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A CHARACTERIZATION OF BOUNDED KRULL PRIME RINGS

HIDETOSHI MARUBAYASHI

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In [9] we defined the concept of non commutative Krull prime rings from the point of view of localizations and we mainly investigated the ideal theory in bounded Krull prime rings (cf. [9], [10]).

The purpose of this paper is to prove the following:

Theorem. *Let R be a prime Goldie ring with two-sided quotient ring Q. Then R is a bounded Krull prime ring if and only if it satisfies the following conditions;*

(1) *R is a regular maximal order in Q (in the sense of Asano).*

(2) *R satisfies the maximum condition for integral right and left v-ideals.*

(3) *R/P is a prime Goldie ring for any minimal prime ideal P of R.*

As corollary we have

Corollary. *Let R be a noetherian prime ring. If R is a regular maximal order in Q, then it is a bounded Krull prime ring.*

In case R is a commutative domain, the theorem is well known and its proof is easy (cf. [11]). We shall prove the theorem by using properties of one-sided v-ideals and torsion theories.

Throughout this paper let R be a prime Goldie ring, not artinian ring, having identity element 1, and let Q be the two-sided quotient ring of R; Q is a simple and artinian ring. We say that R is an *order* in Q. If R_1 and R_2 are orders in Q, then they are called *equivalent* (in symbol: $R_1 \sim R_2$) if there exist regular elements a_1, b_1, a_2, b_2 of Q such that $a_1R_1b_1 \subseteq R_2$, $a_2R_2b_2 \subseteq R_1$. An order in Q is said to be *maximal* if it is a maximal element in the set of orders which are equivalent to R. A right R-submodule I of Q is called a *right R-ideal* provided I contains a regular element of Q and there is a regular element b of Q such that $bI \subseteq R$. I is called *integral* if $I \subseteq R$. Left R-ideals are defined in a similar way. If I is a right (left) R-ideal of Q, then $O_l(I) = \{x \in Q \mid xI \subseteq I\}$ is an order in Q and is equivalent to R. Similary $O_r(I) = \{x \in Q \mid Ix \subseteq I\}$ is an order in Q and is equivalent to R. They are called a *left order* and a *right order* of I respectively.

We define the inverse of I to be $I^{-1}=\{q\in Q\mid IqI\subseteq I\}$. Evidently $I^{-1}=\{q\in Q\mid Iq\subseteq O_l(I)\}=\{q\in Q\mid qI\subseteq O_r(I)\}$. Following [2], we define $I^*=(I^{-1})^{-1}$. If $I=I^*$, then it is said to be a *right (left) v-ideal*. If R is a maximal order, then $I^{-1}=I^{-1-1-1}$ and so I^{-1} is a left (right) v-ideal, and the concept of right (left) v-ideals coincides with one of right (left) v-ideals defined in [9]. So the mapping: $I\to I^*$ of the set of all right (left) R-ideals into the set of all right (left) v-ideals is a $*$-operation in the sense of [9].

Lemma 1. *Let R be a maximal order in Q and let S be any order equivalent to R. Then S is a maximal order if and only if $S=O_l(I)$ for some right v-ideal I of Q.*

Proof. If $S=O_l(I)$ for some right v-ideal I of Q, then it is a maximal order by Satz 1.3 of [1]. Conversely assume that S is a maximal order, then there are regular elements c, d in R such that $cSd\subseteq R$. So SdR is a right R-ideal and is a left S-module. Hence $(SdR)^{-1}$ is a left R-ideal and is a right S-module. Similarly $I=(SdR)^{-1-1}$ is a right v-ideal and is a left S-module so that $O_l(I)\supseteq S$. Hence $S=O_l(I)$.

