this implies $\xi=0$.

Taking the differentials of the mappings in (1), we get the following commutative diagram of linear mappings.

$$
(3)\qquad \begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & T_{x_0}M_{y_0} & & \\
 & & \downarrow & & & & \downarrow d\iota & & \\
0 & \longrightarrow & T^*_B(X\times Y)_{p_0} & \longrightarrow & T_\lambda\Lambda & \longrightarrow & T_{p_0}B & \longrightarrow & 0 \quad \text{(iii)}\\
 & & \downarrow\beta & & \downarrow dp_Y & & \downarrow d\pi_1 & & \\
0 & \longrightarrow & T^*_{y_0}Y & \longrightarrow & T_{p_Y(\lambda)}T^*Y & \longrightarrow & T_{y_0}Y & \longrightarrow & 0 \quad \text{(iv)}\\
 & & \downarrow\gamma & & & & \downarrow & & \\
 & & T^*_{y_0}N_{x_0} & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & & \\
 & & \text{(i)} & & & & \text{(ii)} & &
\end{array}
$$

Here $\lambda\in\Lambda_{p_0}$ is fixed and we have made he following natural identifications:

$$T_\lambda T^*_B(X\times Y)_{p_0} = T^*_B(X\times Y)_{p_0},$$
$$T_{p_Y(\lambda)}T^*_{y_0}Y = T^*_{y_0}Y,$$
$$T_0T^*_{y_0}N_{x_0} = T^*_{y_0}N_{x_0}.$$

The sequence of the column (i) is exact by (2) and the following obvious exact sequence:

$$0\to(T^*_{N_{x_0}}Y)_{y_0}\to T^*_{y_0}Y\to T^*_{y_0}N_{x_0}\to 0.$$

The sequences of the column (ii) and the rows (iii), (iv) are clearly exact. Let $\widetilde{d\iota}\colon T_{x_0}M_{y_0}\to T_\lambda\Lambda$ be a lifting of $d\iota$. Then $\mathrm{Im}(dp_Y\widetilde{d\iota})\subset T^*_{y_0}Y$, whence $\nu_\lambda=\gamma dp_Y\widetilde{d\iota}\colon T_{x_0}M_{y_0}\to T^*_{y_0}N_{x_0}$ is defined. Obviously ν_λ is independent of the choice of the lifting $\widetilde{d\iota}$. It is easy to verify that

$$\begin{aligned}\operatorname{rank}_\lambda dp_Y &= \dim T^*_B(X\times Y)_{p_0}+\dim T_{y_0}Y+\operatorname{rank}\nu_\lambda \\ &= d+n_Y+\operatorname{rank}\nu_\lambda\,.\end{aligned}$$

Computation of rank ν_λ. Define $\hat{\phi}_{y_0}\colon M_{y_0}\to Gr(d, T^*_{y_0}Y)$ by $\hat{\phi}_{y_0}(x)=\phi_{y_0}(x)^0=(T^*_{N_x}Y)_{y_0}(x\in M_{y_0})$. Denote its differential at x_0 by $\Phi_{p_0}\colon T\to \operatorname{Hom}(W^0, T^*_{y_0}N_{x_0})$, where $T=T_{x_0}M_{y_0}$, $W^0=\hat{\phi}_{y_0}(x_0)$. Since $\beta(\lambda)\in W^0$ $(\lambda\in\Lambda_{p_0})$, we can define $\Phi_{p_0,\lambda}\colon T\to T^*_{y_0}N_{x_0}$ $(\lambda\in\Lambda_{p_0})$ by

$$(\Phi_{p_0,\lambda})(\xi)=\Phi_{p_0}(\xi)(\beta(\lambda))\,,\qquad \xi\in T\,.$$

Sublemma. $\nu_\lambda=\Phi_{p_0,\lambda}$.

Proof. There are a neighbourhood U of x_0 in M_{y_0} and a smooth mapping

$$\phi\colon U\to \operatorname{Hom}(W^0, T^*_{y_0}Y)$$

such that $\phi(x_0)=id$ and $\phi(x)W^0=\hat{\phi}_{y_0}(x)$. Then $\Phi_{p_0}=d_{x_0}(\gamma\phi)$, whence $\Phi_{p_0,\lambda}=d_{x_0}(\gamma(\phi\circ\lambda))$, where $\phi\circ\lambda\colon U\to T^*_{y_0}Y$ is defined by

$$(\phi\circ\lambda)(x)=\phi(x)(\beta(\lambda))\in\hat{\phi}_{y_0}(x)\,,\qquad x\in U\,.$$

In view of the natural isomorphism

$$dp_X|_{T^*_B(X\times Y)_{(x,y_0)}}\colon T^*_B(X\times Y)_{(x,y_0)}\xrightarrow{\approx}(T^*_{N_x}Y)_{y_0}\qquad \text{(cf. (2))}\,,$$

$\phi\circ\lambda$ defines a smooth cross-section $\widetilde{\phi\circ\lambda}$ of the vector bundle $T^*_B(X\times Y)|_{U\times\{y_0\}}\to U\times\{y_0\}$. $\widetilde{\phi\circ\lambda}$ is a local lifting of ι in the diagram (1): $\pi\widetilde{\phi\circ\lambda}=\iota$ on $U\times\{y_0\}$. Hence $d\widetilde{\phi\circ\lambda}\colon T\to T_\lambda\Lambda$ is a lifting of $d\iota$ in the diagram (3). Thus

$$\begin{aligned}\nu_\lambda &= \gamma(d_\lambda p_Y)(d_{x_0}\widetilde{\phi\circ\lambda}) \\ &= \gamma d_{x_0}(p_Y\widetilde{\phi\circ\lambda}) \\ &= \gamma d_{x_0}(\phi\circ\lambda) \\ &= d_{x_0}(\gamma(\phi\circ\lambda)) \\ &= \Phi_{p_0,\lambda}\,,\end{aligned}\qquad \text{Q.E.D.}$$

We have now

$$\begin{aligned}\min_{\lambda\in\Lambda_{p_0}}\operatorname{rank}_\lambda(p_Y) &= n_Y+d+\min_{\lambda\in\Lambda_{p_0}}\operatorname{rank}\nu_\lambda \\ &= n_Y+d+\min_{\lambda\in\Lambda_{p_0}}\operatorname{rank}(\Phi_{p_0,\lambda}) \\ &= n_Y+d+\min_{u\in W^0\backslash 0}\operatorname{rank}(\Phi_{p_0}\circ u)\,,\end{aligned}$$

where $\Phi_{p_0}\circ u\in\operatorname{Hom}(T, T^*_{y_0}N_{x_0})$ is defined by $(\Phi_{p_0}\circ u)(\xi)=\Phi_{p_0}(\xi)(u)$ $(\xi\in T)$. It remains to show the following

Sublemma. $\min_{u\in W^0\backslash 0} \operatorname{rank} \Phi_{p_0}\circ u = v_{x_0}(\phi_{y_0})$.

Proof. Put $V=T_{y_0}Y$, $W=T_{y_0}N_{x_0}$, $\phi=\phi_{y_0}$, $\hat{\phi}=\hat{\phi}_{y_0}$ for brevity. There is a natural isomorphism

$$\alpha\colon \operatorname{Hom}(W, V/W) \xrightarrow{\approx} \operatorname{Hom}(W^0, V^*/W^0),$$

since W^0 and W can be naturally identified with $(V/W)^*$ and $(V^*/W^0)^*$ respecively. Note that $V^*/W^0 \cong T^*_{y_0}N_{x_0}$. Then it is easily verified that $\Phi_{p_0} = -\alpha\Phi$, where $\Phi\colon T\to \operatorname{Hom}(W, V/W)$ is the differential of ϕ at x_0. We have

$$\Phi_{p_0}\circ u = -u\circ\Phi \in \operatorname{Hom}(T, W^*) = \operatorname{Hom}(T, V^*/W^0),$$

for $u\in W^0=(V/W)^*$. In fact, for $t\in T$, $w\in W$,

$$\begin{aligned}\langle(\Phi_{p_0}\circ u)(t), w\rangle &= \langle \Phi_{p_0}(t)u, w\rangle \\ &= -\langle \alpha(\Phi(t))u, w\rangle \\ &= -\langle u, \Phi(t)w\rangle \\ &= -\langle (u\circ\Phi)(t), w\rangle .\end{aligned}$$

Thus

$$\begin{aligned}v_{x_0}(\phi) &= \min_{u\in (V/W)^*\backslash 0} \operatorname{rank}(u\circ\phi) \\ &= \min_{u\in W^0\backslash 0} \operatorname{rank} \Phi_{p_0}\circ u . \end{aligned} \qquad \text{Q.E.D.}$$

This completes the proof of Lemma 3.6.

