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ON TIGHT 4-DESIGNS

To the memory to Otto Grün*

NOBORU ITO**

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1. Introduction

Let v, k and λ be positive integers with $v>k$. Let X be a v-set and *Jl* a family of k-subsets of X. *(X, Jl)* is called a 4-(v, k, λ) design (or simply a 4-design) if for each 4-subset T of X there exist precisely λ elements of *Jl* containing T. By a theorem of Fisher-Petrenjuk [2] the number of elements in $\mathcal{A}$ is not less than $\frac{1}{2}v(v-1)$. If it is equal to $\frac{1}{2}v(v-1)$, *(X, Jl)* is called tight.

If $v\geqq 6$ and if *Jl* is the family of all $(v-2)$-subsets of X, *(X, Jï)* is a tight 4-design. Such tight 4-designs are called trivial.

Let *(X, Jl)* be a 4-design. If $v-k\geqq 4$ and if $\mathcal{A}c$ is the family of $(v-k)$-subsets of X each of which is a complement of an element of $\mathcal{A}$ in X, *(X, Jlc)* is a 4-design. *(X, Jl)* and *(X, Jlc)* are called complementary with each other. Furthermore if *(X, Jl)* is tight, *(X, Jlc)* is also tight.

There exist only two known non-trivial tight 4-designs *(X, Jl)* they are a 4-(27, 7, 1) design and a 4-(23, 16, 52) design. They are complementary with each other. We call these designs Witt tight designs, because they are found by Witt [5], [6].

Now the purpose of this paper is to prove the following theorem.

Theorem. *Let (X, Jl) be a non-trivial tight* 4-(v, s, λ) *design. Then (X, Jl) is a Witt tight design.*

Our proof relies on the following theorem of Wilson and Ray-Chaudhuri [4]: Let *(X,Jl)* be a tight 4-(v, k, λ) design. Then a non-negative integer μ is called an intersection number of $(X,\mathcal{A})$, if there exist two distinct elements A and B of*Jl* such that $|A\cap B|=\mu$. There exist precisely two intersection numbers, say, μ_1 and μ_2, where $\mu_2>\mu_1$ μ_1 and μ_2 are the roots of the polynomial

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$$t^2-\left\{\frac{2(k-1)(k-2)}{v-3}-1\right\}t-\left(\frac{4\lambda}{k-3}\right)$$

In particular,

$$(1) \qquad \frac{2(k-1)(k-2)}{v-3}=x=\mu_2+\mu_1-1$$

and

$$(2) \qquad \frac{4\lambda}{k-3}=\mu_2\mu_1-2\lambda=y$$

are positive integers.

Furthermore, since $(X, \mathcal{A})$ is tight, we obtain that

$$(3) \qquad 2\lambda(v-2)(v-3)=k(k-1)(k-2)(k-3)$$

2. Eigen-values of adjacency matrices

Let x_i $(i=1,\cdots,v)$ and A_j $(j=1,\cdots,\frac{1}{2}v(v-1))$ be the elements of X and $\mathcal{A}$ respectively. Let I be the incidence matrix of $(X, \mathcal{A})$. So I is the matrix of size $(v, \frac{1}{2}v(v-1))$ whose (i, j)-component is either 1 or 0 according as x_i belongs to A_j or not. Let N_k be the adjacency matrix of $(X, \mathcal{A})$ $(k=1, 2)$. So the (i, j)-component of N_k is 1 or 0 according as $|A_i\cap A_j|=\mu_k$ or not $(k=1,2)$. Then we obtain that

$$(4) \qquad J=E+N_1+N_2\,,$$

$$(5) \qquad I^tI=kE+\mu_1N_2+\mu_2N_1$$

and

$$(6) \qquad II^t=\tfrac{1}{2}k(v-k)E'+\tfrac{1}{2}k(k-1)J'\,,$$

where E and E' denote the identity matrices of degree $\frac{1}{2}v(v-1)$ and v respectively, and J and J' are the matrices of degree $\frac{1}{2}v(v-1)$ and v with every component 1 respectively. Furthermore, by R. Noda [1], we have that

$$(7) \qquad N_1N_2=N_2N_1\,.$$

(7) is equivalent to the fact that the number of elements A_j of $\mathcal{A}$ such that $|A_i\cap A_j|=\mu_k$ for a given element A_i of $\mathcal{A}$ is independent from the choice of A_i $(k=1, 2)$. For a proof of this fact see [3].