Lemma 2. *Let R, S be maximal orders in Q such that $R\sim S$, and let $\{I_i\}$, I be right R-ideals. Then*

(1) *If $\cap_i I_i$ is a right R-ideal, then $\cap_i I_i^*=(\cap_i I_i^*)^*$.*
(2) *If $\sum_i I_i$ is a right R-ideal, then $(\sum I_i)^*=(\sum I_i^*)^*$.*
(3) *If J is a left R and right S-ideal, then $(IJ)^*=(I^*J)^*=(IJ^*)^*=(I^*J^*)^*$.*
(4) *$(I^{-1}I^*)^*=R$ and $(I^*I^{-1})^*=T$, where $T=O_l(I^*)$.*

Proof. The proofs of (1) and (2) are similar to ones of the corresponding results for commutative rings (cf. Proposition 26.2 of [4]).

To prove (3) assume that $IJ\subseteq cS$, where c is a unit in Q. Then we have $(I^*J)\subseteq cS$ and $(IJ^*)\subseteq cS$, because

$c^{-1}IJ\subseteq S\Rightarrow c^{-1}I\subseteq J^{-1}\Rightarrow c^{-1}IJ^*\subseteq J^{-1}J^*\subseteq S\Rightarrow IJ^*\subseteq cS$, and $c^{-1}I\subseteq J^{-1}\Rightarrow c^{-1}I^*=(c^{-1}I)^*\subseteq J^{-1}\Rightarrow c^{-1}I^*J\subseteq J^{-1}J\subseteq S\Rightarrow I^*J\subseteq cS$. Hence $(IJ)^*$ contains $(IJ^*)^*$ and $(I^*J)^*$ by Proposition 4.1 of [9]. The converse inclusions are clear. Therefore we have $(IJ)^*=(I^*J)^*=(IJ^*)^*$. From these it is clear that $(IJ)^*=(I^*J^*)^*$.

To prove (4), assume that $I^{-1}I^*\subseteq cR$, where c is a unit in Q. Then we have $c^{-1}I^{-1}\subseteq I^{-1}$ so that $c^{-1}\subseteq O_l(I^{-1})=R$ and thus $R\subseteq cR$. Hence $(I^{-1}I^*)^*\supseteq R$ by Proposition 4.1 of [9]. The converse inclusion is clear. Therefore $(I^{-1}I^*)^*=R$. Similarly $(I^*I^{-1})^*=T$.

Let R be a maximal order in Q. We denote by $F_r^*(R)$ $(F_l^*(R))$ the set of right (left) v-ideals and let $F^*(R)=F_r^*(R)\cap F_l^*(R)$. It is clear that $F_r^*(R)$ becomes a lattice by the definition; if $I, J\in F_r^*(R)$, then $I\cup{}^*J=(I+J)^*$, and the meet "$\cap$" is the set-theoretic intersection. Similarly $F_l^*(R)$ and $F^*(R)$ also become lattices. For any $I\in F_r^*(R)$ and $L\in F_l^*(R)$, we define the product "$\circ$" of I and L by $I\circ L=(IL)^*$. It is clear that $I\circ L\in F_l^*(S)\cap F_r^*(T)$, where $S=O_l(I)$ and $T=O_r(L)$. In particular, the semi-group $F^*(R)$ becomes an abelian group (cf. Theorem 4.2 of [2]). For convenience we write $F_r'(R)$ for the sublattice of $F_r^*(R)$ consisting of all integral right v-ideals. Similarly we write $F_l'(R)$ and $F'(R)$ for the corresponding sublattices of $F_l^*(R)$ and $F^*(R)$ respectively. Let M and N be subsets of Q. Then we use the following notations: $(M:N)_r=\{x\in R\mid Nx\subseteqq M\}$, $(M:N)_l=\{x\in R\mid xN\subseteqq M\}$. When N is a single element q of Q, then we denote by $q^{-1}M$ the set $(M:N)_r$.

