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THREE DIMENSIONAL HOMOLOGY HANDLES AND CIRCLES

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This paper will extend the known propertes of the Alexander polynomials of classical knot complements to the properties of the Alexander polynomials of arbitrary (possibly non-orientable) compact 3-manifolds with infinite cyclic first homology groups. In particular, the Alexander polynomial will always have a reciprocal property. The existence of the corresponding manifolds and the other related results will be shown.

1. Statement of results

Throughout this paper, spaces will be considered in the PL category.

DEFINITION 1.1. A compact 3-manifold M is called a *homology orientable handle* if M has the homology of an orientable handle: $H_*(M; Z)\approx H_*(S^1\times S^2; Z)$. Likewise, M is a *homology non-orientable handle* if $H_*(M; Z)\approx H_*(S^1\times_\tau S^2; Z)$, a *homology orientable circle* if $H_*(M; Z)\approx H_*(S^1; Z)$ and $\partial M=S^1\times S^1$, and a *homology non-orientable circle* if $H_*(M; Z)\approx H_*(S^1; Z)$ and $\partial M=S^1\times_\tau S^1$.

It is easily seen that if M is a homology orientable (or non-orientable, respectively) handle or circle then M is orientable (or non-orientable, respectively) as a manifold. [Note that, in case $\partial M\neq\phi$, $H_3(M, \partial M; Z)\approx H_2(\partial M; Z)$.]

By $\mathcal{C}(S^1\times S^2)$, $\mathcal{C}(S^1\times_\tau S^2)$, $\mathcal{C}(S^1\times B^2)$ and $\mathcal{C}(S^1\times_\tau B^2)$, we denote the class of homology orientable handles, the class of homology non-orientable handles, the class of homology orientable circles and the class of homology non-orientable circles, respectively.

The following Theorem 1.2 implies that a compact connected 3-manifold M with $H_1(M; Z)=Z$ belongs to one of the four classes $\mathcal{C}(S^1\times S^2)$, $\mathcal{C}(S^1\times_\tau S^2)$, $\mathcal{C}(S^1\times B^2)$ and $\mathcal{C}(S^1\times_\tau B^2)$ if ∂M contains no 2-spheres.

Theorem 1.2. *Let M be a compact connected 3-manifold with $H_1(M; Z)=Z$. If $\partial M=\phi$, then $H_*(M; Z)$ is isomorphic to either $H_*(S^1\times S^2; Z)$ or $H_*(S^1\times_\tau S^2; Z)$. If $\partial M\neq\phi$, then under the assumption that ∂M contains no 2-spheres, $H_*(M; Z)\approx H_*(S^1; Z)$ and ∂M is homeomorphic to either $S^1\times S^1$ or $S^1\times_\tau S^1$.*

If ∂M contains a 2-sphere, then we will attach a 3-cell to eliminate it. This modification is never essential [for example, the orientability of the resulting manifold M' coincides with that of the original manifold M and $\pi_1(M)=\pi_1(M')$.]. So we may assume that ∂M contains no 2-spheres.

Now suppose M belongs to one of the above four classes. Since the first cohomotopy group $\pi^1(M)=[M, S^1]$ is naturally isomorphic to the group of homomorphisms $\mathrm{Hom}[\pi_1(M), \pi_1(S^1)]=\mathrm{Hom}[H_1(M; Z), H_1(S^1; Z)]$, we can choose a map $f: M\to S^1$ which induces an isomorphism $f_*: H_1(M; Z)\to H_1(S^1; Z)$. The *infinite cyclic covering* $p: \tilde{M}\to M$ associated with epimorphism $f_\#: \pi_1(M)\to\pi_1(S^1)=\pi$ is then the covering induced from the exponential map $R^1\to S^1$ along $f: M\to S^1$ (See [3, §1].). We denote by t a generator of the covering transformation group π which is an infinite cyclic multiplicative group.

Let $\Lambda=Z[\pi]$ be the integral group ring of π. Since Λ is a Noetherian ring, it is not difficult to see that $H_1(\tilde{M}; Z)$ is a finitely generated (i.e. Noetherian) module over Λ. [Note that the simplicial oriented chain group $C_\#(\tilde{M}; Z)$ (for some triangulation of M) forms a finitely generated free Λ-module.]

Let $\mathfrak{E}(t)$ be a relation matrix of $H_1(\tilde{M}; Z)$. That is, for an exact sequence of Λ-modules $\mathfrak{F}_1\to\mathfrak{F}_2\to H_1(\tilde{M}; Z)\to 0$ with free modules $\mathfrak{F}_1$, $\mathfrak{F}_2$ of finite ranks, let $\mathfrak{E}(t)$ be a matrix representing the homomorphism $\mathfrak{F}_1\to\mathfrak{F}_2$. If $r=\mathrm{rank}\ \mathfrak{F}_2\geq 1$, then the *first elementary ideal* $E(\mathfrak{E}(t))$ of $\mathfrak{E}(t)$ is the ideal over Λ generated by the determinants of $r\times r$ submatrices of $\mathfrak{E}(t)$. (In case $\mathfrak{E}(t)$ contains no $r\times r$ submatrices, we have $E(\mathfrak{E}(t))=0$.) If $r=0$, then let $E(\mathfrak{E}(t))=\Lambda$.

DEFINITION 1.3. Any generator $A(t)$ of the smallest principal Λ-ideal containing $E(\mathfrak{E}(t))$ is called the *Alexander polynomial* of M. [Note that $A(t)$ is an invariant of $\pi_1(M)$ in the sense that if $\pi_1(M)$ and $\pi_1(M')$ are isomorphic, then $A(t)\doteq$ *) $A'(t^\varepsilon)$, where $A(t)$, $A'(t)$ are the Alexander polynomials of M, M', respectively, and $\varepsilon=1$ or -1. See Magnus-Karrass-Solitar [7, p 157].]

The Alexander polynomial $A(t)$ of M is restricted to some extent. Actually the following is shown.

Theorem 1.4. *For $M\in C(S^1\times S^2)$ or $M\in C(S^1\times B^2)$, we have $A(t)\doteq A(t^{-1})$ and $|A(1)|=1$. For $M\in C(S^1\times_\tau S^2)$, we have $A(t)\doteq A(-t^{-1})$ and $|A(1)|=1$. For $M\in C(S^1\times_\tau B^2)$, we have $A(t)\doteq(t^m+1/t+1)A_0(t)$, where $m\geq 1$ is the odd number determined by the group $H_1(M, \partial M; Z)=Z_m$ and $A_0(t)$ is an integral polynomial satisfying $A_0(t)\doteq A_0(-t^{-1})$ and $|A_0(1)|=1$.*

REMARK 1.5. From Theorem 1.4, we see that if M is orientable then $A(t)$ is the complete invariant of M up to units $\pm t^i$. If M is a closed knot complement (i.e. the exterior for some tame knot in S^3) then M belongs to $C(S^1\times B^2)$

*) $\doteq$ means "equal up to units of Λ". This notation will be also used in the following sense: For two elements A and A' of $\Gamma=\Lambda\otimes Q$, $A\doteq A'$ means that A equals to A' up to units of Γ.

and $A(t)$ was called the *knot polynomial* and Theorem 1.4 is well-known (See for example R.H. Crowell and R.H. Fox [2].).

The converse of Theorem 1.4 is also true. That is,

Theorem 1.6. *Let $f(t)$ be an integral polynomial with $|f(1)|=1$. If $f(t)\doteq f(t^{-1})$, then in both $C(S^1\times S^2)$ and $C(S^1\times B^2)$ there exist manifolds whose Alexander polynomials are $f(t)$. If $f(t)\doteq f(-t^{-1})$, then in $C(S^1\times_\tau S^2)$ there exists a manifold whose Alexander polynomial is $f(t)$. If $f(t)=(t^m+1/t+1)f_0(t)$ for some odd number $m\geq 1$ and some integral polynomial $f_0(t)$ with $f_0(t)\doteq f_0(-t^{-1})$, then in $C(S^1\times_\tau B^2)$ there exists a manifold M with $H_1(M, \partial M; Z)=Z_m$ whose Alexander polynomial is $f(t)$.*

SUPPLEMENTS 1.7. Let $f(t)=a_0+a_1t+\cdots+a_mt^m(a_0a_m\neq 0, m\geq 0)$ be an integral polynomial with $|f(1)|=1$. If $f(t)\doteq f(t^{-1})$ or $f(t)\doteq f(-t^{-1})$ is satisfied, then it is not difficult to see that m is always even and that the following explicit formulae are obtained:

$$f(t)=t^m f(t^{-1}) \quad \text{if} \quad f(t)\doteq f(t^{-1})$$
$$f(t)=(-1)^{m/2}t^m f(-t^{-1}) \quad \text{if} \quad f(t)\doteq f(-t^{-1})\,.$$

2. Proofs

Let M be a compact connected 3-manifold with $H_1(M; Z)=Z$ and $p\colon \tilde{M}\to M$ be the infinite cyclic covering associated with natural epimorphism $\gamma\colon \pi_1(M)\to\pi$.