3.4. Proof of Theorem 2.4. We apply Corollary 3.4. The condition (i) is evidently satisfied. Taking $k=k_B+d$, the condition (ii)′ is also satisfied by virtue of Lemma 3.6. Thus

$$\operatorname{reg} \tilde{L}_B \geqq -m - \frac{1}{4}(n_X+n_Y-2k),$$

where $m=-\frac{1}{4}(p+q)$ by Proposition 3.1. Hence

$$\operatorname{reg} \tilde{L}_B \geqq \frac{1}{2}k_B,$$

which is clearly equivalent to

$$\operatorname{reg} L_B \geqq \frac{1}{2}k_B . \qquad \text{Q.E.D.}$$

REMARK 3.7. By Lemma 3.6 and Remark 3.5, we have

$$k_B = \min_{(x,y)\in B} v_y(\psi_x),$$

where $\psi_x: N_x \to Gr(q, T_xX)$ is defined by

$$\psi_x(y) = T_xM_y .$$

4. Parametrices for L_B

In this section the projections π_1, π_2 will be assumed proper. Then L_B extends to an operator $C^\infty(Y) \to C^\infty(X)$, which will also be denoted by L_B.

An operator $P: C^\infty(X) \to C^\infty(Y)$ is called *a parametrix for L_B* if $I - L_BP$ and $I - PL_B$ have smooth kernels.

Theorem 4.1. *L_B has a parametrix if the following conditions are satisfied:*
(i) $k_B + d = n_X = n_Y \geqq 2$;
(ii) *for each $(x, \xi) \in T^*X \backslash 0$ and $(y, \eta) \in T^*Y \backslash 0$,*

$$\#\{y \in N_x;\ \xi|_{T_xM_y} = 0\} \leqq 1 ,$$
$$\#\{x \in M_y;\ \eta|_{T_yN_x} = 0\} \leqq 1 .$$

Proof. By Theorem 5.1.2 of [1] and Corollary 3.2, we have only to show that $C_B = T_B^*(X \times Y)' \backslash 0$ is the graph of a diffeomorphism of $T^*Y \backslash 0$ to $T^*X \backslash 0$.

Lemma 4.2. *Suppose $n_X = n_Y$. Then C_B is the graph of a diffeomorphism if and only if the following conditions hold:*
(i)′ *the natural projection $C_B \to T^*X \backslash 0$ is a submersion;*
(ii)′ *the correspondence given by C_B between $T^*X \backslash 0$ and $T^*Y \backslash 0$ is one-to-one.*

Proof. It is trivial that the conditions are necessary. Assume that (i)′ and (ii)′ hold. Let C_B^0 be one of the connected components of C_B. The image of $C_B^0 \to T^*X \backslash 0$ is open by (i)′ and connected. On the other hand, since π_1 is proper, the mapping $C_B/\mathbf{R}^+ \to (T^*X \backslash 0)/\mathbf{R}^+$ induced by $C_B \to T^*X \backslash 0$ is also proper, whence it follows immediately that the image of $C_B^0 \to T^*X \backslash 0$ is closed. Since $n_X \geqq 2$, $T^*X \backslash 0$ is connected. Thus $C_B^0 \to T^*X \backslash 0$ is surjective. As for $C_B^0 \to T^*Y \backslash 0$, it is a submersion by Remark 3.5. Hence the same argument shows that $C_B^0 \to T^*Y \backslash 0$ is surjective. (ii)′ implies then C_B^0 is the graph of a diffeomorphism. But (ii)′ forces C_B to be C_B^0. Q.E.D.

Lemma 4.3. i) *Let $(x, \xi) \in T^*X \backslash 0$. Then the mapping $\phi_{(x,\xi)}$: $\{(y, \xi);\ (x, \xi, y, \eta) \in C_B\} \to \{y \in N_x;\ \xi|_{T_xM_y} = 0\}$ defined by $\phi_{(x,\xi)}(y, \eta) = y$ is bijective.*

ii) *Let $(y, \eta) \in T^*Y \backslash 0$. Then the mapping $\psi_{(y,\eta)}$: $\{(x, \xi);\ (x, \xi, y, \eta) \in C_B\} \to \{x \in M_y;\ \eta|_{T_yN_x} = 0\}$ defined by $\psi_{(y,\eta)}(x, \xi) = x$ is bijective.*

Proof. i) We note that $\phi_{(x,\xi)}$ is well defined. In fact $(x, \xi, y, \eta) \in C_B$ implies that $(\xi, -\eta) \in T^*_{(x,y)}(X \times Y)$ vanishes on $T_{(x,y)}B$ which includes $T_xM_y \times (0)$, whence $\xi|_{T_xM_y} = 0$.

Let $(x, \xi, y, \eta), (x, \xi, y, \eta') \in C_B$. Then

$$\pi_2^*\eta = \pi_1^*\xi = \pi_2^*\eta' \in T^*_{(x,y)}B .$$

Since π_2: $B \to Y$ is a submersion, we have $\eta = \eta'$. Hence $\phi_{(x,\xi)}$ is injective.

Let $y \in N_x$ with $\xi|_{T_x M_y} = 0$. Then $\pi_1^*\xi \in T^*_{(x,y)}B$ is zero on the tangent space of the fiber of π_2. Hence there is an $\eta \in T_y^* Y$ with $\pi_1^*\xi = \pi_2^*\eta$. Then $(x, \xi, y, \eta) \in C_B$. Hence $\phi_{(x,\xi)}$ is surjective.

(ii) can be proved in the same way. Q.E.D.

By Lemma 3.6, (i) implies (i)'. By Lemma 4.3, (ii) implies (ii)'. Hence by Lemma 4.2, C_B is the graph of a diffeomorphism. This completes the proof of Theorem 4.1.

REMARK. It is probable that the conditions (i) and (ii) are also necessary for L_B to have a parametrix.

EXAMPLE. Let $Y = \mathbf{RP}^n$ be the real projective space and $X = (\mathbf{RP}^n)^*$ the dual projective space. Put

$$B = \{(x, y) \in X \times Y;\ y \in x\} ,$$

and fix smooth positive densities $\{d\mu_x;\ x \in X\}$. Then it is easily seen that $k_B + d = n$, and the condition (ii) of Theorem 4.1 is satisfied. Hence, if $n \geqq 2$, L_B has a parametrix.

REMARK. It is known that if we choose appropriate densities then L_B is an isomorphism (cf. [5]).

5. Mean value operators on symmetric spaces of non-compact type

In Section 5 and Section 6, we shall study the mean value operator of example 1.3 more closely when G/K is a symmetric space. In this section we consider the case G/K is of non-compact type.