Lemma 3. *Let R be a maximal order in Q. Then*

(1) *If $I\in F_r^*(R)$ and $q\in Q$, then $q^{-1}I=(I^{-1}q+R)^{-1}$ and so $q^{-1}I\in F_r'(R)$.*

(2) *If $I\in F_r^*(R)$ and J is a right R-ideal, then $(I:J)_r\in F'(R)$ or 0.*

(3) *If $I\in F_r^*(R)$ and $J\in F_l^*(R)$, then $(I\circ J)^{-1}=J^{-1}\circ I^{-1}$.*

(4) *If I, $J\in F_r^*(R)$ and $L\in F_l^*(R)$, then $(I\cup{}^*J)\circ L=I\circ L\cup{}^*J\circ L$.*

Proof. (1) Since $(I^{-1}q+R)q^{-1}I\subseteqq R$, we get $(I^{-1}q+R)^{-1}\supseteqq q^{-1}I$. Let x be any element of $(I^{-1}q+R)^{-1}$. Then $(I^{-1}q+R)x\subseteqq R$ so that $x\in R$ and $I^{-1}qx\subseteqq R$. Let $S=O_l(I)$. Then it is a maximal order equivalent to R by Lemma 1. It is evident that $Sqx+I$ is a left S-ideal and that $II^{-1}(Sqx+I)\subseteqq I$. Thus, by Lemma 2, we have

$qx\in S(Sqx+I)\subseteqq(II^{-1})^*\circ(Sqx+I)^*=(II^{-1}(Sqx+I))^*\subseteqq I$. Hence $x\in q^{-1}I$ and so $q^{-1}I=(I^{-1}q+R)^{-1}$. It is clear that $q^{-1}I\in F_r'(R)$ by Corollary 4.2 of [9].

(2) If $(I:J)_r\neq 0$, then it is an R-ideal of Q and $J(I:J)_r\subseteqq I$. So $J((I:J)_r)^*\subseteqq(J(I:J)_r)^*\subseteqq I$. Hence $((I:J)_r)^*\subseteqq(I:J)_r$ so that $((I:J)_r)^*=(I:J)_r$.

(3) It is clear that $O_l(I\circ J)\supseteqq O_l(I)$ and so $O_l(I\circ J)=O_l(I)$ by Lemma 1. Since $(I\circ J)\circ(J^{-1}\circ I^{-1})=S$, where $S=O_l(I)$, we get $(I\circ J)^{-1}\supseteqq J^{-1}\circ I^{-1}$. Let x be any element of $(I\circ J)^{-1}$. Then $IJx\subseteqq(I\circ J)x\subseteqq S$. Let $T=O_r(J)$. Then $Tx+J^{-1}I^{-1}$ is a left T-ideal and $IJ(Tx+J^{-1}I^{-1})\subseteqq S$. Hence $I\circ J\circ(Tx+J^{-1}I^{-1})^*\subseteqq S$ by Lemma 2. By multiplying $J^{-1}\circ I^{-1}$ to the both side of the inequality we have $x\in(Tx+J^{-1}I^{-1})^*\subseteqq J^{-1}\circ I^{-1}$. Therefore we get $(I\circ J)^{-1}=J^{-1}\circ I^{-1}$.

(4) From Lemma 2, we have: $(I\cup{}^*J)\circ L=[(I+J)^*L]^*=[(I+J)L]^*=(IL+JL)^*=[(IL)^*+(JL)^*]^*=I\circ L\cup{}^*J\circ L$.

Let R be a maximal order. We consider the following condition:

(A): $F_r'(R)$ and $F_l'(R)$ both satisfy the maximum condition.

If R is a maximal order satisfying the condition (A), then $F^*(R)$ is a direct product of infinite cyclic groups with prime v-ideals as their generators by Theorem 4.2 of [2]. It is evident that an element P in $F'(R)$ is a prime element in the lattice if and only if it is a prime ideal of R.

Following [1], R is said to be *regular* if every integral one-sided R-ideal contains a non-zero R-ideal.