Lemma 2.1. $H_2(\tilde{M}, \partial\tilde{M}; Z_2)\approx Z_2$.

Proof. It suffices to establish the duality

$$H^0(\tilde{M}; Z_2)\approx H_2(\tilde{M}, \partial\tilde{M}; Z_2)\,.$$

This duality is an analogy of the partial Poincaré duality theorem [3, Theorem 2.3], because $H_1(M; Z_2)=Z_2$ which implies that $H_1(\tilde{M}; Z_2)$ is finitely generated over Z_2 (See J.W. Milnor [8] or [3, Proposition 3.4].).

First, note that there is a duality $H^1_c(\tilde{M}; Z_2)\approx H_2(\tilde{M}, \partial\tilde{M}; Z_2)$— even if $\tilde{M}$ is non-orientable.

Second, the isomorphism $H^0(\tilde{M}; Z_2)\approx H^1_c(\tilde{M}; Z_2)$ is obtained from the same argument as in [3], since $H_1(\tilde{M}; Z_2)$ is finitely generated over Z_2. This proves Lemma 2.1.

2.2. Proof of Theorem 1.2. If $\partial M=\phi$ and M is orientable, then by the Poincaré duality we obtain that $H_*(M; Z)\approx H_*(S^1\times S^2; Z)$. If $\partial M=\phi$ and M is non-orientable, we know that $H_3(M; Z)=0$ and $H^3(M; Z)=Z_2$. Since the Euler characteristic $\chi(M)$ is equal to 0, it follows that $H_2(M; Z)$ is a torsion group. Hence $H_2(M; Z) \approx H^3(M; Z) = Z_2$. This implies that $H_*(M; Z) \approx H_*(S^1 \times_\tau S^2; Z)$. In case $\partial M \neq \phi$, the infinite cyclic covering $p: \tilde{M} \to M$ is used. Since $H_1(\tilde{M}; Z_2)$ is finitely generated over Z_2 and by Lemma 2.1 $H_2(\tilde{M}, \partial\tilde{M}; Z_2) \approx Z_2$, it follows from the following part of the homology exact sequence of the pair $(\tilde{M}, \partial\tilde{M})$:

$$H_2(\tilde{M}, \partial\tilde{M}; Z_2) \to H_1(\partial\tilde{M}; Z_2) \to H_1(\tilde{M}; Z_2)$$

that $H_1(\partial\tilde{M}; Z_2)$ is finitely generated over Z_2.

For each component N of ∂M let $\gamma^*: \pi_1(N) \to \pi$ be the composite $\pi_1(N) \xrightarrow[\text{inclusion}]{} \pi_1(M) \to \pi$. γ^* is a non-trivial homomorphism. Otherwise, by [3, Lemma 4.1] $\partial\tilde{M}$ must contain infinite many copies of N as components. Because N is not 2-sphere by assumption, $H_1(\partial\tilde{M}; Z_2)$ is not finitely generated over Z_2. This is a contradiction.

Therefore γ^* is non-trivial and hence each component $\tilde{N}$ of the preimage $p^{-1}(N)$ is an infinite cyclic covering space over N (See [3, Corollary 4.2].). Using that $H_1(\partial\tilde{M}; Z_2)$ is finitely generated, we obtain that $H_*(\tilde{N}; Z_2)$ is finitely generated over Z_2. This implies that $\chi(N)=0$ (See J.W. Milnor [8].). Hence $\chi(\partial M)=0$. By the formula $\chi(\partial M)=2\chi(M)$, $\chi(M)=0$. From this we see that $H_2(M; Z)$ is a torsion group. However, $H_2(M; Z)$ is free since $\partial M \neq \phi$. Thus, we have $H_*(M; Z) \approx H_*(S^1; Z)$. Furthermore, by the Poincaré duality over Z_2, $H_1(M, \partial M; Z_2) \approx H^2(M; Z_2) = 0$. This implies that $\tilde{H}_0(\partial M; Z_2)=0$. That is, ∂M is connected. By using $\chi(\partial M)=0$, we obtain that ∂M is homeomorphic to either $S^1 \times S^1$ or $S^1 \times_\tau S^1$. This completes the proof.

From now on we will assume M belongs to one of the four classes $\mathcal{C}(S^1 \times S^2)$, $\mathcal{C}(S^1 \times B^2)$, $\mathcal{C}(S^1 \times_\tau S^2)$ and $\mathcal{C}(S^1 \times_\tau B_2)$, unless otherwise stated.

Lemma 2.3. *$\tilde{M}$ is orientable.*

(The author wishes to thank the referee for pointing out the following simple proof. The original proof was more complicated cf. [3, Corollary 3.5])

Proof. First we note that $\tilde{M}$ is orientable if and only if the first Stiefel-Whitney class $w_1(\tilde{M})$ vanishes (See for example E.H. Spanier [11, p 349].).

Second, from the following short exact sequence of simplicial chain Λ-modules (for some triangulation of M)

$$0 \to C_\sharp(\tilde{M}; Z_2) \xrightarrow{t-1} C_\sharp(\tilde{M}; Z_2) \xrightarrow{p} C_\sharp(M; Z_2) \to 0\,,$$

we obtain the exact sequence

$$H_1(\tilde{M}; Z_2) \xrightarrow{p_*} \underset{\substack{\| \\ Z_2}}{H_1(M; Z_2)} \underset{\approx}{\to} \underset{\substack{\| \\ Z_2}}{H_0(\tilde{M}; Z_2)} \xrightarrow{t-1} \underset{\substack{\| \\ Z_2}}{H_0(\tilde{M}; Z_2)} \underset{\approx}{\xrightarrow{p_*}} \underset{\substack{\| \\ Z_2}}{H_0(M; Z)} \to 0\,.$$

This implies that the homomorphism $p_*: H_1(\tilde{M}; Z_2) \to H_1(M; Z_2)$ is trivial. Using the field Z_2, the dual homomorphism $p^*: H^1(M; Z_2) \to H^1(\tilde{M}; Z_2)$ is trivial. Therefore $w_1(\tilde{M}) = p^*(w_1(M)) = 0$. This completes the proof.

REMARK 2.4. Since $\tilde{M}$ is orientable and $H_1(\tilde{M}; Q)$ is finitely generated over Q, there is a duality $H^0(\tilde{M}; Z) \approx H_2(\tilde{M}, \partial\tilde{M}; Z) \approx Z$ by the partial Poincaré duality theorem [3, Theorem 2.3]. Then t induces the automorphism of $H_2(\tilde{M}, \partial\tilde{M}; Z) = Z$ of degree 1 or -1 according as the original manifold M is orientable or non-orientable. In fact, the short exact sequence $0 \to C_\sharp(\tilde{M}, \partial\tilde{M}; Z) \xrightarrow{t-1} C_\sharp(\tilde{M}, \partial\tilde{M}; Z) \xrightarrow{p} C_\sharp(M, \partial M; Z) \to 0$ induces the exact sequence $H_2(\tilde{M}, \partial\tilde{M}; Z) \xrightarrow{t-1} H_2(\tilde{M}, \partial\tilde{M}; Z) \xrightarrow{p_*} H_2(M, \partial M; Z) \to 0$. [In case M is orientable, this sequence is easily obtained. In case M is non-orientable, use the facts that $H_2(M, \partial M; Z) = Z_2$ and $H_1(\tilde{M}, \partial\tilde{M}; Z)$ is torsion-free. Note that the torsion product Tor $[H_1(\tilde{M}, \partial\tilde{M}; Z), G]$ vanishes for all finitely generated groups G, since $H_2(\tilde{M}, \partial\tilde{M}; G) \approx G$ by the partial Poincaré duality theorem [3, Theorem 2.3, Case(4)].]. In case M is orientable, the sequence is replaced by the exact sequence $Z \xrightarrow{t-1} Z \xrightarrow{p_*} Z \to 0$. Hence $t-1: Z \to Z$ is the trivial homomorphism. This implies that t induces the identity homomorphism. In case M is non-orientable, the above sequence implies the exact sequence $0 \to Z \xrightarrow{t-1} Z \xrightarrow{p_*} Z_2 \to 0$. This asserts that t is the automorphism of degree -1.