5.1. Let θ be the Cartan involution of $\mathfrak{g}$ for G/K and $\mathfrak{p}$ the (-1)-eigenspace of θ. Put

$$(X, Y) = -B(X, \theta Y) \qquad (X, Y \in \mathfrak{g}) ,$$

where $B(\ ,\ )$ is the Killing form of $\mathfrak{g}$. $(\ ,\ )$ gives a positive definite inner product on $\mathfrak{g}$. Let $\mathfrak{h}_\mathfrak{p}$ be a maximal abelian subspace in $\mathfrak{p}$ and extend $\mathfrak{h}_\mathfrak{p}$ to a maximal abelian subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ containing $\mathfrak{h}_\mathfrak{p}$. Put $\mathfrak{h}_\mathfrak{k} = \mathfrak{h} \cap \mathfrak{k}$. Then $\mathfrak{h} = \mathfrak{h}_\mathfrak{k} + \mathfrak{h}_\mathfrak{p}$. $\mathfrak{h}^\mathbf{C} = \mathfrak{h} \otimes \mathbf{C}$ is a Cartan subalgebra of $\mathfrak{g}^\mathbf{C} = \mathfrak{g} \otimes \mathbf{C}$. Define a real vector space $\mathfrak{h}_\mathbf{R}$ by

$$\mathfrak{h}_\mathbf{R} = \sqrt{-1}\,\mathfrak{h}_\mathfrak{k} + \mathfrak{h}_\mathfrak{p} \subset \mathfrak{g}^\mathbf{C}$$

and introduce in $\mathfrak{h}_\mathbf{R}$ the inner product corresponding to $(\ ,\ )$ under the natural isomorphism: $\mathfrak{h}_{\mathbf{R}} \cong \mathfrak{h}$. Denote the orthogonal projection $\mathfrak{h}_{\mathbf{R}} \to \mathfrak{h}_{\mathfrak{p}}$ by $\alpha \mapsto \bar{\alpha}$. We identify $\mathfrak{h}_{\mathbf{R}}$ naturally with its dual space using the inner product. Let $\Delta \subset \mathfrak{h}_{\mathbf{R}}$ be the root system of $\mathfrak{g}^{\mathbf{C}}$ with respect to $\mathfrak{h}^{\mathbf{C}}$. Let

$$\Delta_{\mathfrak{p}} = \{\alpha \in \Delta;\ \bar{\alpha} \neq 0\}\ ,$$
$$\tilde{\Delta} = \{\bar{\alpha};\ \alpha \in \Delta_{\mathfrak{p}}\} \subset \mathfrak{h}_{\mathfrak{p}}\,.$$

For $\alpha \in \Delta$, $\gamma \in \tilde{\Delta}$, put

$$\mathfrak{g}^{\alpha} = \{X \in \mathfrak{g}^{\mathbf{C}};\ [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}^{\mathbf{C}}\}\ ,$$
$$\tilde{\mathfrak{g}}^{\gamma} = \sum_{\bar{\alpha}=\gamma} \mathfrak{g}^{\alpha}\,,$$

where $\alpha(H)=(\alpha, H)$. Fix linear orderings in $\mathfrak{h}_{\mathbf{R}}$ and $\mathfrak{h}_{\mathfrak{p}}$ which are *compatible*, that is, $\bar{\alpha}>0$ implies $\alpha>0$ $(\alpha \in \mathfrak{h}_{\mathbf{R}})$. Let Δ^{+}, $\Delta_{\mathfrak{p}}^{+}$, $\tilde{\Delta}^{+}$ denote the sets of positive roots of Δ, $\Delta_{\mathfrak{p}}$, $\tilde{\Delta}$ respectively. Then

$$\tilde{\Delta}^{+} = \{\bar{\alpha};\ \alpha \in \Delta_{\mathfrak{p}}^{+}\}\ .$$

For $\gamma \in \tilde{\Delta}^{+}$ we put

$$\mathfrak{k}^{\gamma} = \mathfrak{k} \cap (\tilde{\mathfrak{g}}^{\gamma} + \tilde{\mathfrak{g}}^{-\gamma})\,,$$
$$\mathfrak{p}^{\gamma} = \mathfrak{p} \cap (\tilde{\mathfrak{g}}^{\gamma} + \tilde{\mathfrak{g}}^{-\gamma})\,.$$

We define also

$$\mathfrak{k}^{0} = \{X \in \mathfrak{k};\ [X, \mathfrak{h}_{\mathfrak{p}}] = 0\}$$
$$\mathfrak{p}^{0} = \mathfrak{h}_{\mathfrak{p}}\,.$$

Then we get orthogonal decompositions:

$$\mathfrak{k} = \mathfrak{k}^{0} + \sum_{\gamma \in \tilde{\Delta}^{+}} \mathfrak{k}^{\gamma}\,,$$
$$(5) \qquad \mathfrak{p} = \mathfrak{p}^{0} + \sum_{\gamma \in \tilde{\Delta}^{+}} \mathfrak{p}^{\gamma}\,.$$

Obviously we have

Lemma 5.1. *Let $H \in \mathfrak{h}_{\mathfrak{p}}$, $\gamma \in \tilde{\Delta}^{+}$. Then*
(i) $\mathrm{ad}(H)|_{\mathfrak{k}^{\gamma}}=0$ if $\gamma(H)=0$;
(ii) $\mathrm{ad}(H)|_{\mathfrak{k}^{\gamma}}$ *is an isomorphism of $\mathfrak{k}^{\gamma}$ onto $\mathfrak{p}^{\gamma}$ if $\gamma(H) \neq 0$.*

By virtue of the identity

$$(\mathrm{ad}(H)|_{\mathfrak{k}^{\gamma}+\mathfrak{p}^{\gamma}})^{2} = \gamma(H)^{2}\mathrm{id}_{\mathfrak{k}^{\gamma}+\mathfrak{p}^{\gamma}}$$

$(H \in \mathfrak{h}_{\mathfrak{p}}, \gamma \in \tilde{\Delta}^{+})$, Lemma 5.1 implies immediately the following

Lemma 5.2. *Let $H \in \mathfrak{h}_{\mathfrak{p}}$, $\gamma \in \tilde{\Delta}^{+} \cup \{0\}$. Put $\lambda_{H}=\mathrm{Ad}(\exp H)|_{\mathfrak{k}^{\gamma}+\mathfrak{p}^{\gamma}}$. Then*
(i) $\lambda_{H}=\mathrm{id}_{\mathfrak{k}^{\gamma}+\mathfrak{p}^{\gamma}}$ *if $\gamma(H)=0$;*
(ii) *if $\gamma(H) \neq 0$, λ_{H} is given by*

$$\begin{pmatrix} X \\ \\ Y \end{pmatrix} \mapsto \begin{pmatrix} \cosh \gamma(H) & \dfrac{\sinh \gamma(H)}{\gamma(H)} \mathrm{ad}(H) \\ \dfrac{\sinh \gamma(H)}{\gamma(H)} \mathrm{ad}(H) & \cosh \gamma(H) \end{pmatrix} \begin{pmatrix} X \\ \\ Y \end{pmatrix}$$

for $X \in \mathfrak{k}^\gamma$, $Y \in \mathfrak{p}^\gamma$.

Thus for $\gamma \in \tilde{\Delta}^+$ and $H \in \mathfrak{h}_\mathfrak{p}$ with $\gamma(H) \neq 0$, $p(\lambda_H|_{\mathfrak{k}^\gamma})$ is an isomorphism of $\mathfrak{k}^\gamma$ onto $\mathfrak{p}^\gamma$, $p: \mathfrak{g} \to \mathfrak{p}$ being the orthogonal projection.