Lemma 4. *Let R be a regular maximal order satisfying the condition (A) and let P be a non-zero prime ideal of R. Then P is a minimal prime ideal of R if and only if it is a prime v-ideal.*

Proof. Assume that P is a minimal prime ideal. Let c be any regular element in P. Then since $(cR)^*=cR$ and R is regular, we get $P\supseteqq cR\supseteqq(P_1^{n_1})^*\circ\cdots\circ(P_k^{n_k})^*$, where P_i is a prime v-ideal. Hence $P\supseteqq P_i$ for some i and so $P=P_i$. Conversely assume that $P\supsetneqq P_0\neq 0$, where P_0 is a prime ideal. Then since $P_0^*(P_0^{-1}P_0)=(P_0^*P_0^{-1})P_0\subseteqq RP_0=P_0$ and $P_0^{-1}P_0\nsubseteqq P_0$, we have $P_0^*\subseteqq P_0$ and thus $P_0^*=P_0$. It follows that P_0 is a maximal element in $F'(R)$ by [2, p. 11], a contradiction. Hence P is a minimal prime ideal of R.

REMARK. Let R be a maximal order satisfying the condition (A). Then it is evident from the proof of the lemma that prime v-ideals are minimal prime ideals of R.

Let I be any right ideal of R. Then we denote by $\sqrt{I}$ the set $\cup\{(s^{-1}I:R)_r \mid s\notin I, s\in R\}$. Following [3], if $\sqrt{I}$ is an ideal of R, then we say that I is *primal* and that $\sqrt{I}$ is the *adjoint ideal* of it. A right ideal I of R is called *primary* if $JA\subseteqq I$ and $J\nsubseteqq I$ implies that $A^n\subseteqq I$ for some positive integer n, where J is a right ideal of R and A is an ideal of R. We shall apply these concepts for integral right v-ideals.

Lemma 5. *Let R be a maximal order satisfying the condition (A) and let I be a meet-irreducible element in $F_r'(R)$. Then I is primal, and $\sqrt{I}$ is a minimal prime ideal of R or 0, and $\sqrt{I}=(x^{-1}I: R)_r$ for some $x\notin I$.*

Proof. If $\sqrt{I}=0$, then the assertion is evident. Assume that $\sqrt{I}\neq 0$. By Lemma 3, $(s^{-1}I:R)_r$ is a v-ideal or 0. Hence the set $S=\{(s^{-1}I:R)_r \mid s\notin I, s\in R\}$ has a maximal element. Assume that $(s^{-1}I:R)_r$ and $(t^{-1}I:R)_r$ are maximal elements in S. Then $(sR+I)(s^{-1}I:R)_r\subseteqq I$ implies that $(sR+I)^*(s^{-1}I:R)_r\subseteqq I$ by Lemma 2 and so $(s^{-1}I:R)_r\subseteqq(I:(sR+I)^*)_r$. The converse inclusion is clear. Thus we have $(s^{-1}I:R)_r=(I:(sR+I)^*)_r$. Similarly $(t^{-1}I:R)_r=(I:(tR+I)^*)_r$. Since I is irreducible in $F_r'(R)$, we have $I\subsetneqq(sR+I)^*\cap(tR+I)^*=J$. Let x be any element in J but not in I. Then it follows that $(x^{-1}I:R)_r\supseteqq(s^{-1}I:R)_r$, $(t^{-1}I:R)_r$ so that $\sqrt{I}=(x^{-1}I:R)_r=(s^{-1}I:R)_r$, which is a v-ideal. Hence I is primal. If $AB\subseteqq\sqrt{I}$ and $A\nsubseteqq\sqrt{I}$, where A and B are ideals of R, then $xAB\subseteqq I$ and $xA\nsubseteqq I$. Let y be any element in xA but not in I. Then $yB\subseteqq I$ and so $B\subseteqq(y^{-1}I:R)_r\subseteqq\sqrt{I}$. Thus $\sqrt{I}$ is a prime ideal of R. It follows that $\sqrt{I}$ is minimal from the remark to Lemma 4.

A right ideal of R is said to be *bounded* if it contains a non-zero ideal of R.