Lemma 2.5. *There exists a PL map $f: M \to S^1$ such that*
(1) $f_*: H_1(M; Z) \approx H_1(S^1; Z)$,
(2) *For some point $p \in S^1$, $F = f^{-1}(p)$ is a proper connected two-sided surface in M with connected complement $M - F$,*
(3) *F and $M - F$ are orientable,*
(4) *$[F] \in H_2(M, \partial M; Z)$ is a generator.* (*Note $H_2(M, \partial M; Z) = Z$ or Z_2 according as M is orientable or non-orientable.*)

Proof. By [3, Corollary 1.3], there is a PL map $f: M \to S^1$ satisfying (1) and (2). By Lemma 2.3, $\tilde{M}$ is orientable. Hence F and $M - F$ are orientable, since $M - F$ is canonically embedded in $\tilde{M}$. (3) is then satisfied. (4) follows from the fact that F intersects a circle representing a generator of $H_1(M; Z) = Z$ transversally at a single point (See [3, Corollary 1.3].). This shows Lemma 2.5.

Note that if $A(t)$ is the Alexander polynomial of M then $A(t^{-1})$ can be also considered as the Alexander polynomial of M by replacing one generator of the infinite cyclic covering transformation group with the other generator.

Lemma 2.6. (Calculating the Alexander polynoimal of M.)
(I) *Since $H_1(\tilde{M}; Q)$ is a finitely generated torsion Γ-module and Γ is a principal*

ideal domain, $H_1(\tilde{M}; Q)$ decomposes into cyclic Γ-modules: $H_1(\tilde{M}; Q)\approx\Gamma/(f_1(t))_Q\oplus\Gamma/(f_2(t))_Q\oplus\cdots\oplus\Gamma/(f_s(t))_Q$. Then for $\varepsilon=1$ or -1 $A(t^\varepsilon)\doteq f_1(t)f_2(t)\cdots f_s(t)$ as elements of Γ.

(II) *Since $H_1(\tilde{M}; Q)$ is finitely generated over Q, the isomorphism $t: H_1(\tilde{M}; Q)\to H_1(\tilde{M}; Q)$ represents a rational square matrix B. Then for $\varepsilon=1$ or -1 $A(t^\varepsilon)\doteq\det(tE-B)$ as elements of Γ, where E is the unit matrix.*

(III) *Let F be a surface in M described in Lemma 2.5 and M^* be the manifold obtained from M by splitting along F. Since $\tilde{M}$ can be constructed from the countable copies $\{M_i\}_{i=-\infty}^{\infty}$ of M^* by pasting next to next, (called Neuwirth construction [3, §1], L.P. Neuwirth [9]), it follows from the Mayer-Vietoris sequence that the sequence $H_1(F; Q)\otimes\Gamma\xrightarrow{r}H_1(M^*; Q)\otimes\Gamma\to H_1(\tilde{M}; Q)\to 0$ is exact as Γ-modules, where $r(x)=t(i_1)_*(x)-(i_2)_*(x)$ and $i_1, i_2: F\to M^*$ are the suitable identifications onto two copies of F. Since M^* is orientable, we have $H_1(F; Q)\approx H_1(M^*; Q)$ by Poincaré duality. Thus, $(i_1)_*, (i_2)_*: H_1(F; Q)\to H_1(M^*; Q)$ represent rational square matrices A_1, A_2, respectively, and r represents a matrix tA_1-A_2. Then for $\varepsilon=1$ or -1 $A(t^\varepsilon)\doteq\det(tA_1-A_2)$ as elements of Γ.*

(IV) *Let $(x_1, x_2, \cdots, x_n: r_1, r_2, \cdots, r_m)_\varphi$ be a presentation of $\pi_1(M)$ and $\bar{\gamma}: Z[\pi_1(M)]\to Z[\pi]=\Lambda$ be the ring homomorphism naturally extending the group epimorphism $\gamma: \pi_1(M)\to\pi$. Now we consider the Alexander (Jacobian) matrix $(\bar{\gamma}\varphi(\partial r_i/\partial x_j))$ (See R.H. Crowell and R.H. Fox [2].). By $E(\pi_1(M))$ we denote the Λ-ideal generated by the determinants of $(n-1)\times(n-1)$ submatrices of $(\bar{\gamma}\varphi(\partial r_i/\partial x_j))$. Then for $\varepsilon=1$ or -1 $A(t^\varepsilon)$ is a generator of the smallest principal ideal containing $E(\pi_1(M))$.*

Proof. If $H_1(\tilde{M};Q)\approx\Gamma/(f_1(t))_Q\oplus\cdots\oplus\Gamma/(f_s(t))_Q$ then the matrix $\begin{pmatrix} f_1(t) & & 0 \\ & \ddots & \\ 0 & & f_s(t)\end{pmatrix}$ is a relation matrix of $H_1(\tilde{M}; Q)$ over Γ. Hence from the uniqueness of the elementary ideal over Γ and Definition 1.3 we obtain $(A(t^\varepsilon))_Q=E(\mathfrak{C}(t^\varepsilon))\otimes Q=(f_1(t)\cdots f_s(t))_Q$ for $\varepsilon=1$ or -1. So $A(t^\varepsilon)\doteq f_1(t)f_2(t)\cdots f_s(t)$. This proves (I). Moreover, by S. Lang [5, p 401], we have $(\det(tE-B))_Q=(f_1(t)\cdots f_s(t))_Q$. This proves (II). For (III) since tA_1-A_2 is a relation matrix, we also obtain $(A(t))_Q=(\det(tA_1-A_2))_Q$, which proves (III). For (IV) it suffices to prove for some particular presentation of $\pi_1(M)$, since $E(\pi_1(M))$ does not depend upon a choice of presentations of $\pi_1(M)$ (cf.[2]). So we may choose a presentation $(x_1, x_2, \cdots, x_n: r_1, r_2, \cdots, r_m)_\varphi$ so that $\gamma\varphi(x_1)=t$, $\gamma\varphi(x_i)=1$ for $i\geq 2$ (In fact, choose a pre-abelian presentation (Magnus-Karrass-Solitar [7, p 140]).). It is not hard to see that the sequence $\Lambda[r_1^*, r_2^*, \cdots, r_m^*]\xrightarrow{d_2}\Lambda[x_1^*, x_2^*, \cdots, x_n^*]\xrightarrow{d_1}\Lambda$ is semi-exact (i.e. $d_1d_2=0$) as Λ-modules, where $\Lambda[r_1^*, \cdots, r_m^*]$ and $\Lambda[x_1^*, \cdots, x_n^*]$ are the free Λ-modules with bases $r_1^*, \cdots, r_m^*$ and $x_1^*, \cdots, x_n^*$, respectively, and d_2 is defined by $d_2(r_i^*)=\sum_{j=1}^n\bar{\gamma}\varphi(\partial r_i/\partial x_j)x_j^*$ and d_1 is defined by $d_1(x_j^*)=\gamma\varphi(x_j)-1$. [Remember the *fundamental formula* $r_i - 1 = \sum_{j=1}^{n}(\partial r_i/\partial x_j)(x_j - 1)$.] Since $d_1(x_1^*) = \gamma\varphi(x_1) - 1 = t - 1$ and $d_1(x_j^*) = \gamma\varphi(x_j) - 1 = 1 - 1 = 0$, $j \geq 2$, it follows that $\bar{\gamma}\varphi(\partial r_i/\partial x_1) = 0$, $i = 1, 2, \cdots, m$ and Ker $d_1 = \Lambda[x_2^*, \cdots, x_n^*]$. Then d_2 defines a map d_2': $\Lambda[r_1^*, \cdots, r_m^*] \to \Lambda[x_2^*, \cdots, x_n^*]$. By a result of R.H. Crowell [1, p 39], $H_1(\tilde{M}; Z)$ is Λ-isomorphic to Ker d_1/Im d_2; so, in this case, the sequence $\Lambda[r_1^*, \cdots, r_m^*] \xrightarrow{d_2'} \Lambda[x_2^*, \cdots, x_n^*] \to H_1(\tilde{M}; Z) \to 0$ is an exact sequence of Λ-modules. Hence $\mathfrak{G}(t) = (\bar{\gamma}\varphi(\partial r_i/\partial x_j))_{j\geq 2, i\geq 1}$ is a relation matrix of $H_1(\tilde{M}; Z)$. So, $A(t^\varepsilon)$ ($\varepsilon = 1$ or -1) is a generator of the smallest principal ideal containing the first elementary ideal $E(\mathfrak{G}(t))$. On the other hand, clearly, $E(\mathfrak{G}(t)) = E(\pi_1(M))$, since $\bar{\gamma}\varphi(\partial r_i/\partial x_1) = 0$, $i = 1, 2, \cdots, m$. This completes the proof.

Lemma 2.7. $|A(1)| = 1$.