Put $\mathfrak{m}_H = \sum' \mathfrak{p}^\gamma$ $(H \in \mathfrak{h}_\mathfrak{p})$, where γ runs over the set $\tilde{\Delta}^+ \backslash \tilde{\Delta}_H^+$, $\tilde{\Delta}_H^+ = \{\gamma \in \tilde{\Delta}^+;$ $\gamma(H) = 0\}$. We have $p\,\mathrm{Ad}(a^{-1})\mathfrak{k} = \mathfrak{m}_H$, where $a = \exp(H)$ $(H \in \mathfrak{h}_\mathfrak{p})$. In fact, Lemma 5.2 implies

$$p\,\mathrm{Ad}(a^{-1})\mathfrak{k} = \begin{cases} \mathfrak{p}^\gamma & \gamma \in \tilde{\Delta}^+ \backslash \tilde{\Delta}_H^+ \\ 0 & \gamma \in \tilde{\Delta}_H^+ \cup \{0\} . \end{cases}$$

Fix now an $a \in G$. Since there is an $H \in \mathfrak{h}_\mathfrak{p}$ such that $a \in K \exp(H) K$, we may assume $a = \exp(H)$. By Proposition 2.8 we have

$$k_{B_a} = \min_{u \in (\mathfrak{m}_H^\perp)^* \backslash 0} \mathrm{rank}(u \circ \Phi) ,$$

where $\mathfrak{m}_H = \sum'' \mathfrak{p}^\gamma$, γ ranging over the set $\tilde{\Delta}_H^+ \cup \{0\}$ and $\Phi: \mathfrak{k} \times \mathfrak{m}_H \to \mathfrak{m}_H^\perp$ is defined by $\Phi(X, Y) = p_0([X, Y])$ $(X \in \mathfrak{k},\ Y \in \mathfrak{m}_H)$, $p_0: \mathfrak{p} \to \mathfrak{m}_H^\perp$ being the orthogonal projection.

Lemma 5.3. $k_{B_a} = \min\limits_{Z \in \mathfrak{m}_H^\perp \backslash 0} \dim \mathrm{ad}(Z)(\sum' \mathfrak{k}^\gamma)$, *where* γ *ranges over the set* $\tilde{\Delta}^+ \backslash \tilde{\Delta}_H^+$.

Proof. Obviously

$$k_{B_a} = \min_{Z \in \mathfrak{m}_H^\perp \backslash 0} \mathrm{rank}\, \Phi_Z ,$$

where $\Phi_Z: \mathfrak{k} \times \mathfrak{m}_H \to \mathbf{R}$ $(Z \in \mathfrak{m}_H^\perp)$ is defined by

$$\Phi_Z(X, Y) = ([X, Y], Z) \qquad X \in \mathfrak{k},\ Y \in \mathfrak{m}_H .$$

The equality $([X, Y], Z) = (Y, \mathrm{ad}(Z)X)$ $(X \in \mathfrak{k})$ implies $\mathrm{rank}\, \Phi_Z = \dim p_1 \mathrm{ad}(Z)\mathfrak{k}$, where $p_1: \mathfrak{p} \to \mathfrak{m}_H$ is the natural projection. Since

$$[\mathfrak{m}_H^\perp, \textstyle\sum' \mathfrak{k}^\gamma] \subset \mathfrak{m}_H ,$$
$$[\mathfrak{m}_H^\perp, \textstyle\sum'' \mathfrak{k}^\gamma] \subset \mathfrak{m}_H^\perp ,$$

where in the summation $\sum''$, γ runs over the set $\tilde{\Delta}_H^+ \cup \{0\}$, we have

$$p_1 \mathrm{ad}(Z)\mathfrak{k} = \mathrm{ad}(Z)(\textstyle\sum' \mathfrak{k}^\gamma) . \qquad \text{Q.E.D.}$$

5.2. Now we assume a is *regular*, that is, $\dim KaK/K = \max_{g\in G} \dim KgK/K$. It is easily verified that $\exp(H)$ $(H\in\mathfrak{h}_\mathfrak{p})$ is regular if and only if $\tilde{\Delta}_H^+=\emptyset$. Then $\mathfrak{m}_H^\perp=\mathfrak{h}_\mathfrak{p}$. Lemma 5.1 and 5.3 imply immediately

$$k_{B_a} = \sum_{\gamma\in\tilde{\Delta}^+} \dim \mathfrak{p}^\gamma - s\,,$$

where $s = \max_{H'\in\mathfrak{h}_\mathfrak{p}\setminus 0} \sum_{\gamma\in\tilde{\Delta}_H^+} \dim \mathfrak{p}^\gamma$. Let $\mathcal{V}$ be the set of hyperplanes V of $\mathfrak{h}_\mathfrak{p}$ such that $\tilde{\Delta}_V = V\cap\tilde{\Delta}$ spans V. Then obviously

$$s = \max_{V\in\mathcal{V}} s_V\,,$$

where $s_V = \sum_{\gamma\in\tilde{\Delta}_V^+} \dim \mathfrak{p}^\gamma$ $(\tilde{\Delta}_V^+ = \tilde{\Delta}^+\cap V)$. It is obvious that for each $V\in\mathcal{V}$ there are such compatible orderings in $\mathfrak{h}_\mathbf{R}$ and $\mathfrak{h}_\mathfrak{p}$ that $\{\alpha_1, \cdots, \alpha_l\}$, $\{\bar{\alpha}_1, \cdots, \bar{\alpha}_r\}$ and $\{\bar{\alpha}_2, \cdots, \bar{\alpha}_r\}$ are the sets of the simple positive roots of Δ, $\tilde{\Delta}$ and $\tilde{\Delta}_V$ respectively. Here $l=\dim\mathfrak{h}_\mathbf{R}$, $r=\dim\mathfrak{h}_\mathfrak{p}$ and the ordering in V is induced from that of $\mathfrak{h}_\mathfrak{p}$. We may assume that the compatible orderings in $\mathfrak{h}_\mathbf{R}$ and $\mathfrak{h}_\mathfrak{p}$ chosen before have the above property, since the number s_V is independent of the choice of them. Thus

$$\begin{aligned} r_V &= \sum_{\gamma\in\tilde{\Delta}_V^+} \dim \mathfrak{p}^\gamma \\ &= \sum_{\gamma\in\tilde{\Delta}_V^+} \dim_\mathbf{C} \tilde{\mathfrak{g}}^\gamma \\ &= \sum_{\alpha\in\Delta_V^+} \dim_\mathbf{C} \mathfrak{g}^\alpha \\ &= {}^\#\Delta_V^+ \\ &= {}^\#\{\alpha\in\Delta^+;\ \bar{\alpha} = m_2\bar{\alpha}_2+\cdots+m_r\bar{\alpha}_r\neq 0\}\,, \end{aligned}$$

where $\Delta_V^+=\{\alpha\in\Delta^+;\ \bar{\alpha}\in\tilde{\Delta}_V^+\}$. Put

$$\tilde{s} = \max_{1\leqq j\leqq r} {}^\#\{\alpha\in\Delta^+;\ \bar{\alpha} = m_1\bar{\alpha}_1+\cdots+m_r\bar{\alpha}_r\neq 0,\ m_j = 0\}\,,$$

which is clearly independent of the choice of the orderings and satisfies $s\leqq\tilde{s}$, since $s_V\leqq\tilde{s}$ $(V\in\mathcal{V})$. On the other hand, putting $V_j=\{\sum_{k=1}^r a_k\bar{\alpha}_k;\ a_k\in\mathbf{R},\ a_j=0\}\in\mathcal{V}$, we have $\tilde{s} = \max_{1\leqq j\leqq r} s_{V_j}\leqq s$. Hence $\tilde{s}=s$. Therefore

$$\begin{aligned} k_{B_a} &= \sum_{\gamma\in\tilde{\Delta}^+} \dim \mathfrak{p}^\gamma - s \\ &= \min_{1\leqq j\leqq r} {}^\#\{\alpha\in\Delta^+;\ \bar{\alpha} = m_1\bar{\alpha}_1+\cdots+m_r\bar{\alpha}_r,\ m_j\neq 0\}\,. \end{aligned}$$

In conclusion we have proved

Theorem 5.4. *Suppose a is regular. Then $k_{B_a}=k(G/K)$, where*

$$k(G/K) = \min_{1\leqq j\leqq r} {}^{\#}\{\alpha\in\Delta^+;\ \bar{\alpha} = m_1\bar{\alpha}_1+\cdots+m_r\bar{\alpha}_r,\ m_j\neq 0\}\ ,$$

which is independent of a.

