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A NOTE ON A FIXED-POINT-FREE AUTOMORPHISM AND A NORMAL P-COMPLEMENT

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1. Introduction

Let A be a group of automorphisms of a group G, and denote by $C_G(A)$ the subgroup of G consisting of all the elements fixed by A. If $C_G(A)=1$ then A is said to be fixed-point-free. The purpose of this note is to prove the following two theorems.

The first theorem is an extension of a result of F. Gross ([2], Theorem 3.5).

Theorem 1. *Let A be a group of automorphisms of a finite group G and p a prime divisor of $|G|$. Suppose that either A is cyclic and fixed-point-free or $(|A|, |G|)=1$ and $C_G(A)$ is a p-group. If a Sylow p-subgroup P of G is of the form*

$$P = P_1 \times P_2 \times \cdots \times P_l$$

where P_i is a direct product of m_i cyclic subgroups of order p^{n_i} with $n_1 < n_2 < \cdots < n_l$ and if each m_i is less than any prime divisor of $|A|$, then G has a normal p-complement.

If an abelian p-group P is of the form as in the theorem above, we denote $\sum_{i=1}^{l} m_i$ by $m(P)$, and $\max_{1 \leq i \leq l} m_i$ by $\tilde{m}(P)$.

For a p-group P, $ZJ(P)$ denotes the center of the Thompson subgroup of P and we define $(ZJ)^i(P)$ recursively by the rule

$$(ZJ)^0(P) = 1, \quad (ZJ)^1(P) = ZJ(P), \quad \text{and} \quad ZJ(P/(ZJ)^{i-1}(P)) = (ZJ)^i(P)/(ZJ)^{i-1}(P)\,.$$

In a case of p odd Theorem 1 can be extended as follows.

Theorem 2. *Let G be a finite group, p an odd prime divisor of $|G|$ and P a Sylow p-subgroup of G. Suppose that G has a group A of automorphisms satisfying the same assumption as in Theorem 1. If each $\tilde{m}((ZJ)^i(P)/(ZJ)^{i-1}(P))$ is less than any prime divisor of $|A|$, then G has a normal p-complement.*

For the proof of Theorem 1, Lemma 1 in the next section is fundamental. The other arguments are similar to those in Gross [2]. The proof of Theorem 2 is based on the celebrated theorem of Glauberman and Thompson.

The notation is the same as in [1], and all groups are assumed to be finite.

2. Preliminaries and some lemmas

The following propositions are well known and will be used later.

Proposition 1 ([1], Theorems 6.2.2, 10.1.2 and Lemma 10.1.3). *Let A be a group of automorphisms of a group G such that either $(|A|, |G|)=1$, or A is cyclic and fixed-point-free. Then we have*

(i) *For any $p\in\pi(G)$ G has an A-invariant Sylow p-subgroup.*

(ii) *If H is an A-invariant normal subgroup of G, then $C_{G/H}(A)=HC_G(A)/H$. In particular if A is fixed-point-free then A induces a fixed-point-free group of automorphisms of G/H.*

Proposition 2 ([1], Theorem 5.3.1). *Let A be a p'-group of automorphisms of a p-group P which stabilizes some normal series of P. Then $A=1$.*

In the following lemmas we assume that a group G has a group A of automorphisms such that either

(*) A is cyclic and fixed-point-free, or

(**) $(|A|, |G|)=1$ and $C_G(A)$ is a p-group.

We remark that if H is an A-invariant subgroup of G then A induces a group of automorphisms of H satisfying the assumption (*) or (**) and if H is an A-invariant normal subgroup of G then the same holds for G/H.

Lemma 1. *Let G be a group of order $p^a q^b$ with $p\neq q$ primes. If a Sylow p-subgroup P of G is abelian and normal, and $m(P)$ is less than any prime divisor of $|A|$, then G has a normal p-complement.*

Proof. Suppose G is a minimal counter-example to the lemma. Let Q be an A-invariant Sylow q-subgroup of G and let $H=QA$ the semi-direct product of Q by A. Then H acts on P.

(a) Suppose that P has a proper H-invariant subgroup $P_0\neq 1$. Then P_0 is an A-invariant normal subgroup of G. By the minimality of G, G/P_0 and P_0Q have normal p-complements P_0Q/P_0 and Q respectively. Then, since Q char P_0Q and $P_0Q\lhd G$, Q is normal in G, which is a contradiction.

Thus P has no non-trivial H-invariant subgroup. In particular P is an elementary abelian group of order $p^{m(P)}$ and H acts irreducibly on P

(b) Let $Q_0=C_Q(P)$. Since $N_G(Q_0)$ contains P and Q we have $G=N_G(Q_0)$. If $Q_0\neq 1$ then G/Q_0 has normal p-complement Q/Q_0, and hence Q is normal in G, which is a contradiction.

Thus we have $C_Q(P)=1$ and Q acts faithfully on P.

(c) Suppose that Q has a non-trivial A-invariant subgroup Q_1. Then PQ_1 has a normal p-cmoplement Q_1 and hence $Q_1 \leqq C_Q(P)$, which is a contradiction. Thus Q has no non-trivial A-invariant subgroup. In particular Q is abelian.