Proof. Let $\bar{\varepsilon}$: $\Lambda \to Z$ be the augmentation sending t to 1. From the short exact sequence $0 \to C_\sharp(\tilde{M}; Z) \xrightarrow{t-1} C_\sharp(\tilde{M}; Z) \xrightarrow{p} C_\sharp(M; Z) \to 0$ of Λ-modules, we obtain the isomorphism $t-1$: $H_1(\tilde{M}; Z) \xrightarrow{\approx} H_1(\tilde{M}; Z)$. Hence $H_1(\tilde{M}; Z) \otimes_{\bar{\varepsilon}} Z = 0$. If $\mathfrak{G}(t)$ is a relation matrix of $H_1(\tilde{M}; Z)$ then $\mathfrak{G}(1)$ is a relation matrix of $0 = H_1(\tilde{M}; Z) \otimes_{\bar{\varepsilon}} Z$. This implies $E(\mathfrak{G}(1)) = Z$. Hence $Z = E(\mathfrak{G}(1)) = \bar{\varepsilon}(E(\mathfrak{G}(t)) \subset \bar{\varepsilon}(A(t)) = (A(1))$. Thus $A(1) = \pm 1$. This completes the proof.

2.8. Proof of Theorem 1.4. Let $\mu \in H_2(\tilde{M}, \partial\tilde{M}; Z)$ be a generator. By [3, Theorem 2.3], there is a duality

$$\cap\,\mu\colon H^1(\tilde{M}; Q) \approx H_1(\tilde{M}, \partial\tilde{M}; Q),$$

where $\cap$ denotes the cap product operation. In case M is orientable, then by Remark 2.4 we obtain the equality $t[(tu)\cap\mu] = u\cap(t\mu) = u\cap\mu$. Hence the following diagram is commutative:

$$\begin{array}{ccccc} H^1(\tilde{M}; Q) & \xrightarrow[\cap\,\mu]{\approx} & H_1(\tilde{M}, \partial\tilde{M}; Q) & \xleftarrow[\text{inclusion}]{\approx} & H_1(\tilde{M}; Q) \\ \approx\downarrow t & & \approx\downarrow t^{-1} & & \approx\downarrow t^{-1} \\ H^1(\tilde{M}; Q) & \xrightarrow[\cap\,\mu]{\approx} & H_1(\tilde{M}, \partial\tilde{M}; Q) & \xleftarrow[\text{inclusion}]{\approx} & H_1(\tilde{M}; Q) \end{array}$$

[In case $\partial M \neq \phi$, by Poincaré duality $H_1(M, \partial M; Z) = H^2(M; Z) = 0$. Hence the inclusion homomorphism $H_1(\partial M; Z) \to H_1(M; Z)$ is onto. This implies that $\partial\tilde{M}$ is connected (See [3, Lemma 4.1].). Thus the inclusion homomorphism $H_1(\tilde{M}; Q) \to H_1(\tilde{M}, \partial\tilde{M}; Q)$ is an isomorphism.)

If $H^1(\tilde{M}; Q)$ is Γ-isomorphic to $\Gamma/(f_1(t))_Q \oplus \cdots \oplus \Gamma/(f_r(t))_Q$ then the above diagram implies that $H_1(M; Q)$ is Γ-isomorphic to $\Gamma/(f_1(t^{-1}))_Q \oplus \cdots \oplus \Gamma/(f_s(t^{-1}))_Q$. On the other hand, since $H^1(\tilde{M}; Q) = \text{Hom}[H_1(\tilde{M}; Q), Q]$, $H_1(\tilde{M}; Q)$ and $H^1(\tilde{M}; Q)$ are Γ-isomorphic. Thus,

$$(f_1(t)\cdots f_s(t))_Q = (f_1(t^{-1})\cdots f_s(t^{-1}))_Q .$$

Using Lemma 2.6 and Gauss lemma, we showed that $A(t)\doteq A(t^{-1})$ as elements of Λ.

In case M is non-orientable and $\partial M=\phi$ then the isomorphism $H^1(\tilde{M}; Q)\approx_\Gamma\Gamma/(f_1(t))_Q\oplus\cdots\oplus\Gamma/(f_s(t))_Q$ implies the isomorphism $H_1(\tilde{M}; Q)\approx_\Gamma\Gamma/(f_1(-t^{-1}))_Q\oplus\cdots\oplus\Gamma/(f_s(-t^{-1}))_Q$, because the duality $\cap\mu$: $H^1(\tilde{M}; Q)\approx H_1(\tilde{M}; Q)$ has the equality $(tu)\cap\mu=-t^{-1}[u\cap\mu]$ by Remark 2.4. Since $H_1(\tilde{M}; Q)$ and $H^1(\tilde{M}; Q)$ are Γ-isomorphic, we obtain $(f_1(t)\cdots f_s(t))_Q=(f_1(-t^{-1})\cdots f_s(-t^{-1}))_Q$. Using Lemma 2.6 and Gauss lemma, we showed that $A(t)\doteq A(-t^{-1})$ as elements of Λ.

In case M is non-orientable and $\partial M\neq\phi$, then we have $H_1(M, \partial M; Z)=Z_m$ for some odd number $m\geq 1$. [Note that $H_1(M, \partial M; Z)\otimes Z_2=H_1(M, \partial M; Z_2)=H^2(M; Z_2)=0$.] Now we consider the following exact sequence:

$$0\to H_2(\tilde{M}, \partial\tilde{M}; Q)\to H_1(\partial\tilde{M}; Q)\to H_1(\tilde{M}; Q)\xrightarrow{j_*} H_1(\tilde{M}, \partial\tilde{M}; Q) .$$

Since M and ∂M are non-orientable and $\partial\tilde{M}$ contains m copies of $R^1\times S^1$ as components [3, Corollary 4.2], we have $H_2(\tilde{M}, \partial\tilde{M}; Q)=\Gamma/(t+1)_Q$ and $H_1(\partial\tilde{M}; Q)=\Gamma/(t^m+1)_Q$. Accordingly, the above sequence induces the following exact sequence of Γ-modules: $0\to\Gamma/(t^m+1/t+1)_Q\to H_1(\tilde{M}; Q)\to\operatorname{Im} j_*\to 0$. Let $g_0(t)$ be the characteristic polynomial of the Q-linear isomorphism t: $\operatorname{Im} j_*\to\operatorname{Im} j_*$. By Lemma 2.6, we may regard $A(t)$ as the characteristic polynomial of the Q-linear isomorphism t: $H_1(\tilde{M}; Q)\to H_1(\tilde{M}; Q)$. So, the equality $A(t)\doteq(t^m+1/t+1)g_0(t)$ holds (See for example S. Lang [5, p 402].). Next since the following square

$$\begin{array}{ccc} H_1(\tilde{M}; Q) & \xrightarrow{j_*} & H_1(\tilde{M}, \partial\tilde{M}; Q) \\ \approx\uparrow\cap\mu & & \approx\uparrow\cap\mu \\ H^1(\tilde{M}, \partial\tilde{M}; Q) & \xrightarrow{j^*} & H^1(\tilde{M}; Q) \end{array}$$

is commutative, we obtain the isomorphism $\cap\mu$: $\operatorname{Im} j^*\approx\operatorname{Im} j_*$. The isomorphism $\operatorname{Im} j^*\approx_\Gamma\Gamma/(g_1(t))_Q\oplus\cdots\oplus\Gamma/(g_s(t))_Q$ implies the isomorphism $\operatorname{Im} j_*\approx_\Gamma\Gamma/(g_1(-t^{-1}))_Q\oplus\cdots\oplus\Gamma/(g_s(-t^{-1}))_Q$, since $(tu)\cap\mu=(-t^{-1})[u\cap\mu]$. However, $\operatorname{Im} j^*=\operatorname{Hom}[\operatorname{Im} j_*, Q]$ asserts that $\operatorname{Im} j^*$ and $\operatorname{Im} j_*$ are isomorphic as Γ-modules. Therefore

$$g_0(t)\doteq g_1(t)\cdots g_s(t)\doteq g_1(-t^{-1})\cdots g_s(-t^{-1})\doteq g_0(-t^{-1}) .$$

If we denote $A(t)=(t^m+1/t+1)A_0(t)$, where $A_0(t)=c\, g_0(t)$ for some non-zero rational number $c\in Q$, then we have $A_0(t)\in\Lambda$ and $A_0(t)\doteq A_0(-t^{-1})$ as elements of Λ. Combined with Lemma 2.7, the proof is completed.