5.3. Next we consider the case a is no longer regular but non-degenerate. $a\in G$ is called *non-degenerate* if the following condition holds:

(4) Let $\tilde{X}\xrightarrow{\pi} X$ be the universal covering of X and $\tilde{X}=X_1\times\cdots\times X_N$ the decomposition into the irreducible factors. Then $\pi^{-1}(KaK/K)$ is not included in any subsets of the form $X_1\times\cdots\times X_{i-1}\times F_i\times X_{i+1}\times\cdots\times X_N$ $(1\leqq i\leqq N)$ with ${}^{\#}F_i<\infty$.

Theorem 5.5. *Suppose $X=G/K$ is a symmetric space of non-compact type and $a\in G$ is non-degenerate. Then*

$$\operatorname{reg} M^a>0\ .$$

We prepare three lemmas.

Lemma 5.6. *If $a=\exp(H)$ $(H\in\mathfrak{h}_{\mathfrak{p}})$ is non-degenerate, then $\Delta\backslash\Delta_H$ spans $\mathfrak{h}_{\mathbf{R}}$, where $\Delta_H=\{\alpha;\ \alpha(H)=0\}$.*

Proof. Let W be the subspace of $\mathfrak{h}_{\mathbf{R}}$ spanned by $\Delta\backslash\Delta_H$. Assume $W\neq\mathfrak{h}_{\mathbf{R}}$. Put $V=\{H'\in\mathfrak{h}_{\mathbf{R}};\ (H', H)=0\}$. Since $\Delta\subset W\cup V$, we have $\mathfrak{h}_{\mathbf{R}}=V+W$. Put $U=V\cap W$, $\Delta_U=\Delta\cap U=\Delta_H\cap U$, $\Delta_W=\Delta\cap W$. Then $(\alpha, \beta)=0$ for all $\alpha\in\Delta_H\backslash\Delta_U$, $\beta\in\Delta_W\backslash\Delta_U$. In fact we have $\pm\alpha+\beta\notin V\cup W$; otherwise $\alpha\in W$ or $\beta\in V$, a contradiction. Hence the α-series of β consists of β alone. Thus we have $(\alpha, \beta)=0$.

It follows then that $\Delta_H\backslash\Delta_U\subset W^{\perp}$, since $\Delta_W\backslash\Delta_U=\Delta\backslash\Delta_H$ spans W. Hence we have a non-trivial orthogonal decomposition: $\Delta=\Delta'\cup\Delta''$, where $\Delta'=\Delta_W$, $\Delta''=\Delta_H\backslash\Delta_U$. Let $\mathfrak{g}=\mathfrak{g}'+\mathfrak{g}''$ be the corresponding decomposition. Then Lemma 5.1 implies $p\operatorname{Ad}(a^{-1})\mathfrak{k}''=0$ $(\mathfrak{k}''=\mathfrak{k}\cap\mathfrak{g}'')$, where p: $\mathfrak{g}''\to\mathfrak{p}''$ $(\mathfrak{p}''=\mathfrak{p}\cap\mathfrak{g}'')$ is the natural projection.

Let $X_i=G_i/K_i$ and $\mathfrak{g}_i$ be the Lie algebra of G_i. Then $\mathfrak{g}=\mathfrak{g}_1\oplus\cdots\oplus\mathfrak{g}_N$. Since X_i is of non-compact type, $\mathfrak{g}_i$ is simple. Hence $\mathfrak{g}''$ contains some $\mathfrak{g}_i$. Note that the tangent space at K_i of the projection of $\pi^{-1}(KaK/K)$ on the i-th factor X_i is isomorphic to $p\operatorname{Ad}(a^{-1})\mathfrak{k}_i(\mathfrak{k}_i=\mathfrak{k}''\cap\mathfrak{g}_i)$, which is zero. Thus

$$\pi^{-1}(KaK/K)\subset X_1\times\cdots\times X_{i-1}\times F_i\times X_{i+1}\times\cdots\times X_N$$

with ${}^{\#}F_i<\infty$, whence a is not non-degenerate. Q.E.D.

Lemma 5.7. *For $Z\in\mathfrak{m}_H^{\perp}$, the condition:*

(6) $[Z, \mathfrak{k}^{\gamma}] = 0,\ \gamma\in\tilde{\Delta}^+\backslash\tilde{\Delta}_H^+$

implies

$$[Z, \mathfrak{g}^{\alpha}] = 0, \ \alpha \in \Delta \backslash \Delta_H .$$

Proof. For $\alpha \in \Delta \backslash \Delta_H$, we have non-zero elements X_{α}, $X_{-\alpha}$ respectively of $\mathfrak{g}^{\alpha}$, $\mathfrak{g}^{-\alpha}$ such that

$$(\mathfrak{g}^{\alpha}+\mathfrak{g}^{-\alpha}) \cap \mathfrak{k} = \mathbf{R} \cdot (X_{\alpha}+X_{-\alpha}) .$$

Obviously (6) is equivalent to

$$[Z, X_{\alpha}+X_{-\alpha}] = 0 , \qquad \alpha \in \Delta \backslash \Delta_H .$$

Put $Z = Z_0 + \sum_{\beta \in \Delta_H} Z_{\beta}$ ($Z_0 \in \mathfrak{h}^{\mathbf{C}}$, $Z_{\beta} \in \mathfrak{g}^{\beta}$). We have

$$[Z_{\beta}, X_{\alpha}+X_{-\alpha}] \ni \mathfrak{g}^{\alpha+\beta} + \mathfrak{g}^{-\alpha+\beta} .$$

Here $\mathfrak{g}^0 = \mathfrak{h}^{\mathbf{C}}$, $\mathfrak{g}^{\gamma} = 0$ if $\gamma \notin \Delta$. Note that for β, $\beta' \in \Delta_H \cup \{0\}$ with $\beta \neq \beta'$ we have $\alpha+\beta \neq \alpha+\beta'$, $-\alpha+\beta'$. In fact $\alpha+\beta = -\alpha+\beta'$ would imply $\alpha(H)=0$, which contradicts $\alpha \in \Delta \backslash \Delta_H$. Thus

$$[Z_{\beta}, X_{\alpha}] = 0 , \qquad \beta \in \Delta_H \cup \{0\}, \ \alpha \in \Delta \backslash \Delta_H .$$

In particular

$$[Z, \mathfrak{g}^{\alpha}] = 0 , \qquad \alpha \in \Delta \backslash \Delta_H . \qquad \text{Q.E.D.}$$

Lemma 5.8. *For $Z \in \mathfrak{g}^{\mathbf{C}}$, the condition:*

$$[Z, \mathfrak{g}^{\alpha}] = 0 , \qquad \alpha \in \Delta \backslash \Delta_H ,$$

implies $Z \in \mathfrak{h}_H + \sum_{\beta \in \Delta \cap \mathfrak{h}_H} \mathfrak{g}^{\beta}$, where

$$\mathfrak{h}_H = \{H' \in \mathfrak{h}_{\mathbf{R}};\ \alpha(H') = 0 \quad \text{for all } \alpha \in \Delta \backslash \Delta_H\} .$$

Proof. For any subspace $V \subset \mathfrak{g}^{\mathbf{C}}$ we put

$$C(V) = \{X \in \mathfrak{g}^{\mathbf{C}};\ [V, X] = 0\} .$$

Let $\mathfrak{h}_{\alpha} = \{H \in \mathfrak{h}^{\mathbf{C}};\ \alpha(H)=0\}$ ($\alpha \in \mathfrak{h}_{\mathbf{R}}$). Then

$$C(\mathfrak{g}^{\alpha}) = \mathfrak{h}_{\alpha} + \sum{}' \mathfrak{g}^{\beta} ,$$

where β runs over the set $\{\beta \in \Delta;\ \alpha+\beta \notin \Delta \cup \{0\}\}$. In fact let $Z = Z_0 + \sum_{\beta \in \Delta} Z_{\beta}$ ($Z_{\beta} \in \mathfrak{g}^{\beta}$), $\mathfrak{g}^{\alpha} = \mathbf{C}.X_{\alpha}$. Evidently $[Z, \mathfrak{g}^{\alpha}]=0$ implies

$$[Z_{\beta}, X_{\alpha}] = 0 , \qquad \beta \in \Delta \cup \{0\} .$$

If $\beta \neq 0$ and $Z^{\beta} \neq 0$, then we must have $\alpha+\beta \in \Delta \cup \{0\}$. If $\beta=0$, then $[Z_0, X_{\alpha}] = \alpha(Z_0)X_{\alpha}$, whence $Z^0 \in \mathfrak{h}_{\alpha}$. Thus

$$C(\mathfrak{g}^{\alpha})\subset\mathfrak{h}_{\alpha}+\sum'\mathfrak{g}^{\beta}.$$

The other inclusion is obvious.