Lemma 6. *Let R be a maximal order satisfying the condition* (A) *and let I be an irreducible element in $F_r'(R)$. If I is bounded, then it is primary and* $(\sqrt{I})^n \subseteq I$ *for some positive integer n.*

Proof. Since $I \in F_r'(R)$ and is bounded, $(I:R)_r$ is non-zero and is a v-ideal. Write $(I:R)_r = (P_1^{n_1})^* \circ \cdots \circ (P_k^{n_k})^*$, where P_i are prime v-ideals. For any i $(1 \leqq i \leqq k)$, we let $B_i = (P_1^{n_1})^* \circ \cdots \circ (P_{i-1}^{n_{i-1}})^* \circ (P_{i+1}^{n_{i+1}})^* \circ \cdots \circ (P_k^{n_k})^*$. Then $B_i \nsubseteq I$ and $B_i P_i^{n_i} \subseteq (I:R)_r \subseteq I$, because $F^*(R)$ is an abelian group. Thus $P_i^{n_i} \subseteq \sqrt{I}$ and so $P_i = \sqrt{I}$ $(1 \leqq i \leqq k)$ by Lemma 5. Therefore $(\sqrt{I})^{n_1 + \cdots + n_k} \subseteq I$. It is evident that I is primary.

If A is an ideal of R, then we denote by $C(A)$ those elements of R which are regular mod (A).

Lemma 7. *Let R be a maximal order satisfying the condition* (A). *Let P be a prime v-ideal. Then*

(1) $C(P) = C((P^n)^*)$ *for every positive integer n.*

(2) $C(P) \subseteq C(0)$.

Proof. (1) We shall prove by the induction on n (>1). Assume that $C(P) = C((P^{n-1})^*)$. If $cx \in (P^n)^*$, where $c \in C(P)$ and $x \in R$, then $cx(P^{-1})^{n-1} \subseteq (P^n)^*(P^{-1})^{n-1} \subseteq P$ by Lemma 2. Since $cx \in (P^{n-1})^*$, we get $x \in (P^{n-1})^*$ and so $x(P^{-1})^{n-1} \subseteq R$. Hence $x(P^{-1})^{n-1} \subseteq P$. Then we have $(xR + P^n)(P^{-1})^{n-1} P^{n-1} \subseteq P^n$ so that $x \in (P^n)^*$ by Lemma 2. Conversely suppose that $cx \in P$, $c \in C((P^n)^*)$, $x \in R$. Then $cxP^{n-1} \subseteq (P^n)^*$ and so $xP^{n-1} \subseteq (P^n)^*$. Since $(xP + P^n)P^{n-1}(P^{-1})^{n-1} \subseteq (P^n)^*(P^{-1})^{n-1} \subseteq P$, we get $x \in P$ by Lemma 2. Therefore $C(P) = C((P^n)^*)$.

(2) If $0 \neq \cap_n (P^n)^*$, then it is a v-ideal by Lemma 2. Write $\cap_n (P^n)^* = (P_1^{n_1})^* \circ \cdots \circ (P_k^{n_k})^*$, where P_i are prime v-ideals. This is a contradiction, because $F^*(R)$ is an abelian group and P, P_i are minimal prime ideals of R. Hence $0 = \cap_n (P^n)^*$. Therefore (2) follows from (1).

If P is a prime ideal of a ring S, then the family $T_P = \{I\colon \text{right ideal} \mid s^{-1}I \cap C(P) \neq \phi \text{ for any } s \in S\}$ is a right additive topology (cf. Ex. 4 of [12, p. 18]). The following lemma is due to Lambek and Michler if S is right noetherian. However, only trivial modifications to their proof are needed to establish the more general result.

Lemma 8. *Let P be a prime ideal of S and let $\bar{S} = S/P$ be a right prime Goldie ring. Then the torsion theory determined by the S-injective hull $E(\bar{S})$ of $\bar{S}$ coincides with one determined by the right additive topology T_P, that is, a right ideal I of S is an element in T_P if and only if* $Hom_S(S/I, E(\bar{S})) = 0$ (Corollary 3.10 of [8]).