From (6) we see that the eigen-value distribution of II^t consists of $\frac{1}{2}k^2(v-1)$ with multiplicity 1 and $\frac{1}{2}k(v-k)$ with multiplicity $v-1$. In particular, II^t is non-singular. Now there exists an orthogonal matrix U of degree $\frac{1}{2}v(v-1)$ such that IU has the following shape:

$$(8) \qquad IU=(I^*0)\,,$$

where 0 denotes the zero matrix of size $(v, \frac{1}{2}v(v-3))$ and I^* is a non-singular matrix of degree v. From (8) it follows that

$$IUU^tI^t = II^t = I^*I^{*t}.$$

Hence IP and I^*I^{*t} have the same eigen-value distribution. On the other hand, from (8) it also follows that

$$(9) \qquad U^tI^tIU = \begin{pmatrix} I^{*t}I^* & 0 \\ 0 & 0 \end{pmatrix},$$

where 0's denote zero matrices. So from (9) we see that the eigen-value distribution of PI consists of $\frac{1}{2}k^2(v-1)$ with multiplicity 1, $\frac{1}{2}k(v-k)$ with multiplicity $v-1$ and 0 with multiplicity $\frac{1}{2}v(v-3)$.

Since N_1 and N_2 are commuting real symmetric matrices, there exists an orthogonal matrix V such that $U'^tN_1U'$and $U'^tN_2U'$are diagonal matrices:

$$U'^tN_1U' = [\alpha_1, \cdots, \alpha_{\frac{1}{2}v(v-1)}] ,$$

and

$$U'^tN_2U' = [\beta_1, \cdots, \beta_{\frac{1}{2}v(v-1)}] .$$

Then by (4) and (5) we may assume that

$$(10) \qquad 1+\alpha_1+\beta_1 = \tfrac{1}{2}v(v-1),$$

$$(11) \qquad 1+\alpha_i+\beta_i = 0 \quad \text{for} \quad 2 \leqq i \leqq \tfrac{1}{2}v(v-1),$$

$$(12) \qquad k+\mu_1\alpha_1+\mu_2\beta_1 = \tfrac{1}{2}k^2(v-1),$$

$$(13) \qquad k+\mu_1\alpha_i+\mu_2\beta_i = \tfrac{1}{2}k(v-k) \quad \text{for} \quad 2 \leqq i \leqq v,$$

and

$$(14) \qquad k+\mu_1\alpha_j+\mu_2\beta_j = 0 \quad \text{for} \quad v+1 \leqq j \leqq \tfrac{1}{2}v(v-1).$$

Let A be a fixed element of $\mathcal{A}$. Let b_k be the number of elements of $\mathcal{A}$ which intersect with A in μ_k elements of X $(k=1, 2)$. Then we obtain that

$$(15) \qquad \alpha_1 = b_1 \quad \text{and} \quad \beta_1 = b_2 .$$

After M. Kano [1] we set

$$(16) \qquad a = \mu_2 - \mu_1 .$$

Now from (10)—(16) we obtain that

$$(17) \qquad 1+b_1+b_2 = \tfrac{1}{2}v(v-1),$$

$$(18) \qquad k+\mu_1b_1+\mu_2b_2 = \tfrac{1}{2}k^2(v-1),$$

$$(19) \qquad \alpha_i = \{-\tfrac{1}{2}k(v-k)+(k-\mu_2)\}/a,$$

$$\beta_i = \{\tfrac{1}{2}k(v-k)-(k-\mu_1)\}/a,$$

where $2\leqq i\leqq v$, and

$$(20)\qquad \alpha_j=(k-\mu_2)/a\,, \quad \beta_j=-(k-\mu_1)/a\,,$$

where $v+1\leqq j\leqq \frac{1}{2}v(v-1)$.