Thus

$$C(\sum_{\alpha\in\Delta\backslash\Delta_H}\mathfrak{g}^{\alpha})=\bigcap_{\alpha\in\Delta\backslash\Delta_H}\mathfrak{h}_{\alpha}+\sum''\mathfrak{g}^{\beta},$$

where β runs over the set $\{\beta\in\Delta;\ \pm\alpha+\beta\notin\Delta\cup\{0\},\ \alpha\in\Delta\backslash\Delta_H\}$. But $\pm\alpha+\beta\notin\Delta\cup\{0\}$ implies that the α-series of β consists of β alone, whence $(\alpha,\beta)=0$. Thus $\sum''\mathfrak{g}^{\beta}\subset\sum_{\beta\in\Delta\cap\mathfrak{h}_H}\mathfrak{g}^{\beta}$. Q.E.D.

Proof of Theorem 5.5. We may assume $a=\exp(H)$ $(H\in\mathfrak{h}_{\mathfrak{p}})$. By Lemma 5.3, it suffices to show that $Z\in\mathfrak{m}_H^{\perp}$ is zero if it satisfies (6). By Lemmas 5.7 and 5.8, (6) implies $Z\in\mathfrak{h}_H+\sum_{\beta\in\Delta\cap\mathfrak{h}_H}\mathfrak{g}^{\beta}$. But by virtue of Lemma 5.6 we have $\mathfrak{h}_H=0$, whence $Z=0$. Q.E.D.

6. Mean value operators on symmetric spaces of compact type

In this section we assume $X=G'/K'$ is a symmetric space of compact type.

6.1. Let G/K be the symmetric space of non-compact type dual to X. We retain the previous notations for G/K. Let $\mathfrak{g}'$ and $\mathfrak{k}'$ be the Lie algebras of G' and K' respectively. There are identifications: $\mathfrak{g}'=\mathfrak{k}+i\mathfrak{p}\subset\mathfrak{g}^{C}$, $\mathfrak{k}=\mathfrak{k}'$ $(i=\sqrt{-1})$. (5) gives an orthogonal decomposition:

$$i\mathfrak{p}=\sum_{\gamma\in\tilde{\Delta}^{+}\subset\{0\}}i\mathfrak{p}^{\gamma}.$$

As before we have the following

Lemma 6.1. *Let $H\in\mathfrak{h}_{\mathfrak{p}}$, $\gamma\in\tilde{\Delta}^{+}\cup\{0\}$. Put*

$$\lambda_{iH}=\mathrm{Ad}(\exp(iH))|_{\mathfrak{k}^{\gamma}+i\mathfrak{p}^{\gamma}}.\quad \textit{Then}$$

(i) $\lambda_{iH}=\mathrm{id}_{\mathfrak{k}^{\gamma}+i\mathfrak{p}^{\gamma}}$, *if* $\gamma(H)=0$;

(ii) *if* $\gamma(H)\neq0$, λ_{iH} *is given by*

$$\begin{pmatrix}X\\ \\ Y\end{pmatrix}\mapsto\begin{pmatrix}\cos\gamma(H) & \dfrac{\sin\gamma(H)}{\gamma(H)}\mathrm{ad}(iH)\\ \\ \dfrac{\sin\gamma(H)}{\gamma(H)}\mathrm{ad}(iH) & \cos\gamma(H)\end{pmatrix}\begin{pmatrix}X\\ \\ Y\end{pmatrix},$$

$X\in\mathfrak{k}^{\gamma}$, $Y\in i\mathfrak{p}^{\gamma}$.

Hence, for $\gamma\in\tilde{\Delta}^{+}$ and $H\in\mathfrak{h}_{\mathfrak{p}}$ with $\gamma(H)\notin\pi\mathbf{Z}$, $\bar{p}\lambda_{iH}|_{\mathfrak{k}^{\gamma}}$ is an isomorphism of $\mathfrak{k}^{\gamma}$ onto $i\mathfrak{p}^{\gamma}$, $\bar{p}$: $\mathfrak{g}'\rightarrow i\mathfrak{p}$ being the orthogonal projection.

Put $\overline{\mathfrak{m}}_H = \sum' i\mathfrak{p}^\gamma$ where γ runs over the set $\tilde{\Delta}^+ \backslash \tilde{\Delta}^+_{(H)}$, $\tilde{\Delta}^+_{(H)} = \{\gamma \in \tilde{\Delta}^+;\ \gamma(H) \in \pi\mathbf{Z}\}$. We have

$$\bar{p}\,\mathrm{Ad}(a^{-1})\mathfrak{k} = \overline{\mathfrak{m}}_H\,,$$

where $a = \exp(iH)$. In fact, Lemma 6.1 implies

$$\bar{p}\,\mathrm{Ad}(a^{-1})\mathfrak{k}^\gamma = \begin{cases} \mathfrak{p}^\gamma & \gamma \in \tilde{\Delta}^+ \backslash \tilde{\Delta}^+_{(H)} \\ 0 & \gamma \in \tilde{\Delta}^+_{(H)} \cup \{0\}\,. \end{cases}$$

By Proposition 2.8, we have

$$k_{B_a} = \min_{u \in (\overline{\mathfrak{m}}_H^\perp)^* \backslash 0} \mathrm{rank}\, v\overline{\Phi}\,,$$

where $\overline{\mathfrak{m}}_H^\perp = \sum'' i\mathfrak{p}^\gamma$, γ running over the set $\tilde{\Delta}^+_{(H)} \cup \{0\}$ and $\overline{\Phi}\colon \mathfrak{k} \times \overline{\mathfrak{m}}_H \to \overline{\mathfrak{m}}_H^\perp$ is defined by

$$\overline{\Phi}(X, Y) = \bar{p}_0([X, Y])\,, \qquad X \in \mathfrak{k}\,,\ Y \in \overline{\mathfrak{m}}_H\,,$$

$\bar{p}_0\colon i\mathfrak{p} \to \overline{\mathfrak{m}}_H^\perp$ being the orthogonal projection.

Lemma 6.2. $k_{B_a} = \min_{Z \in \overline{\mathfrak{m}}_H^\perp \backslash 0} \dim \mathrm{ad}(Z)(\sum' \mathfrak{k}^\gamma)$, *where* γ *runs over the set* $\tilde{\Delta}^+ \backslash \tilde{\Delta}^+_{(H)}$.