Lemma 9. *Let R be a maximal order satisfying the condition* (A) *and let P*

be a prime v-ideal such that $\bar{R}=R/P$ is a prime Goldie ring. If I is any element in $F_r'(R)$ such that $R\supsetneqq I\supseteqq P$, then $I\cap C(P)=\phi$.

Proof. It is enough to prove the lemma when I is a maximal element $F_r'(R)$. Since $I^{-1}\supsetneqq R$, $P\circ I^{-1}\cap R\supseteqq P$. If $P\circ I^{-1}\cap R=P$, then $P^{-1}=(P\circ I^{-1})^{-1}\cup {}^*R$, because the mapping: $J\rightarrow J^{-1}$ is an inverse lattice isomorphism between $F_r^*(R)$ and $F_l^*(R)$. By Lemma 3, $P^{-1}=I\circ P^{-1}\cup {}^*R$. On the other hand $P\subseteqq I$ implies that $R\subseteqq I\circ P^{-1}$. Hence $P^{-1}=I\circ P^{-1}$ and so $R=I$, a contradiction. Thus we have $P\circ I^{-1}\cap R\supsetneqq P$. Let a be any element in $P\circ I^{-1}\cap R$ but not in P. Then $aI\subseteqq(P\circ I^{-1})I\subseteqq P\circ I^{-1}\circ I=P$ so that $I\subseteqq a^{-1}P\subsetneqq R$. Since $a^{-1}P$ is a right v-ideal by Lemma 3, we get $I=a^{-1}P$. Then $\mathrm{Hom}(R/I, E(\bar{R}))\neq 0$, because $R/I=R/a^{-1}P\cong(aR+P)/P\subseteqq\bar{R}$. Now assume that $I\cap C(P)\neq\phi$ and let c be any element in $I\cap C(P)$. Then $cR+P\in T_P$ by Lemma 3.1 of [6]. Hence $I\in T_P$ and thus $\mathrm{Hom}(R/I, E(\bar{R}))=0$ by Lemma 8. This is a contradiction and so $I\cap C(P)=\phi$.

For convenience, we write $M(p)$ for the family of minimal prime ideals of R. If R is a regular maximal order satisfying the condition (A), then we know from Lemma 4 that a prime ideal P is an element in $M(p)$ if and only if it is a prime element in $F'(R)$.

Lemma 10. *Let R be a regular maximal order satisfying the condition (A), $P\in M(p)$ and let $I\in F_r'(R)$. If $\bar{R}=R/P$ is a prime Goldie ring, then $I\cup{}^*P=R$ if and only if I contains an ideal B such that $B\subsetneqq P$.*

Proof. Assume that $I\supseteqq B$, where B is an ideal not contained in P. Then $I\supseteqq B^*$ and $B^*\cup{}^*P=R$, because P is a maximal element in $F'(R)$ (cf. [2, p. 11]). Therefore $I\cup{}^*P=R$. Conversely assume that the family $S=\{I\in F_r'(R)\,|\,I\cup{}^*P=R, I\neq R \text{ and } I\not\supseteqq B \text{ for any ideal } B \text{ not contained in } P\}$ is not empty and let I be a maximal element in S. If I is irreducible in $F_r'(R)$, then there exists P' in $M(p)$ such that $I\supseteqq P'^n$ by Lemmas 5 and 6. Since $I\in S$, we have $P=P'$. If $n=1$, then $R=I\cup{}^*P=I$, a contradiction. We may assume that $I\not\supseteqq P^{n-1}$ and $n>1$. Then $(P^{n-1})^*=(I\cup{}^*P)\circ(P^{n-1})^*=I\circ(P^{n-1})^*\cup{}^*(P^n)^*\subseteqq I^*=I$ by Lemmas 2 and 3. This is a contradition. If I is reducible, then $I=I_1\cap I_2$, where $I_i\in F_r'(R)$ and $I\subsetneqq I_i$ $(i=1,2)$. There are non zero ideals B_i $(\not\subseteqq P)$ such that $I_i\supseteqq B_i$. Thus I contains the ideal B_1B_2 not contained in P, a contradiction. Hence $S=\phi$. This implies that if $I\cup{}^*P=R$, then I contains an ideal not contained in P.