Since the α_i and the β_i are algebraic integers, the numbers in (19) and (20) are rational integers. Put

$$(21)\qquad e=(k-\mu_2)/a.$$

Now we show that there exist two non-negative rational integers λ_1 and λ_2 such that

$$(22)\qquad N_1^2=b_1E+\lambda_1N_1+\lambda_2N_2\,.$$

In fact, it sufficesto show the existence of two rational numbers λ_1 and λ_2 which satisfy (22). Then it suffices to solve

$$(23)\qquad \begin{cases}\alpha_2^2=b_1+\lambda_1\alpha_2+\lambda_2\beta_2\\ \alpha_{v+1}^2=b_1+\lambda_1\alpha_{v+1}+\lambda_2\beta_{v+1}\end{cases}$$

in λ_1 and λ_2. For, since by (4) the trace of N_1^2 is equal to $\frac{1}{2}b_1v(v-1)$, we obtain (22). In particular, we obtain that

$$(24)\qquad b_1^2=b_1+\lambda_1b_1+\lambda_2b_2\,.$$

By (11) and (23) we see that α_2 and α_{v+1} are the roots of the equation

$$X^2-(\lambda_1-\lambda_2)X-(b_1-\lambda_2)=0\,.$$

So we obtain that

$$(25)\qquad \alpha_2+\alpha_{v+1}=\lambda_1-\lambda_2\quad \alpha_2\alpha_{v+1}=-(b_1-\lambda_2)$$

Since, by (19) and (20), $-\alpha_2>\alpha_{v+1}>0$, by (25) we obtain that

$$(26)\qquad \lambda_2>\lambda_1\,.$$

Since α_2 and α_{v+1} are rational integers,

$$(27)\qquad d=(\lambda_1-\lambda_2)^2+4(b_1-\lambda_2)$$

is a square. Further we obtain that

$$(28)\qquad \alpha_2=\tfrac{1}{2}(\lambda_1-\lambda_2-\sqrt{d})\quad \alpha_{v+1}=\tfrac{1}{2}(\lambda_1-\lambda_2+\sqrt{d})\,.$$

From (20), (21) and (28) we obtain that

$$(29)\qquad \sqrt{d}=\lambda_2-\lambda_1+2e\,.$$

By (1), (16) and (21) we obtain that

$$(30) \qquad 2k = x+1+a(2e+1) .$$

Put

$$(31) \qquad \nu = \lambda_2 - \lambda_1 .$$

Eliminate b_2 from (10), (15) and (24). Then we obtain that

$$(32) \qquad b_1^2+(\nu-1)b_1 = \tfrac{1}{2}\lambda_2(v+1)(v-2) .$$

From (27), (29) and (31) we obtain that

$$(33) \qquad b_1 = e\nu+e^2+\lambda_2 .$$

Since the trace of N_1 is equal to 0, from (15), (19), (20), (21) and (28) it follows that

$$(34) \qquad b_1 = (v-1)\nu-\tfrac{1}{2}(v^2-5v+2)e$$

Eliminate b_2 from (10), (12) and (15). In the resulting equation express μ_1 and μ_2 in terms of a, e and k using (16) and (21). Then we obtain that

$$(35) \qquad ab_1 = \tfrac{1}{2}(v-1)k(v-k)-\tfrac{1}{2}ea(v+1)(v-2)$$

Eliminate λ_2 and b_1 from (32), (33) and (34). Then after canceling $\nu(\nu-1)$ we obtain that

$$(36) \qquad \nu^2-\{(v-5)e+\tfrac{1}{2}v-1)\}\,\nu \\ + ti(v^2\text{-}9v+22)e+\pm (v^2-5v+2)\}e = 0 .$$