Lemma 2.9. *Let $f(t)$ be an integral polynomial with $|f(1)|=1$. If $f(t)\doteq f(t^{-1})$, then there exists $M\in C(S^1\times B^2)$ whose Alexander polynomial is $f(t)$. If*

$f(t) \doteq f(-t^{-1})$, then there exists $M \in \mathcal{C}(S^1 \times_\tau B^2)$ with $H_1(M, \partial M; Z)=0$ whose Alexander polynomial is $f(t)$.

Proof. If $f(t) \doteq f(t^{-1})$ then it is easy to obtain $M \in \mathcal{C}(S^1 \times B^2)$ whose Alexander polynomial is $f(t)$, because it is well-known in the classical knot theory (See H. Seifert [10].) that there exists a tame knot $K^1 \subset S^3$ whose Alexander polynomial is $f(t)$. In fact, we may take M to be the exterior (i.e. the closed knot complement) of $K^1 \subset S^3$.

So it suffices to prove for the non-orientable case. The method of the proof is somewhat analogous to the method of J. Levine [6], by which he gave an alternative proof of a characterization of the knot polynomials due to H. Seifert [10].

Now we may assume $f(t)=\sum_{i=-s}^{s} a_i t^i$ $(s>0)$ $\sum_i a_i=1$ and $a_i=(-1)^i a_{-i}$. [If $s=0$, then we can take $S^1 \times_\tau B^2 \in \mathcal{C}(S^1 \times_\tau B^2)$.]

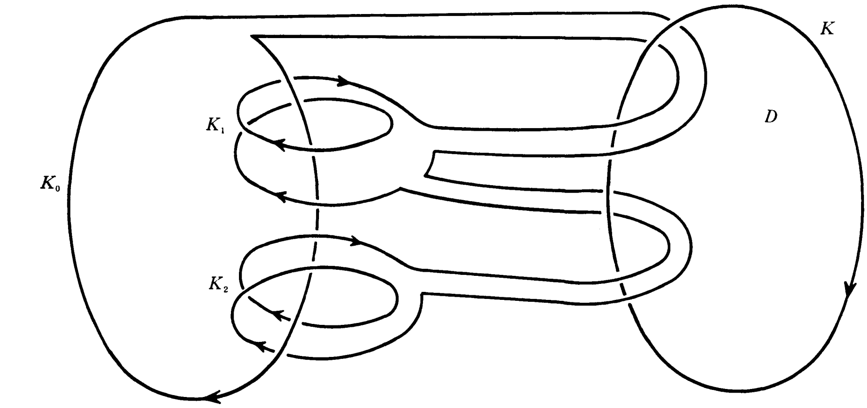
Take an oriented disk D in an oriented 3-sphere S^3 and let $K=\partial D$. Also, let $K_0, K_1, \cdots, K_s$ be $s+1$ trivial knots, disjoint each other and from D, and with linking numbers as follows:

$$L(K_0, K_i) = a_i \qquad \text{for} \quad i = 1, 2, \cdots, s$$
$$L(K_i, K_j) = 0 \qquad \text{for} \quad i.j \neq 0,\ i \neq j\,.$$

We construct a new knot K' by connecting up the $\{K_i\}$ in the following manner (cf. [6]): Choose two points p_i and q_i on each K_i and mutually disjoint oriented arcs $\{A_j\}$ in S^3-K, beginning at q_{i-1} and ending at p_i so that each A_i is disjoint from the $\{K_i\}$ except for the points q_{i-1}, p_i. Next, thicken A_i to be a band B_i which we identify with $I \times A_i$, meeting K_{i-1} along $I \times q_{i-1}$ and K_i along $I \times p_i$, but otherwise disjoint from the $\{K_i\}$; furthermore, the $\{B_i\}$ should be mutually disjoint. Then define the knot K' by $K'=(\cup_{i=0}^{s} K_i \cup \cup_{i=1}^{s} B_i)-\cup_{i=1}^{s}(\text{Int } I) \times A_i$. K' is a knot disjoint from K and we may orient K' coherently with the $\{K_i\}$. The oriented knot K' is called a *complete fusion along the arcs* $\{A_i\}$ and is denoted by $K'=K_0 \# K_1 \# \cdots \# K_s$.

We pose one additional restriction on the construction of K'. That is, each A_i passes once around K in the sense that A_i should intersect D transversally at a single point with positive orientation. We illustrated K' for the case $f(t)=2t^{-2}+2t^{-1}-3-2t+2t^2$ in Fig. 1.

Choose a tubular neighborhood $T(K)$ of K in S^3 so that $D_0=cl(S^3-T(K)) \cap D$ is a proper disk of $X=cl(S^3-T(K))$. Note that X is PL homeomorphic to $S^1 \times B^2$. Now split X along D_0 and re-attach the resulting manifold by an *orientation-preserving* homeomorphism between the resulting two copies of D_0. Thus, we obtain a manifold X_τ which is PL homeomorphic to $S^1 \times_\tau B^2$. By a suitable move of the homeomorphism, we can assume that $K' \subset X$ is deformed into a knot $K_\tau' \subset X_\tau$.



$f(t)=2t^{-2}+2t^{-1}-3-2t+2t^2$

Fig. 1

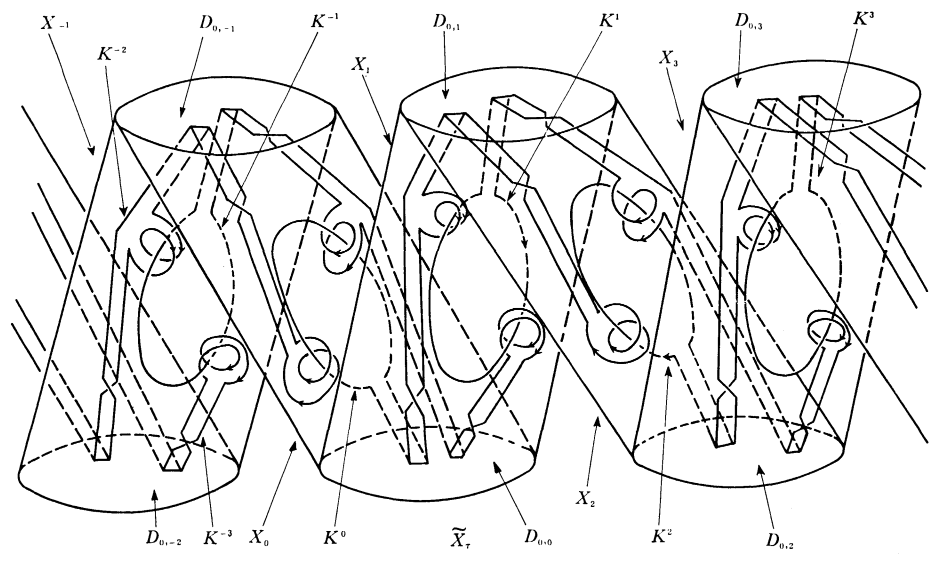


Fig. 2

$X_\tau - D_0$ lifts to an infinite sequence $\{X_i\}$, $-\infty < i < \infty$, of copies of $X_\tau - D_0$; we may assume they are numbered so that X_i is separated from X_{i+1} by a lifting $D_{0,i}$ of D_0 and $\partial X = D_{0,i} - D_{0,i-1}$. For every pair of integers i, m, where $0 \leq i \leq s$ and $-\infty < m < \infty$, let $K_{i,m}$ be the lifting of K_i lying in X_m. The $\{K_{i,m}\}$ are mutually disjoint. Since the universal covering space $\widetilde{X}_\tau$ is orientable, we let $\widetilde{X}_\tau$ be oriented so that $L(K_{0,0}, K_{i,0}) = a_i$ for $i=1, 2, \cdots, s$. Then we have

$$L(K_{i,m}, K_{j,n}) = \begin{cases} (-1)^m a_i & \text{if } \ m=n, \ j=0 \\ (-1)^m a_j & \text{if } \ m=n, \ i=0\,. \end{cases}$$

Since each A_i intersects D_0 transversally at a single point, A_i lifts to a sequence $\{A_{i,m}\}$, $-\infty<m<\infty$ of arcs, where $A_{i,m}$ joints $K_{i-1,m-1}$ to $K_{i,m}$. Thus K_τ' lifts to a sequence K^m of knots, where K^m is a complete fusion $K^m = K_{0,m} \# K_{1,m+1} \# \cdots \# K_{s,m+s}$ along the arcs $\{A_{i,m+i}\}_{1\le i\le s}$ (See Fig. 2.).

The linking numbers of the $\{K^m\}$ and K^0 are given as follows:

$$L(K^0, K^m) = \begin{cases} (-1)^m a_m & \text{if } \ 0<|m|\le s \\ 0 & \text{if } \ |m|>s\,, \end{cases}$$

because $L(K^0, K^m)=\sum_i L(K_{i,i}, K_{i-m,i})$ and $a_{-m}=(-1)^m a_m$.