Let P be a prime ideal of a ring S. If S satisfies the Ore condition with respect to $C(P)$, then we denote by S_P the quotient ring with respect to $C(P)$.

Lemma 11. *Let R be a regular maximal order satisfying the condition (A) and let P be an element in $M(p)$ such that $\bar{R}=R/P$ is a prime Goldie ring. Then*

(1) *R satisfies the Ore condition with respect to* $C(P)$.

(2) $R_P = \varinjlim B^{-1}$, *where B ranges over all non zero ideals not contained in P.*

(3) R_P *is a noetherian, local and Asano order.*

Proof. (1) It is clear that $T = \varinjlim B^{-1}(B(\nsubseteq P)$: ideal) is an overring of R. Let c be any element in $C(P)$. Then c is regular by Lemma 7 and so $cR \in F_r'(R)$. Since $(cR \cup {}^*P) \cap C(P) \neq \phi$, we have $cR \cup {}^*P = R$ by Lemma 9 and so cR contains an ideal not contained in P by Lemma 10. Hence $c^{-1} \in T$. So for any $r \in R$, $c \in C(P)$, there exists an ideal $B\ (\nsubseteq P)$ such that $c^{-1}rB \subseteqq R$. It is evident that $B \cap C(P) \neq \phi$. Let d be any element in $B \cap C(P)$. Then we have $c^{-1}rd = s$ for some s in R, that is, $rd = cs$. This implies that R satisfies the right Ore condition with respect to $C(P)$. The other Ore condition is shown to hold by a symmetric proof.

(2) is evident from (1).

(3) We let $P' = PR_P$. Then clearly $P' = R_P P$ and $P = P' \cap R$. So we may assume that $\bar{R} = R/P \subseteqq \bar{R}_P = R_P/P'$ as rings. By (1), $\bar{R}_P$ is the quotient ring of $\bar{R}$. Since $\bar{R}$ is a prime Goldie ring, $\bar{R}_P$ is the simple artinian ring. Hence P' is a maximal ideal of R_P. Let V' be any maximal right ideal of R_P. Suppose that $V' \not\supseteqq P'$. Then $V' + P' = R_P$. Write $1 = v + pc^{-1}$, where $v \in V'$, $p \in P$ and $c \in C(P)$. Then $c = vc + p$ and so $vc = c - p \in C(P) \cap V'$. This implies that $V' = R_P$, a contradiction and so $V' \supseteqq P'$. Hence P' is the Jacobson radical of R_P. The ideal $P^{-1}P$ properly contains P so that $C(P) \cap P^{-1}P \neq \phi$. It follows that $P^{-1}PR_P = R_P$. Similarly $R_P P P^{-1} = R_P$. Hence P' is an invertible ideal of R_P. Therefore R_P/P'^n is an artinian ring for any n, because $\bar{R}_P$ is an artinian ring. Let I' be any essential right ideal of R_P. It is clear that $I' = (I' \cap R)R_P$. Let c be any regular element of $I' \cap R$. Then, since $cR \in F_r'(R)$ and R is regular, cR contains a non zero v-ideal $(P^n)^* \circ (P_1^{n_1})^* \circ \cdots \circ (P_k^{n_k})^*$, where $P_i \in M(p)$. So we get $I' \supseteqq R_P P^n = P'^n$. Therefore essential right ideals of R_P satisfies the maximum condition. Since R_P is finite dimensional in the sense of Goldie, R_P is right noetherian. Similarly R_P is left noetherian. Hence R_P is a noetherian, local and Asano order by Proposition 1.3 of [7].

After all these preparations we now prove the following theorem which is the purpose of this paper:

Theorem. *A prime Goldie ring R is a bounded Krull prime ring if and only if it satisfies the following conditions:*

(1) *R is a regular maximal order,*

(2) *R satisfies the maximum condition for integral right and left v-ideals,*

(3) R/P *is a prime Goldie ring for any* $P \in M(p)$.