(d) We consider the action of H on P. We may regard P as a vector space of dimension $m(P)$ over $K_0=GF(p)$, where $GF(p)$ is a finite field of p elements. Then P is an irreducible $K_0[H]$-module, and as is well known there is an extension field $K=GF(p^r)$ of K_0 and a vector space V over K such that V is an absolutely irreducible $K[H]$-module and if we regard V as a vector space over K_0 then V is isomorphic to P as $K_0[H]$-module. If V is of dimension s over K then $m(P)=rs$.

Now we take a splitting field L of Q which contains K and let $V_L=L\otimes_K V$. Then V_L is an irreducible $L[H]$-module, and since Q is abelian any irreducible $L[Q]$-submodule of V_L is of dimension 1. By the theorem of Clifford ([1], Theorem 3.4.1) V_L is the direct sum of the Wedderburn components $V_1, \cdots, V_t$ with respect to Q. Since t divides s and also divides $|H: Q|=|A|$, if $t>1$ then $m(P)$ is not less than some prime divisor of $|A|$, which contradicts the assumption. Thus $t=1$ and V_L is a direct sum of irreducible $L[Q]$-submodules $W_1, \cdots, W_s$ which are all isomorphic as $L[Q]$-modules. Let $\lambda: Q \rightarrow L^\sharp$ be the linear representation of Q obtained by W_i. Then, since Q acts faithfully on P, λ is faithful. For $\phi \in A$, $W_1\phi^{-1}$ is an irreducible $L[Q]$-submodule of V_L and hence isomorphic to W_1. Therefore $\lambda(x)=\lambda(x^\phi)$ for $x \in Q$ and we have $\lambda(x^{-1}x^\phi)=1$. Hence $x^\phi=x$ for any $\phi \in A$ and any $x \in Q$, which is a contradiction.

Lemma 2. *Suppose that G has an A-invariant abelian normal p-subgroup P_0 such that $m(P_0)$ is less than any prime divisor of $|A|$. Let P be a Sylow p-subgroup of G. Then $G=PC_G(P_0)$.*

Proof. Let $q(\neq p) \in \pi(G)$ and Q an A-invariant Sylow q-subgroup of G. Then P_0Q satisfies the assumption of Lemma 1. Hence $Q \leqq C_G(P_0)$. Thus $G/C_G(P_0)$ is a p-group, which proves our lemma.

Lemma 3. *Suppose that a Sylow p-subgroup P of G has a chain*

$$1=P_0<P_1<\cdots<P_l=P$$

such that P_i char P, P_i/P_{i-1} is abelian and $m(P_i/P_{i-1})$ is less than any prime divisor of $|A|$. Then $N_G(P)=PC_G(P)$, and if P is abelian G has a normal p-complement.

Proof. We may assume P is A-invariant. Let Q be an A-invariant Sylow q-subgroup of $N_G(P)$, where $q\neq p$. Then $N_G(P)/P_{i-1}$ has an A-invariant abelian normal subgroup P_i/P_{i-1} satisfying the assumption of Lemma 2. Hence Q acts trivially on P_i/P_{i-1}. Thus Q stabilizes the normal series of P in the lemma. Therefore $Q \leqq C_G(P)$ and $N_G(P)/C_G(P)$ is a p-group. Thus we have $N_G(P)=$

$PC_G(P)$. If P is abelian then $P \leqq Z(N_G(P))$, and hence by a theorem of Burnside G has a normal p-complement.

3. Proofs of the theorems

Proof of Theorem 1. It will suffice to show that P has a chain of characteristic subgroups as in Lemma 3.

Now $P/\Omega_1(P)$ is isomorphic to

$$P_1/\Omega_1(P_1) \times P_2/\Omega_1(P_2) \times \cdots \times P_l/\Omega_1(P_l)$$

where $P_i/\Omega_1(P_i)$ is a direct product of m_i cyclic subgroups of order p^{n_i-1}. Thus by induction on $|P|$ we may assume that there is a chain

$$\Omega_1(P) = K_0 < K_1 < \cdots < K_r = P$$

such that K_i char P, K_i/K_{i-1} is abelian and $m(K_i/K_{i-1})$ is less than any prime divisor of $|A|$. Now let $L_i = \mho^{n_i-1}(P) \cap \Omega_1(P)$ for $i=1, 2, \cdots, l$ and let $L_{l+1}=1$. Then L_i char P and we have a chain

$$1 = L_{l+1} < L_l < \cdots < L_1 = \Omega_1(P)$$

where $m(L_i/L_{i+1}) = m_i$. Thus we have a chain of subgroups of P as in Lemma 3.

Proof of Theorem 2. Let G be a minimal counter-example to the theorem. By a theorem of Glauberman and Thompson ([1], Theorem 8.3.1.) we have $G = N_G(ZJ(P))$. Since $G/ZJ(P)$ satisfies the assumption of our theorem it has a normal p-complement $H/ZJ(P)$. Then by Theorem 1 H has a normal p-complement K, which is also a normal p-complement of G. This is a contradiction.

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