From (36) we obtain that

$$(37) \qquad \{2\nu-ev-e^2+4e\}\{(e+1)v-2\nu-e^2-6e-1\} \\ = (e+2)(e+1)e(e-1) = (e^2+e-2)(e^2+e).$$

3. Cases with small e

It is convenient to eliminate cases with small e first. For this purpose (37) can play a rather useful role. First we show that both terms of the left-hand side of (37) are positive for $e>l$. Deny this. Then the sum of the negatives of both terms of the left-hand side of (37) is smaller than that of the right-hand side of (37). Since

$$(e^2+e)-(e^2+e-2)= 2 ,$$

we have the following three cases:

(i) $ev+e^2-2\nu-4e = 2\nu+e^2+6e+1-(e+1)v+1,$

(ii) $ev+e^2-2\nu-4e = 2\nu+e^2+6e+1-(e+1)v,$

and

(iii) $ev+e^2-2\nu-4e = 2\nu+e^2+6e+1-(e+1)v-1\ .$

Case (i): We have that $4\nu=(2e+1)v-10e-2$. So from (37) we obtain that

(38) $v^2-2(2e^2+2e+1)v+12(e^2+e)=0\ .$

Put $c=e^2+e$. Then $c\geqq 6$, because $e\geqq 2$. From (38) it follows that

$$\{v-(2c+1)\}^2= 4c^2-8c+1\,.$$

So put $4c^2-8c+1=f^2$, where f is a positive integer. Then we obtain that

(39) $2c-f=\dfrac{8c-1}{2c+f}\,.$

Hence the right-hand side of (39) does not exceed 3. If it equals 1, it leads to the absurdity that $6c=1$. If it equals 2, it leads to the absurdity that 3=0. If it equals 3, then we get that $c=2$, which is against the assumption.

Case (ii) and (iii) can be handled with in the similar manner as in case (i).

Case where $e=1$. We notice that this is the case for all trivial designs. From (37) we obtain that

(40) $(2\nu-v+3)(v-\nu-4)=0\ .$

If $\nu=\frac{1}{2}(v-3)$, then from (34) and (33) we obtain that $b_1=\frac{1}{2}(v+1)$, and that $\lambda_2=1$. Then $\lambda_1=0$. From (31) $\nu=1$. Hence $v=5$, which is against the assumption. Thus we obtain that $\nu=v-4$. Then from (34) and (33) we obtain that

(41) $b_1=\frac{1}{2}(v-2)(v-3)\,,$

and that

(42) $\lambda_2=\frac{1}{2}(v-3)(v-4)\,.$

From (17) $b_2=2(v-2)$. Thus from (18) we obtain that

(43) $\mu_1(v-2)= k(k-2)\,.$

From (43) and (3) it follows that

(44) $2\lambda(v-3)=\mu_1(k-1)(k-3)\,.$

From (44) and (1) we obtain that

(45) $\qquad 4\lambda(k-2)=\mu_1 x(k-3)\,.$

Using (21), from (45) and (2) we obtain that

$$\mu_2+1 = a(\mu_2+a)\,,$$

which implies that $a=1$. Now (35) implies that $k=v-2$. Thus $(X, \mathcal{A})$ is trivial.

Case where $e=2$. We notice that this is the case for Witt tight designs. From (37) we obtain that

(46) $\qquad (\nu-v+2)(3v-2\nu-17) = 12\,.$

We let the first factor of the left-hand side of (46) run all positive divisors of 12 from 1 to 12. It gives us the values of ν and v in each case. Then we can calculate the values of remaining parameters. Anyway we have the following six cases: $v=27$; $v=23$, $\nu=23$, $b_1=90$, $b_2=162$; $v=23$, $\nu=24$, $b_1=112$, $b_2=140$; $v=24$; $v=27$; $v=38$.

We may assume that $k\leqq\frac{1}{2}v$. Then using (1) and (3) we can eliminate all but the second and third cases with $k=7$. Then $a=2$, $\mu_2=3$ and $\mu_1=1$. A Witt tight design yields the third case and the uniqueness is known. The second case is eliminated by (18).