Let φ_0: $S^1\times B^2 \to \tilde{X}_\tau$ be a tubular neighborhood of K^0 with $L(K^0, \varphi_0(S^1\times q)) = a_0$ for some point $q\in\partial B^2=S^1$. For each m, $-\infty<m<\infty$, define an embedding φ_m: $S^1\times B^2\to\tilde{X}_\tau$ to be the composite $S^1\times B^2 \xrightarrow{\varphi_0} \tilde{X}_\tau \xrightarrow{t^m} \tilde{X}_\tau$, where t is a generator of the covering transformation group π. Then φ_m determines a tubular neighborhood of K^m such that $L(K^m, \varphi_m(S^1\times q))=(-1)^m a_0$. Let $\tilde{T}$ be the submanifold of $\tilde{X}_\tau$ obtained by removing the interiors of $\varphi_m(S^1\times B^2)$, $-\infty<m<\infty$.

Define a manifold $\tilde{M}$ to be obtained from $\tilde{T}$ by attaching to each component of $\partial\tilde{T}$ a copy of $B^2\times S^1$ by means of the maps $\varphi_m|S^1\times S^1$. Since $t|\tilde{T}$ has a canonical extension to a homeomorphism from $\tilde{M}$ to $\tilde{M}$, we can regard the group $\pi=\{t^m\}$ as the properly discontinuous action on $\tilde{M}$. Then define a manifold M to be the orbit space $\tilde{M}/\pi$. Note that the projection $\tilde{M}\to M$ forms an infinite cyclic covering with its transformation group π.

We shall show that $H_1(M; Z)=Z$ and the Alexander polynomial of M is $f(t)$.

Note that $H_1(\tilde{T}; Z)$ is a free Λ-module generated by $[\varphi_0(p\times S^1)]$ ($p\in S^1$). This follows from the exact sequence of Λ-modules:

$$\begin{array}{ccccccc} H_2(\tilde{X}_\tau; Z) & \to & H_2(\tilde{X}_\tau, \tilde{T}; Z) & \to & H_1(\tilde{T}; Z) & \to & H_1(\tilde{X}_\tau; Z) \\ \| & & & & & & \| \\ 0 & & & & & & 0 \end{array}$$

and the fact that , by excision, $H_2(\tilde{X}_\tau, \tilde{T}; Z)$ is the free Λ-module generated by $[\varphi_0(p\times B^2)]$.

Now consider the exact sequence

$$H_2(\tilde{M}, \tilde{T}; Z) \xrightarrow{\Delta} H_1(\tilde{T}; Z) \to H_1(\tilde{M}; Z) \to H_1(\tilde{M}, \tilde{T}; Z)\,.$$

By excision, $H_1(\tilde{M}, \tilde{T}; Z)=0$ and $H_2(\tilde{M}, \tilde{T}; Z)$ is the free Λ-module generated by $[B^2\times q]$, where the boundary of $B^2\times q$ is $\varphi_0(S^1\times q)$. It follows that the image of Δ is the submodule of $H_1(\tilde{T}; Z)$ generated by $[\varphi_0(S^1\times q)]$.

We shall show that $[\varphi_0(S^1\times q)]=f(t)\,[\varphi_0(p\times S^1)]$.

Let $[\varphi_0(S^1\times q)]=g(t)[\varphi_0(p\times S^1)]$ in $H_1(\tilde{T}; Z)$ for some element

$g(t)=\sum_i c_i t^i \in \Lambda$. If $m \neq 0$, $(-1)^m c_m = \sum_i c_i L(t^i[\varphi_0(p\times S^1)], K^m)$

$$= L([\varphi_0(S^1\times q)], K^m)$$
$$= L(K^0, K^m)$$
$$= \begin{cases} (-1)^m a_m & \text{if } |m| \leq s \\ 0 & \text{if } |m| > s. \end{cases}$$

If $m=0$, $c_0 = c_0 L(\varphi_0(p\times S^1), K^0)$

$$= \sum_i c_i L(t^i[\varphi_0(p\times S^1)], K^0)$$
$$= L([\varphi_0(S^1\times q)], K^0) = a_0 .$$

Thus, we showed that $H_1(\tilde{M}; Z)=\Lambda/(f(t))$.

From the short exact sequence of simplicial chain Λ-modules $0\to C_\sharp(\tilde{M}; Z) \xrightarrow{t-1} C_\sharp(\tilde{M}; Z) \xrightarrow{p} C_\sharp(M; Z)\to 0$, we obtain the homology exact sequence of Λ-modules

$$\to H_1(\tilde{M}; Z) \xrightarrow{p_*} H_1(M; Z) \to H_0(\tilde{M}; Z) \to 0 .$$

[Note that $H_0(\tilde{M}; Z) \underset{\approx}{\xrightarrow{p_*}} H_0(M; Z)$.] This sequence induces the exact sequence of abelian groups:

$$H_1(\tilde{M}; Z)\otimes_{\bar{\varepsilon}} Z \xrightarrow{p_*} H_1(M; Z)\otimes_{\bar{\varepsilon}} Z \to H_0(\tilde{M}; Z)\otimes_{\bar{\varepsilon}} Z \to 0 ,$$

where $\bar{\varepsilon}\colon \Lambda\to Z$ is the augmentation. Note that $H_1(\tilde{M}; Z)\otimes_{\bar{\varepsilon}} Z=\Lambda/(f(t))\otimes_{\bar{\varepsilon}} Z= Z/(1)=0$, because $f(1)=1$. Therefore $H_1(M;Z)=H_1(M;Z)\otimes_{\bar{\varepsilon}} Z\approx H_0(\tilde{M};Z)\otimes_{\bar{\varepsilon}} Z =Z$. Since $\partial\tilde{M}$ is connected, the inclusion homomorphism $H_1(\partial M; Z)\to H_1(M; Z)$ is onto (See [3, Corollary 4.2].). So, $H_1(M, \partial M; Z)=0$. This completes the proof.

Lemma 2.10. *Given an odd integer $m\geq 1$, then there exists $M\in \mathcal{C}(S^1\times_\tau B^2)$ with $H_1(M, \partial M; Z)=Z_m$ whose Alexander polynomial is $t^m+1/t+1$.*

Proof. Consider an oriented 2-sphere D with m holes and let $C_1, C_2, \cdots, C_m$ be the components of ∂D with the induced orientations. Choose an orientation-reversing auto-homeomorphism $h\colon D\to D$ sending C_1 to C_2, C_2 to C_3, $\cdots$, C_{m-1} to C_m and C_m to C_1. Let $\tilde{M}=D\times R^1$ and define an auto-homeomorphism t: $\tilde{M}\to\tilde{M}$ by $t(x, y)=(h(x), y+1)$. If $\tilde{M}$ is oriented, then t is an orientation-reversing auto-homeomorphism. Since the group $\pi=\{t^i\}$ is a properly discontinuous action on $\tilde{M}$, the quotient projection $\tilde{M}\to\tilde{M}/\pi=M$ is an infinite cyclic covering with its transformation group π. Note that M is non-orientable. Form a direct computation, it is not difficult to see that $H_1(\tilde{M}, \partial\tilde{M}; Z)= \Lambda/(t^m-1/t-1)$. Let $\mu\in H_2(\tilde{M}, \partial\tilde{M}; Z)=Z$ be a generator. Then the duality $\cap\,\mu\colon H^1(\tilde{M}; Z)\approx H_1(\tilde{M}, \partial\tilde{M}; Z)$ determines the module $H^1(\tilde{M}; Z)=\Lambda/((-t^{-1})^m$

$-1/(-t^{-1})-1)$. Since m is odd, we obtain that $H_1(\tilde{M}; Z)=\Lambda/(t^m+1/t+1)$. Using that $H_1(\tilde{M}; Z)\otimes_{\bar{\varepsilon}}Z=0$, where $\bar{\varepsilon}\colon \Lambda\to Z$ is the augmentation, the exact sequence $H_1(\tilde{M}; Z)\to H_1(M; Z)\to H_0(\tilde{M}; Z)\to 0$ induces the isomorphism $H_1(M; Z)=H_1(M; Z)\otimes_{\bar{\varepsilon}}Z\approx H_0(\tilde{M}; Z)\otimes_{\bar{\varepsilon}}Z=Z$. Hence we showed that $M\in\mathcal{C}(S^1\times_\tau B^2)$ whose Alexander polynomial is $t^m+1/t+1$. Since $\partial\tilde{M}$ consists of m components, it follows from [3, Corollary 4.2] that $H_1(M; Z)/\mathrm{Im}[H_1(\partial M; Z)\to H_1(M; Z)]\approx Z_m$. Using the homology sequence of the pair $(M, \partial M)$, we obtain that $H_1(M, \partial M; Z)\approx Z_m$. This proves Lemma 2.10.

