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TWO CONGRUENCE PROPERTIES OF LEGENDRE POLYNOMIALS

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1. Let $P_n(x)$ be the Legendre polynomial ($n\geqq 0$, cf. Whittaker-Watson [2]) and p be an odd prime number. In this article we give two congruence properties of $P_n(x)$ modulo any power of p.

Our principal results are as follows:

Theorem 1. *For any $m\geqq 1$, we have*

$$P_{mp^{\nu+1}-1}(x)\equiv P_{mp^{\nu}-1}(x^p) \mod p^{\nu+1}\,.$$

Theorem 2.

$$P_{mp^{\nu+1}}(x)\equiv P_{mp^{\nu}}(x^p) \mod p^{\nu+1}\,.$$

The $P_n(x)$ is defined as follows:

$$(1) \qquad (1-2xh+h^2)^{-1/2}=\sum_{n=0}^{\infty}P_n(x)h^n\,,$$

and can be written explicitly as

$$(2) \qquad P_n(x)=\sum_{r=0}^{m}(-1)^r\frac{(2n-2r)!}{2^n\cdot r!(n-r)!(n-2r)!}x^{n-2r}\,,$$

where $m=n/2$ or $(n-1)/2$ whichever is an integer.

We shall use the results of Honda [1] freely.

2. Proof of Theorem 1

Put
$$f(h)=\sum_{n=1}^{\infty}\frac{P_{n-1}(x)}{n}h^n\,.$$

Then we have

$$f'(h)=(1-2xh+h^2)^{-1/2}\,.$$

Now transform the variable h to t by

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$$h=\frac{2t(1-xt)}{1-t^2}.$$

Then

$$\frac{dt}{\sqrt{1-2hx+h^2}}=\frac{2dt}{1-t^2}$$

and $h=2t+\cdots\in \boldsymbol{Z}[x][[t]]$. As is easily verified, the special element which kills $\int_0^t \frac{dt}{1-t^2}$ (mod p) is $p-T$ (cf. Honda [1]). Hence $f(h)$ is of type $p-T$, regarding $\boldsymbol{Z}_p[x]$ as a discrete valuation ring $\mathfrak{o}$ in the notation of [1, §2]. (In this case σ is given by $(k(x))^\sigma=k(x^p)$ for $k(x)\in \boldsymbol{Z}_p[x]$.) This proves

$$pf(h)\equiv f(h^p) \mod p,$$

i.e.,

$$P_{mp^{\nu+1}-1}(x)\equiv P_{mp^\nu-1}(x^p) \mod p^{\nu+1}.$$

3. Proof of Theorem 2

Put

$$g(h)=\sum_{n=1}^{\infty}\frac{P_n(x)}{n}h^n.$$

Then

$$g'(h)dh=\{(1-2hx+h^2)^{-1/2}-1\}dh/h.$$

By the transformation

$$h=2t(1-xt)/(1-t^2)=2t+\cdots,$$

we get

$$g'(h)dh=2(x-t)dt/(1-t^2)(1-xt).$$

Furthermore we transform t to u by

$$1+u=\frac{\sqrt{1-t^2}}{1-xt}$$

and get

$$g'(h)dh=\frac{2du}{1+u}.$$

Since

$$h=2xu+\cdots,$$

the formal group whose transformer is $g(h)$ is of type $p-T$. This proves

$$pg(h) \equiv g(h^p) \mod p\,,$$

namely

$$P_{mp^{\nu+1}}(x) \equiv P_{mp^\nu}(x^p) \mod p^{\nu+1}$$

which completes the proof of Theorem 2.

References

[1] T. Honda: *On the theory of commutative formal groups*, J. Math. Soc. Japan **22** (1970), 213–246.

[2] E.T. Whittaker and G.N. Watson: A Course of Modern Analysis, Cambridge, 1963.