Cases where $3\leqq e\leqq 6$. From now on we are expecting no designs. Furthermore the checking procedure is just the same as in the above case. The following two lemmas are sufficient to take care of all cases where $3\leqq e\leqq 6$.

Lemma 1. *k cannot be a prime for $e>2$.*

Proof. From (1), (2) and (3) we obtain that

(47) $\qquad y(v-2)=kx\,.$

(i) The case where k divides $v-2$. Put $v-2=ck$, where c is a positive integer. Since $(X, \mathcal{A})$ is non-trivial, $c\geqq 2$. From (1) it follows that

(48) $\qquad 2(k-1)(k-2)=x(ck-1)\,.$

Thus $x=dk-4$, where d is a positive integer. So from (48) we obtain that

$$\left(c-\frac{2}{d}\right)k=\frac{4c}{d}+1-\frac{6}{d}\,.$$

If $d\geqq 4$, then $(c-\frac{1}{2})k<c+1$. Since $k\geqq 4$, this is a contradiction. If $d=3$ or 2, we obtain a similar contradiction. If $d=1$, then

$$k = 4+\frac{3}{c-2}.$$

So $c=3$ or 5. If $c=3$, then $k=7$ and $v=23$. From (17) and (18) it follows that $\mu_1=2$. Since $e\geqq 3$, from (21) it follows that $a=1$. But the case $a=1$ can occur only for trivial designs. In fact, from (1), (2), (3) and (16) it follows that

$$(49) \qquad a^2 = (\mu_2-\mu_1)^2 = (\mu_2+\mu_1)^2-4\mu_2\mu_1 = \frac{4(k-1)(k-2)(v-k-1)(v-k-2)}{(v-2)(v-3)^2}+1$$

(49) implies that

$$(50) \qquad (a^2-1)(v-2)(v-3)^2 = 4(k-1)(k-2)(v-k-1)(v-k-2)$$

If $a=1$, (50) implies that $k=v-2$. Then we have the trivial designs. If $c=5$, then $k=5$ and $v=27$. But then (3) shows that λ cannot be an integer.

(ii) The case where k and $v-2$ are relatively prime. By (47) $v-2$ divides x. So put $x=z(v-2)$. From (1) it follows that

$$2(k-1)(k-2) = z(v-2)(v-3).$$

Since $v-2>k$, we obtain that $z=1$. Thus

$$(51) \qquad x = v-2 \quad \text{and} \quad y = k.$$

Now the equation

$$2(k-1)(k-2) = (v-2)(v-3)$$

can be rewritten as a Pellian equation

$$(2v-5)^2-2(2k-3)^2 = -1 .$$

The solutions of the Pellian equation

$$Y^2-2Z^2 = -1$$

are recursively given by

$$(52) \qquad (Y_1, Z_1) = (1, 1)$$

and

$$(53) \qquad \begin{cases} Y_{i+1} = 3Y_i+4Z_i \\ Z_{i+1} = 2Y_i+3Z_i \end{cases} \qquad (i = 1, 2, \cdots).$$

So there exists an integer m such that

$$(54) \qquad (Y_m, Z_m) = (2v-5, 2k-3).$$

By (2), (51) and (54) we obtain that

(55) $16\lambda = 4k^2-12k= Z_m^2-9.$

(55) implies that

(56) $Z_m = 3, 5, -5 \text{ or } -3 \pmod{16}.$

On the other hand, from (52) and (53) we obtain that

(57) $$(Y_l, Z_l) \pmod{16} = \begin{cases} (1, 1) & \text{for } l \equiv 1 \pmod 8, \\ (7, 5) & \text{for } l \equiv 2 \pmod 8, \\ (-7, -3) & \text{for } l = 3 \pmod 8, \\ (-1, -7) & \text{for } l = 4 \pmod 8, \\ (1, -7) & \text{for } l = 5 \pmod 8, \\ (7, -3) & \text{for } l \equiv 6 \pmod 8, \\ (-7, 5) & \text{for } l \equiv 7 \pmod 8, \\ (-1, 1) & \text{for } l \equiv 0 \pmod 8. \end{cases}$$

From (56) and (57) it follows that $Z_m=2k-3=5$ or $-3 \pmod{16}$. This implies that k is even.