2.11. Proof of Theorem 1.6. Let $f(t)$ be an integral polynomial with $|f(1)|=1$ and $f(t)\doteq f(t^{-1})$. By Lemma 2.9 there exists $M\in\mathcal{C}(S^1\times B^2)$ whose Alexander polynomial is $f(t)$. Let $\bar{M}$ be a closed manifold obtained from M by attaching $S^1\times B^2$ to ∂M so that $H_1(\bar{M}; Z)=Z$. Then $\bar{M}\in\mathcal{C}(S^1\times S^2)$ and we shall show that $f(t)$ is the Alexander polynomial of $\bar{M}$.

$$\begin{aligned}\text{By excision, } H_1(\tilde{\bar{M}}; Z)&\approx_\Lambda H_1(\tilde{\bar{M}}, R^1\times B^2; Z)\\ &\approx_\Lambda H_1(\tilde{M}, \partial\tilde{M}; Z)\\ &\approx_\Lambda H_1(\tilde{M}; Z)\,.\end{aligned}$$

Hence by Lemma 2.6, $f(t)$ is the Alexander polynomial of M.

Next, let $f(t)$ be an integral polynomial with $|f(1)|=1$ and $f(t)\doteq f(-t^{-1})$. By Lemma 2.9 there exists $M\in\mathcal{C}(S^1\times_\tau B^2)$ with $H_1(M, \partial M; Z)=0$ whose Alexander polynomial is $f(t)$. Then let $\bar{M}$ be a closed manifold obtained from M by attaching $S^1\times_\tau B^2$ to ∂M so that $H_1(\bar{M}; Z)=Z$. Using $H_1(\tilde{\bar{M}}; Z)\approx_\Lambda H_1(\tilde{M}; Z)$, we see that $f(t)$ is the Alexander polynomial of $\bar{M}$, by Lemma 2.6.

Now let $f(t)=(t^m+1/t+1)f_0(t)$ be an integral polynomial for some odd number $m\geq 1$ and an integral polynomial $f_0(t)$ with $f_0(t)\doteq f_0(-t^{-1})$ and $|f_0(1)|=1$. By Lemma 2.9, there exists $M_0\in\mathcal{C}(S^1\times_\tau B^2)$ with $H_1(M_0, \partial M_0; Z)=0$ whose Alexander polynomial is $f_0(t)$. By Lemma 2.10, there exists $M_m\in\mathcal{C}(S^1\times_\tau B^2)$ with $H_1(M_m, \partial M_m; Z)=Z_m$ whose Alexander polynomial is $t^m+1/t+1$.

Choose a solid Klein bottle $S^1\times_\tau B^2$ in M_m which represents a generator of $H_1(M_m; Z)=Z$ and let M be the manifold obtained from $cl(M_m-S^1\times_\tau B^2)$ by attaching M_0 to $\partial(S^1\times_\tau B^2)$ by a homeomorphism $\partial M_0\to\partial(S^1\times_\tau B^2)$. Then it is not so difficult to see that M is in $\mathcal{C}(S^1\times_\tau B^2)$ and $H_1(M, \partial M; Z)=Z_m$ and the Alexander polynomial is $f(t)=(t^m+1/t+1)f_0(t)$. [The sequence $0\to H_1(\partial\tilde{M}_0; Q)\to H_1(\tilde{cl}(M_m-S^1\times_\tau B^2); Q)\oplus H_1(\tilde{M}_0; Q)\to H_1(\tilde{M}; Q)\to 0$ is exact and $H_1(\partial\tilde{M}_0; Q)=\Gamma/(t+1)_Q$ and $H_1(\tilde{cl}(M_m-S^1\times_\tau B^2); Q)=\Gamma/(t+1)_Q\oplus\Gamma/(t^m+1/t+1)_Q$. If $A(t)$ is the characteristic polynomial of the isomorphism $t\colon H_1(\tilde{M}; Q)\to H_1(\tilde{M}; Q)$ then we obtain that $(t+1)A(t)\doteq(t+1)\,(t^m+1/t+1)f_0(t)$. Hence $A(t)\doteq(t^m+1/t+1)f_0(t)=f(t)$.] This completes the proof.

3. Further discussions

3.1. A construction of a homology handle or circle having a fiber bundle structure over S^1.

DEFINITION 3.1.1. Let M be a homology handle or circle. M is called a *fibered manifold* (over S^1) if M is a fiber bundle over S^1.

DEFINITION 3.1.2. A *skew-orthogonal matrix* is an integral $(2g)\times(2g)$-matrix S satisfying $S.\tilde{S}=\varepsilon E$, where $\varepsilon=1$ or -1 and E is the unit matrix and $\tilde{S}$ is defined as follows:

If $S=\begin{pmatrix} S_{11}\cdots S_{1g} \\ \vdots \ \ddots \ \vdots \\ S_{g1}\cdots S_{gg}\end{pmatrix}$, $S_{ij}=\begin{pmatrix} a_{ij} & d_{ij} \\ c_{ij} & b_{ij}\end{pmatrix}$ then

$$\tilde{S}=\begin{pmatrix} \tilde{S}_{11}\cdots \tilde{S}_{g1} \\ \vdots \ \ddots \ \vdots \\ \tilde{S}_{1g}\cdots \tilde{S}_{gg}\end{pmatrix},\quad \tilde{S}_{ij}=\begin{pmatrix} d_{ij} & -b_{ij} \\ -c_{ij} & a_{ij}\end{pmatrix}.$$

Note that any integral 2×2-matrix whose determinant is ± 1 is a skew-orthogonal matrix.

Let F be an oriented surface of genus $g\geq 1$ with non-empty connected boundary. Choose a standard basis $\langle a_1, b_1, \cdots, a_g, b_g\rangle$ for $H_1(F; Z)$ with intersection numbers $a_i\cdot b_i=1$, $a_i\cdot b_j=0$ $(i\neq j)$ and $a_i\cdot a_j=b_i\cdot b_j=0$ (all i, j). It is not so difficult to show that, given a skew-orthogonal matrix S, then there is an auto-homeomorphism $h: F\to F$ such that the automorphism $h_*: H_1(F; Z)\to H_1(F; Z)$ represents S with respect to the basis $\langle a_1, b_1, \cdots, a_g, b_g\rangle$ and conversely*). h is orientation-preserving or orientation-reversing according as $\varepsilon=1$ or -1.

Let $\tilde{M}=F\times R^1$ and define the transformation t: $\tilde{M}\to\tilde{M}$ by $t(x, y)=(h(x), y+1)$. Since $\pi=\{t^m\}$ is a properly discontinuous action on $\tilde{M}$, the orbits space $M=\tilde{M}/\pi$ is a compact manifold such that the natural projection $\tilde{M}\to M$ is an infinite cyclic covering projection whose covering transformation group is π. Clearly, M is orientable or non-orientable according as $\varepsilon=1$ or -1. Since t: $H_1(\tilde{M}; Z)\to H_1(\tilde{M}; Z)$ represents S, it follows that $H_1(M; Z)\approx Z\oplus H_1(\tilde{M}; Z)/(E-S)H_1(\tilde{M}; Z)$. Hence $H_1(M; Z)\approx Z$ if and only if $\det(E-S)=\pm 1$. Note that, from construction, M is a fibered manifold with fiber F and such that $H_1(M, \partial M; Z)=0$ (See [3, Lemma 4.1].) and whose Alexander polynomial is $\det(tE-S)$ by Lemma 2.6.

Conversely, if M is a fibered homology circle with $H_1(M, \partial M; Z)=0$ then it is easy to obtain a skew-orthogonal matrix S such that $\det(tE-S)$ is the Alexander polynomial of M.

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*) The author thanks to Professor H. Terasaka for pointing out Definition 3.1.2 and this assertion (which proof can be also obtained from [7, P178]).

Thus we obtain the following.

Lemma 3.1.3. *Given a skew-orthogonal matrix* S *with* $\det(E-S)=\pm 1$, *then there exists a fibered homology circle* M *with* $H_1(M, \partial M; Z)=0$ *whose Alexander polynomial is* $\det(tE-S)$. *Such a manifold may be orientable or non-orientable according as* $\varepsilon=1$ *or* -1.

Conversely, given a fibered homology circle M *with* $H_1(M, \partial M; Z)=0$, *then there exists a skew-orthogonal matrix* S *with* $\det(E-S)=\pm 1$ *and such that* det $(tE-S)$ *is the Alexander polynomial of* M. ε *becomes* 1 *or* -1 *according as* M *is orientable or non-orientable.*

It is clear that Lemma 3.1.3 taken homology handles instead of homology circles also holds.