Proof. Obviously

$$k_{B_a} = \min_{Z \in \overline{\mathfrak{m}}_H^\perp \backslash 0} \mathrm{rank}\, \overline{\Phi}_Z\,,$$

where $\overline{\Phi}_Z\colon \mathfrak{k} \times \overline{\mathfrak{m}}_H \to \mathbf{R}(Z \in \overline{\mathfrak{m}}_H^\perp)$ is defined by

$$\overline{\Phi}_Z(X, Y) = ([X, Y], Z) \qquad X \in \mathfrak{k}\,,\ Y \in \overline{\mathfrak{m}}_H\,.$$

Just as in the case of Lemma 5.3, we have

$$\mathrm{rank}\, \overline{\Phi}_Z = \dim \bar{p}_1\, \mathrm{ad}(Z)\mathfrak{k}\,,$$

$\bar{p}_1\colon i\mathfrak{p} \to \overline{\mathfrak{m}}_H$ being the natural projection. Since

$$[\overline{\mathfrak{m}}_H^\perp, \textstyle\sum' \mathfrak{k}^\gamma] \subset \overline{\mathfrak{m}}_H\,,$$
$$[\overline{\mathfrak{m}}_H^\perp, \textstyle\sum'' \mathfrak{k}^\gamma] \subset \overline{\mathfrak{m}}_H^\perp\,,$$

where, in the summation $\sum''$, γ runs over the set $\tilde{\Delta}^+_{(H)} \cup \{0\}$, we have

$$\bar{p}_1\, \mathrm{ad}(Z)\mathfrak{k} = \mathrm{ad}(Z)(\textstyle\sum' \mathfrak{k}^\gamma)\,. \qquad \text{Q.E.D.}$$

6.2. Suppose $a = \exp(iH) \in G'$ $(H \in \mathfrak{h}_\mathfrak{p})$ is *regular*, that is $\dim K'aK'/K' = \max_{g \in G'} \dim K'gK'/K'$. It is easily verified that a is regular if and only if $\tilde{\Delta}^+_{(H)} = \emptyset$.

Then $\overline{\mathfrak{m}}_H = i\mathfrak{m}_H$ and $\overline{\mathfrak{m}}_H^\perp = i\mathfrak{h}_\mathfrak{p} = i\mathfrak{m}_H^\perp$. Since $\tilde{\Delta}_H^+ = \tilde{\Delta}_{(H)}^+ = \emptyset$, it follows from Lemma 6.2, Lemma 5.3 and Theorem 5.4 that

$$\begin{aligned} k_{B_a} &= \min_{Z \in \mathfrak{h}_\mathfrak{p} \setminus 0} \dim \mathrm{ad}(iZ)(\sum \mathfrak{k}^\gamma) \\ &= \min_{Z \in \mathfrak{h}_\mathfrak{p} \setminus 0} \dim \mathrm{ad}(Z)(\sum \mathfrak{k}^\gamma) \\ &= k(G/K)\,. \end{aligned}$$

Hence we obtain

Theorem 6.3. *Suppose $X = G'/K'$ is a symmetric space of compact type and $a \in G'$ is regular. Let G/K be the symmetric space dual to X. Then $k_{B_a} = k(G/K)$.*

6.3. Finally we consider the case where a is *non-degenerate,* that is, the condition (4) holds for a.

Theorem 6.4. *Suppose $X = G'/K'$ is a symmetric space of compact type and $a \in G'$ is non-degenerate. Then*

$$\mathrm{reg}\, M^a > 0\,.$$

The proof proceeds just as before.

Lemma 6.5. *If $a = \exp(iH)$ $(H \in \mathfrak{h}_\mathfrak{p})$ is non-degenerate, then $\Delta \setminus \Delta_{(H)}$ spans $\mathfrak{h}_\mathbf{R}$, where $\Delta_{(H)} = \{\alpha \in \Delta;\ \alpha(H) \in \pi\mathbf{Z}\}$.*

Proof. Let W be the subspace of $\mathfrak{h}_\mathbf{R}$ spanned by $\Delta \setminus \Delta_{(H)}$ and assume $W \neq \mathfrak{h}_\mathbf{R}$. Put $\Delta_W = \Delta \cap W$, $\Gamma_{(H)} = \{H' \in \mathfrak{h}_\mathbf{R};\ (H, H') \in \pi\mathbf{Z}\}$. Then $(\alpha, \beta) = 0$ for all $\alpha \in \Delta_W \setminus (\Delta_W \cap \Gamma_{(H)}) = \Delta \setminus \Delta_{(H)}$, $\beta \in \Delta_{(H)} \setminus (\Delta_W \cap \Gamma_{(H)})$. In fact we must have $\pm\alpha + \beta \notin \Delta$, since $\alpha \notin \Gamma_{(H)}$, $\beta \in \Gamma_{(H)}$ imply $\pm\alpha + \beta \notin \Delta_{(H)}$ and $\alpha \in W$, $\beta \notin W$ guarantee $\pm\alpha + \beta \notin \Delta \setminus \Delta_{(H)} \subset W$. Hence the α-series of β consists of β alone. Thus $(\alpha, \beta) = 0$.

We have then $\Delta_{(H)} \setminus (\Delta_W \cap \Gamma_{(H)}) \subset W^\perp$ and get a nontrivial orthogonal decomposition $\Delta = \Delta_I \cup \Delta_{II}$ where $\Delta_I = \Delta_W$, $\Delta_{II} = \Delta_{(H)} \setminus (\Delta_W \cap \Gamma_{(H)})$. Let $\mathfrak{g}' = \mathfrak{g}_I \oplus \mathfrak{g}_{II}$ be the corresponding decomposition of $\mathfrak{g}'$. Let $X_i = G_i'/K_i'$ and $\mathfrak{g}_i'$ be the Lie algebra of G_i'. Then $\mathfrak{g}' = \mathfrak{g}_1' \oplus \cdots \oplus \mathfrak{g}_N'$. Let $\Delta = \Delta_1 \cup \cdots \cup \Delta_N$ be the corresponding decomposition. We claim $\Delta_i \subset \Gamma_{(H)}$ for some i. In fact there is a $\mathfrak{g}_i'$ such that $\mathfrak{g}_i' \cap \mathfrak{g}_{II} \neq (0)$. If $\mathfrak{g}_i'$ is simple, $\mathfrak{g}_i' \subset \mathfrak{g}_{II}$ and then $\Delta_i \subset \Gamma_{(H)}$. Suppose $\mathfrak{g}_i'$ is not simple and $\mathfrak{g}_i' \not\subset \mathfrak{g}_{II}$. Then $\mathfrak{g}_i'$ can be identified with $\mathfrak{u} \oplus \mathfrak{u}$, where $\mathfrak{u} = \mathfrak{g}_i' \cap \mathfrak{g}_{II}$ is simple and the $\mathfrak{g}_i'$-component of $\sqrt{-1}\,H$ is of type $\sqrt{-1}(H_i \oplus (-H_i))$. Since $(\alpha, H) = (\alpha, H_i) \in \pi\mathbf{Z}$ for $\alpha \in \Delta \cap (\mathfrak{u} \oplus (0))$, we have $(\alpha, H) \in \pi\mathbf{Z}$ also for $\alpha \in \Delta \cap ((0) \oplus \mathfrak{u})$. Thus $\Delta_i \subset \Gamma_{(H)}$.

Lemma 6.1 implies then $\bar{p}\,\mathrm{Ad}(a^{-1})\mathfrak{k}_i' = 0$ $(\mathfrak{k}_i' = \mathfrak{k}' \cap \mathfrak{g}_i')$, where $\bar{p}\colon \mathfrak{g}_i' \to \mathfrak{p}_i'$ $(\mathfrak{p}_i' = \mathfrak{p} \cap \mathfrak{g}_i')$ is the natural projection. Since the tangent space at $K_i' \in X_i$ of the projection of $\pi^{-1}(K'aK'/K')$ on the i-th factor X_i is isomorphic to $\bar{p}\,\mathrm{Ad}(a^{-1})\mathfrak{k}_i'$, we have

$$\pi^{-1}(K'aK'/K')\subset X_1\times\cdots\times X_{i-1}\times F_i\times X_{i+1}\times\cdots\times X_N$$

with ${}^{\#}F_i<\infty$, whence a is not non-degenerate. Q.E.D.

From now on we fix $a=\exp(iH)$ $(H\in\mathfrak{g}_{\mathfrak{p}})$ which is non-degenerate.