Lemma 2. *The greatest common divisor of $v-2$ and $k-3$ divides* 3.

Proof. Let c denote the number of blocks containing three distinct points of X. Then since $(X, \mathcal{A})$ is tight, we obtain that

$$2c(v-2) = k(k-1)(k-2).$$

Thus the greatest common divisor of $v-2$ and $k-3$ divides that of $2k$ and $k-3$. By (18) if v is even, then k is even.

REMARK. Perhaps the case where $e=3$, $v=47$ and $k=12$ is slightly beyond the powers of Lemmas 1 and 2. In that case, from (1) it follows that $\mu_2+\mu_1=6$. Hence $2\mu_2-a=6$. By (21) $12-3a=\mu_2$. Hence $7a=8$. This is absurd.

From now on we may assume that $e\geqq 7$.

4. Case where $a=2$

This case has been already treated by M. Kano in a similar way [1]. We may assume that $k\leqq \frac{1}{2}v$. Then from (16) and (50) it follows that

(58) $$\frac{2(a^2-1)(v-3)}{v-4} > \frac{2(k-1)(k-2)}{v-3} = x = a-1+2\mu_1.$$

Since $v \geqq k+3=ae+a+\mu_1+3 \geqq 20$, if a=2, then (58) implies that $43>16\mu_1$. Thus $\mu_1 \leqq 2$. Then (2) implies the contradiction that $k \leqq 10$.

5. e-adic expansions of ν and related parameters

From (34) it follows that

$$2\nu = e(v-4)+\frac{2b_1-2e}{v-1}.$$

So further by (33) $\frac{2b_1-{}^{-}_{-}}{\nu-1}$ is a positive rational integer. Put

$$(59) \qquad C=\frac{2b_1-2e}{v-1}.$$

Then

$$(60) \qquad 2\nu = e(v-4)+C.$$

Substituting (60) in (37) we obtain that

$$(61) \qquad v = C+(e+1)^2+\frac{(e+2)(e+1)e(e-1)}{C-e^2}.$$

The numerator of the third term of the right-hand side of (61) equals

$$e^4+e^3+(e-2)e^2+(e-2)e.$$

Now express C in the e-adic form

$$(62) \qquad C=c_4e^4+c_3e^3+c_2e^2+c_1e+c_0,$$

where $0 \leqq c_i < e$ $(i=0, 1, 2, 3, 4)$.

Since the first factor of the left-hand side of (37) is positive, by (33) we can easily see that $C>e^2$. So the third term of the right-hand side of (61) is a positive integer. We put

$$(63) \qquad e^4+e^3+(e-2)e^2+(e-2)e=(C-e^2)D,$$

where

$$D = \nu\text{-}C\text{-}(e+\backslash\gamma = d_4e^4+d_3e^3+d_2e^2+d_1e+d_0$$

with $0 \leqq d_i < e$ $(i=0, 1, 2, 3, 4)$.

First we show that $c_4=0$ in (62). In fact, clearly we have that $c_4 \leqq 1$. If $c_4=1$, then $D=1$. So from (61) and (63) we obtain that

$$(64) \qquad \nu = e^4+2e^3+e^2+2.$$

Then from (59) it follows that

(65) $$2b_1 = e^8+4e^7+4e^6+(e-1)e^5+(e-3)e^4.$$

Now from (35) it follows that

(66) $$(v-1)k^2-v(v-1)k+a\{2b_1+e(v+1)(v-2)\}= 0 .$$

Further from (1) and (30) it follows that

(67) $$2k^2-2vk+(v+1)+a(2e+1)(v-3)= 0 .$$

So from (66) and (67) we obtain that

$$2a\{2b_1+e(v+1)(v-2)\}=(v+1)(v-1)+a(2e+1)(v-3)(v-1),$$

which implies that

(68) $$a\{4b_1+2e(3v-5)-(v-3)(v-1)\} = (v+1)(v-1) .$$

From (64), (65) and (68) we obtain that

$$a(e^4+2e^3-e^2+2e+1)= e^4+2e^3+e^2+3 .$$

So $a=3$ (mod e). This implies that $a=3$ and that $2e^4+4e^3-4e^2+6e=0$. This is a contradiction. Hence we obtain that $c_4=0$.