Theorem 3.1.4.*) *Let* $f(t)=a_0+a_1t+\cdots+a_nt^n(n\geq 0)$ *be an integral polynomial with* $|f(0)|=|f(1)|=1$. *If* $f(t)\doteq f(t^{-1})$, *then in both* $C(S^1\times S^2)$ *and* $C(S^1\times B^2)$ *there exist fibered manifolds whose Alexander polynomials are* $f(t)$. *If* $f(t)\doteq f(-t^{-1})$, *then in* $C(S^1\times_\tau S^2)$ *there exists a fibered manifold whose Alexander polynomial is* $f(t)$. *If* $f(t)=(t^m+1/t+1)f_0(t)$ *for some odd number* $m\geq 1$ *and some integral polynomial* $f_0(t)$ *with* $f_0(t)\doteq f_0(-t^{-1})$, *then in* $C(S^1\times_\tau B^2)$ *there exists a fibered manifold* M *with* $H_1(M, \partial M; Z)=Z_m$ *whse Alexander polynomial is* $f(t)$.

Sketch of Proof. It suffices to show that if $f(t)\doteq f(\varepsilon t^{-1})$, $\varepsilon=1$ or -1 then there is a fibered M in $C(S^1\times B^2)$ or $C(S^1\times_\tau B^2)$ with $H_1(M, \partial M; Z)=0$ whose Alexander polynomial is $f(t)$. Then the desired result will be obtained by a suitable attachment of $S^1\times B^2$ or M_m, constructed in Lemma 2.10, to M, as in 2.11. (Note that M_m is fibered.) By J. Levine [6] (for $\varepsilon=1$) or Lemma 2.9 (for $\varepsilon=-1$), we obtain $M\in C(S^1\times B^2)$ (for $\varepsilon=1$) or $M\in C(S^1\times_\tau B^2)$ (for $\varepsilon=-1$) such that $H_1(\tilde{M}; Z)=\Lambda/(f(t))$. $|f(0)|=1$ implies that $H_1(\tilde{M}; Z)$ is finitely generated free over Z. Hence by [3, Theorem 2.3] there is a duality $\cap\mu$: $H^1(\tilde{M}; Z)\approx H_1(\tilde{M}, \partial\tilde{M}; Z)$, which says that the cup product $\cup: H^1(\tilde{M}, \partial\tilde{M}; Z)\times H^1(\tilde{M}, \partial\tilde{M}; Z)\to H^2(\tilde{M}, \partial\tilde{M}; Z)=Z$ gives a symplectic inner product over Z. That is, there is a basis $\langle e_1, e_1', \cdots, e_s, e_s'\rangle$ for $H^1(\tilde{M}, \tilde{M}\partial; Z)$ such that $e_i\cup e_i'=1$, $e_i\cup e_j'=0$ $(i\neq j)$, $e_i\cup e_j=e_i'\cup e_j'=0$ (all i, j). Then the automorphism t: $H^1(\tilde{M}, \partial\tilde{M}; Z)\to H^1(\tilde{M}, \partial\tilde{M}; Z)$ represents a skew-orthogonal matrix S: $S.\tilde{S}=\varepsilon E$ with respect to the basis $\langle e_1, e_1', \cdots, e_s, e_s'\rangle$. Using $\det(tE-S)\doteq f(t)$ and Lemma 3.1.3, we complete the proof.

3.2. The Genus of a homology handle or circle

Now we will assume that M belongs to one of the four classes: $C(S^1\times S^2)$, $C(S^1\times B^2)$, $C(S^1\times_\tau S^2)$, $C(S^1\times_\tau B^2)$. Given M, there is a PL map $f: M\to S^1$ such that for some point $p\in S^1$, $F=f^{-1}(p)$ is a proper connected 2-sided orientable surface in M and with $f_*: H_1(M; Z)\approx H_1(S^1; Z)$ (See Lemma 2.5.).

*) In the classical knot theory, a corresponding result has been obtained by G. Brude, *Alexanderpolynome Neuwirthschen Knoten,* Topology 5(1966), 321–330.

The pair (f, p) is called a *Seifert pair*.

DEFINITION 3.2.1. The *genus* of M is the minimal number of the genus of $F=f^{-1}(p)$, where the pair (f, p) ranges over all Seifert pairs.

The genus of M is so related to the degree of the Alexander polynomial $A(t)$ of M. In fact, by Lemma 2.6 (III), we obtain:

(3.2.2) *genus*$(M) \geq$ *degree*$(A(t))/2$ *if* $M \in \mathcal{C}(S^1 \times S^2)$ *or* $\mathcal{C}(S^1 \times B^2)$ *or* $\mathcal{C}(S^1 \times_\tau S^2)$, *genus*$(M) \geq \{$*degree*$(A(t))-(m-1)\}/2$ *if* $M \in \mathcal{C}(S^1 \times_\tau B^2)$ *and* $H_1(M, \partial M; Z)=Z_m (m>0)$.

If M is fibered, then the inequality is replaced by the equality.

3.3. Finding a standard type

$S^1 \times S^2$, $S^1 \times B^2$, $S^1 \times_\tau S^2$ and $S^1 \times_\tau B^2$ are called the *standard types* of $\mathcal{C}(S^1 \times S^2)$, $\mathcal{C}(S^1 \times B^2)$, $\mathcal{C}(S^1 \times_\tau S^2)$ and $\mathcal{C}(S^1 \times_\tau B^2)$, respectively. Let $\mathcal{C}$ be any one of the four classes and M_0 be the standard type of $\mathcal{C}$.

Theorem 3.3.1. (1) *In case* $\partial M \approx S^1 \times_\tau S^1$, *then assume* $H_1(M, \partial M; Z)=0$. *Then genus*$(M)=0$ *implies that* M *is PL homeomorphic to* $M_0 \# \bar{S}^3$, *where* $\bar{S}^3$ *is a homology sphere.*
(2) *If* $\pi_1(M)=\pi$, *then* M *is PL homeomorphic to* $M_0 \# \tilde{S}^3$, *where* $\tilde{S}^3$ *is a homotopy sphere.*

The proof of (1) is not difficult. For (2), see [4].

3.4. The Alexander polynomials of groups

For a finitely presented group G with $H_1(G; Z)=Z$, we can define the Alexander polynomial $A(t)$ of G (See Magnus-Karras-Solitar [7, p 157].)*). $A(t)$ is the invariant of G in the sense that if $A_1(t)$ and $A_2(t)$ are arbitrary Alexander polynomials of G then $A_1(t) \doteq A_2(t^\varepsilon)$ for $\varepsilon=1$ or -1. $H_1(G; Z)=Z$ implies $|A(1)|=1$. However, in general, any reciprocal property does not hold. Actually it is not difficult to obtain that any integral polynomial $f(t)$ with $|f(1)|=1$ can be realized as the Alexander polynomial of a finitely presented group. More strongly, $f(t)$ can be realized as the Alexander polynomial of a 4-dimensional homology orientable handle group i.e. $\pi_1(M)$ for a compact 4-manifold M having the homology of $S^1 \times S^3$: $H_*(M; Z) \approx H_*(S^1 \times S^3; Z)$**).

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*) $A(t)$ is in fact defined as the 1st invariant factor in [7]. This can be also defined from a relation matrix of a Λ-module $H_1(\tilde{K}; Z)$ for any finite complex K with $\pi_1(K)=G$, as just in Definition 1.3, since $H_1(\tilde{K}; Z)$ is identified with the abelianized group of the commutator subgroup of G. ($\tilde{K}$ is the infinite cyclic covering space of K.) In this case, (I), (II) and (IV) of Lemma 2.6 taken $\tilde{K}$ instead of $\tilde{M}$ also hold. In particular, $A(t)$ can derive from Fox free calculus [2] of G.

**) D.W. Sumners [12] showed the existence of a locally flat 2-knot with knot group presentation $(\alpha, \beta \mid \alpha^{a_0}\beta\alpha^{a_1}\cdots\beta\alpha^{a_m}\beta^{-m})$ whose Alexander polynomial is $f(t)=a_0+a_1t+\cdots+a_mt^m$. Hence to see this assertion, it suffices to attach $S^1 \times B^3$ to the exterior of this 2-knot so as to obtain a homology handle.

Let for example $f(t)=t^3-2t^2+3t-3$. Since $f(1)=-1$, there is a 4-dimensional homology orientable handle group with Alexander polynomial $f(t)$. On the other hand, Theorem 1.4 says that this polynomial is no Alexander polynomial of a compact 3-manifold group with $H_1=Z$.

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