Lemma 6.6. *For $Z\in\overline{\mathfrak{m}}_H^{\perp}$, the condition:*

$$[Z,\mathfrak{k}^{\gamma}]=0\,,\qquad \gamma\in\tilde{\Delta}^{+}\backslash\tilde{\Delta}^{+}_{(H)}\,, \tag{7}$$

implies

$$[Z,\mathfrak{g}^{\alpha}]=0\,,\qquad \alpha\in\Delta\backslash\Delta_{(H)}\,.$$

Proof. Let $\{X_\alpha;\alpha\in\Delta\}$ be a Weyl basis of $\mathfrak{g}^{\mathbf{C}}\bmod\mathfrak{h}^{\mathbf{C}}$, that is, $X_\alpha\in\mathfrak{g}^{\alpha}$ and the following hold:

$$[X_\alpha, X_{-\alpha}]=\alpha\,,\qquad \alpha\in\Delta\,,$$
$$[X_\alpha, X_\beta]=N_{\alpha,\beta}X_{\alpha+\beta}\,,\qquad \alpha,\beta\in\Delta,\ \alpha+\beta\in\Delta\,,$$

where $N_{\alpha,\beta}\neq 0$. Then

$$N_{\alpha,\beta}=-N_{-\alpha,-\beta}=-N_{\beta,\alpha}\,,\qquad \alpha,\beta\in\Delta\,.$$

Let σ and τ be the conjugations of $\mathfrak{g}^{\mathbf{C}}$ with respect to $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ and $\mathfrak{g}'=\mathfrak{k}+\sqrt{-1}\mathfrak{p}$ respectively. Then we can take $\{X_\alpha\}$ so that $\tau X_\alpha=-X_{-\alpha}$ for $\alpha\in\Delta$ (cf. [4]). Since $\mathfrak{k}=(1+\sigma+\tau+\sigma\tau)\mathfrak{g}^{\mathbf{C}}$, we have for $\gamma\in\tilde{\Delta}^{+}$

$$\begin{aligned}\mathfrak{k}^{\gamma}&=\sum_{\tilde{\alpha}=\gamma}\mathfrak{k}\cap(\mathfrak{g}^{\alpha}+\mathfrak{g}^{-\alpha}+\sigma\mathfrak{g}^{\alpha}+\sigma\mathfrak{g}^{-\alpha})\\&=\sum_{\tilde{\alpha}=\gamma}(\mathbf{R}.(X_\alpha+\sigma X_\alpha-X_{-\alpha}-\sigma X_{-\alpha})+\mathbf{R}.i(X_\alpha-\sigma X_\alpha-X_{-\alpha}+\sigma X_{-\alpha}))\,.\end{aligned}$$

Hence

$$\mathbf{C}.\mathfrak{k}^{\gamma}=\sum_{\tilde{\alpha}=\gamma}(\mathbf{C}.(X_\alpha-X_{-\alpha})+\mathbf{C}.(\sigma X_\alpha-\sigma X_{-\alpha}))\,.$$

Thus (7) implies

$$[Z, X_\alpha-X_{-\alpha}]=0\,,\qquad \alpha\in\Delta\backslash\Delta_{(H)}\,.$$

Put $Z=Z_0+\sum_{\lambda\in\Delta_{(H)}} z_\lambda X_\lambda$ $(Z_0\in\mathfrak{p}^0,\ z_\lambda\in\mathbf{C})$. We have

$$[Z_0, X_\alpha]=0\,,\qquad \alpha\in\Delta\backslash\Delta_{(H)}\,,$$

since the $\mathfrak{g}^{\alpha}$-component of $[Z, X_\alpha-X_{-\alpha}]$ is $[Z_0, X_\alpha]$.

Let $\lambda\in\Delta_{(H)}$, $\alpha\in\Delta\backslash\Delta_{(H)}$. We will show $[z_\lambda X_\lambda, X_\alpha]=0$. We may assume $\lambda+\alpha\in\Delta$.

First we consider the easier case: $\lambda+2\alpha\notin\Delta_{(H)}$. Then $\lambda+\alpha\neq\mu\pm\alpha$ for any $\mu\in\Delta_{(H)}$. Then the $\mathfrak{g}^{\lambda+\alpha}$-component of $[Z, X_\alpha-X_{-\alpha}]$ is $[z_\lambda X_\lambda, X_\alpha]$, whence $[z_\lambda X_\lambda, X_\alpha]=0$.

We assume now $\lambda+2\alpha=\mu\in\Delta_{(H)}$. Then $\mu\neq\pm\lambda$, $\pm\sigma\lambda$. Otherwise $2\alpha(H)=\mu(H)-\lambda(H)=\overline{\mu}(H)-\overline{\lambda}(H)\in 2\pi\mathbf{Z}$, whence $\alpha\in\Gamma_{(H)}$, which contradicts $\alpha\in\Delta\backslash\Delta_{(H)}$. Since the α-series of λ contains λ and $\lambda+2\alpha$, we have $\beta=\lambda+\alpha=\mu-\alpha\in\Delta$. Note that $\alpha=\beta-\lambda=\mu-\beta$. The $\mathfrak{g}^{\beta}$-, $\mathfrak{g}^{-\beta}$-components of $[Z, X_\alpha-X_{-\alpha}]$ are respectively

$$[z_\lambda X_\lambda, X_\alpha]-[z_\mu X_\mu, X_{-\alpha}],$$
$$-[z_{-\lambda}X_{-\lambda}, X_{-\alpha}]+[z_{-\mu}X_{-\mu}, X_\alpha],$$

and the $\mathfrak{g}^{\alpha}$-, $\mathfrak{g}^{-\alpha}$-components of $[Z, X_\beta-X_{-\beta}]$ are respectively

$$[z_{-\lambda}X_{-\lambda}, X_\beta]-[z_\mu X_\mu, X_{-\beta}],$$
$$-[z_\lambda X_\lambda, X_{-\beta}]+[z_{-\mu}X_{-\mu}, X_\beta].$$

Hence $NY=0$, where $Y={}^t(z_\lambda, z_\mu, z_{-\lambda}, z_{-\mu})$ and

$$N=\begin{pmatrix} N_{\lambda,\alpha} & -N_{\mu,-\alpha} & 0 & 0 \\ 0 & 0 & -N_{-\lambda,-\alpha} & N_{-\mu,\alpha} \\ 0 & -N_{\mu,-\beta} & N_{-\lambda,\beta} & 0 \\ -N_{\lambda,-\beta} & 0 & 0 & N_{-\mu,\beta} \end{pmatrix}$$

Since $\det N=(N_{\lambda,\alpha}N_{\mu,-\beta})^2+(N_{\mu,\alpha}N_{\lambda,-\beta})^2\neq 0$, we have $z_\lambda=0$. In particular $[z_\lambda X_\lambda, X_\alpha]=0$.

Thus we have shown $[Z, X_\alpha]=0$, $\alpha\in\Delta\backslash\Delta_{(H)}$. Q.E.D.

Proof of Theorem 6.4. We may assume $a=\exp(iH)$ $(H\in\mathfrak{h}_{\mathfrak{p}})$. By Lemma 6.2 it suffices to show that $Z\in\overline{\mathfrak{m}}_H^{\perp}$ is zero if it satisfies (7). By Lemma 6.6, we have

$$[Z, \mathfrak{g}^{\alpha}]=0, \qquad \alpha\in\Delta\backslash\Delta_{(H)}.$$

Then the same arguments as in the proof of Lemma 5.8 show

$$Z\in\mathfrak{h}_{(H)}+\sum_{\beta\in\Delta\cap\mathfrak{h}_{(H)}}\mathfrak{g}_\beta,$$

where $\mathfrak{h}_{(H)}=\{H'\in\mathfrak{h}_{\mathbf{R}};\ \alpha(H')=0 \text{ for all } \alpha\in\Delta\backslash\Delta_{(H)}\}$. By Lemma 6.5, $\mathfrak{h}_{(H)}=0$. Hence $Z=0$. Q.E.D.

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