Proof. Assume that $R = \cap_i R_i\ (i \in \boldsymbol{I})$ is a bounded Krull prime ring, where R_i is a noetherian, local and Asano order with unique maximal ideal P_i'. (1) is clear from Corollary 1.4 and Lemma 1.6 of [10]. Let I be any right (left) R-ideal. Then $I^*=\cap IR_i(=\cap R_iI)$ by Proposition 1.10 of [10]. Since R_i is noetherian, (2) follows from the condition (K3) in the definition of Krull rings. Let $P_i=P_i'\cap R$. It follows that $\{P_i \mid i\in I\}=M(p)$ by Proposition 1.7 of [10] so that (3) is evident from Proposition 1.1 of [9].

It remains to prove that the conditions (1), (2) and (3) are sufficient. Let P be any element in $M(p)$. Then R satisfies the Ore condidion with respect to $C(P)$ and R_P is a noetherian, local and Asano order by Lemma 11. Hence R_P is an essential overring of R. It is clear that $R\subseteq T=\cap R_P$, where $P\in M(p)$. To prove the converse inclusion let x be any element of T. Then there is an ideal B_P ($\subsetneq P$) such that $xB_P\subseteq R$ by Lemma 11. Let B be the sum of all ideals B_P. If B^* is different from R, then B^* is contained in some P in $M(p)$. But $B^*\subsetneq P$ so that $B^*=R$. Hence we have $x\in(xR+R)\subseteq(xR+R)^*\circ B^*=(xB+B)^*\subseteq R$. Thus we get $R=\cap R_P$. Let c be any regular element in R. Then cR contains a v-ideal $(P_1^{n_1})^*\circ\cdots\circ(P_k^{n_k})^*$, where $P_i\in M(p)$. It follows that $cR_P=R_P$ for every $P\in M(p)$ different to $P_i(1\leqq i\leqq k)$ by Lemma 11. Hence R is a bounded Krull prime ring. This completes the proof of the theorem.

Corollary. *Let R be a regular, noetherian and prime ring. If R is a maximal order, then it is a bounded Krull prime ring.*

OSAKA UNIVERSITY

References

[1] K. Asano: *Zur Arithemetik in Schiefringen* I. Osaka Math. J. **1** (1949), 98–134.
[2] A. Asano and K. Murata: *Arithemetical ideal theory in semigroups*, J. Inst. Polytec. Osaka city Univ. **4** (1953), 9–33.
[3] W.E. Barnes: *Primal ideals and isolated components in noncommutative rings*, Trans Amer Math. Soc. **82** (1956), 1–16.
[4] R. Gilmer: Multiplicative ideal theory II, Queen's Series in Pure and Applied Math. Queen's Univ. Kingston, Ontario, 1968.
[5] A.W. Goldie: *The structure of prime rings under ascending chain conditions*, Proc. London Math. Soc. **8** (1958), 589–608.
[6] A.W. Goldie: *Localization in non-commutative noetherian rings*, J. Algebra **5** (1967), 89–105.
[7] C.R. Hajarnavis and T.H. Lenagan: *Localisation in Asano orders*, J. Algebra **21** (1972), 441–449.
[8] J. Lambek and G. Michler: *The torsion theory at a prime ideal of a right noetherian ring*, J. Algebra **25** (1973), 364–389.
[9] H. Marubayashi: *Non commutative Krull rings*, Osaka J. Math. **12** (1975), 703–714.
[10] H. Marubayashi: *On bounded Krull prime rings*, **13** (1976), 491–501.
[11] P. Samuel: Lectures on unique factorization domains, Tata Inst. of Fundamental Research, Bombay, 1964.
[12] B. Stenström: Rings and modules of quotients, Lecture notes in Math. vol. 237, Berlin-Heidelberg-New York, Springer, 1971.