Secondly we show that $c_3= 0$ in (62). If $c_3\geqq 2$, then we obtain that $D<e$. So from (61) we obtain that

(69) $$v = c_3e^3+(c_2+1)e^2+(c_1+2)e+c_0+1+D$$

Now from (59) and (68) we obtain that

(70) $$a(2C+6e-v+3)= v+1 .$$

Then from (69) and (70) it follows that

$$2C+6e-v+3 = c_3e^3+(c_2-1)e^2+(c_1+4)e+c_0+2-D.$$

Since $a\geqq 3$, this implies that

$$\begin{aligned}&2c_3e^3+2(c_2-1)e^2+2(c_1+4)e+2c_0+4-2D-1\\ &\leqq e_3e^3+(c_2+1)e^2+(c_1+2)e+c_0+1+D ,\end{aligned}$$

which is clearly a contradiction. So we obtain that $c_3\leqq 1$. Now assume that $c_3=1$. If $c_2\geqq 2$, then $D\leqq e$. So from (61) and (70) we obtain that

$$v = e^3+(c_2+1)e^2+(c_1+2)e+(c_0+1)+D$$

and that

$$2C+6e-v+3=\frac{v+1}{a}$$
$$=e^3+(c_2-1)e^2+(c_1+4)e+c_0+2-D\,.$$

Since $a \geqq 3$, this implies that

$$2e^3+2(c_2-1)e^2+2(c_1+4)e+2(c_0+2)-2D-1$$
$$\leqq e^3+(c_2+1)e^2+(c_1+2)e+(c_0+1)+D\,,$$

which is clearly a contradiction. If $c_2=1$, then $D \leqq e+1$. This leads to a contradiction as above. If $c_2=0$, then $D \leqq e+3$. Since $a \geqq 3$, we get a contradiction as above. Hence we obtain that $c_3=0$.

Thirdly we show that $c_2=2$ in (62). If $c_2 \geqq 4$, then $D<e^2$. From (61) and (70) we obtain that

$$v = (c_2+1)e^2+(c_1+2)e+(c_0+1)+D$$

and that

$$2C+6e-v+3=\frac{v+1}{a}$$
$$=(c_2-1)e^2+(c_1+4)e+(c_0+2)-D\,.$$

Since $a \geqq 3$, this implies that

$$3(c_2-1)e^2+3(c_1+4)e+3(c_0+2)-3D-1$$
$$\leqq (c_2+1)e^2+(c_1+2)e+c_0+1+D\,,$$

which is clearly a contradiction. Hence we obtain that $c_2 \leqq 3$. If $c_2=1$, then $C-e^2=c_1e+c_0$. From (61) we obtain that

$$v = 2e^2+(c_1+2)e+c_0+1+D$$

and that

$$2C+6e+3-v=(c_1+4)e+c_0+2-D. \tag{71}$$

Since $c_1e+c_0 \leqq e^2-1$, from (63) we obtain that

$$D \geqq e^2+2e\,.$$

Since (71) is positive, we obtain that $c_1 \geqq e-3$. If $c_1=e-1$, then

$$v=3e^2+e+c_0+1+D \tag{72}$$

and

$$2C+6e+3=2e^2+4e+2c_0+3\,. \tag{73}$$

Since (72) is smaller than (73), this is a contradiction. If $c_1=e-2$, then

$$v = 3e^2+c_0+1+D$$

and

$$2C+6e+3 = 2e^2+2e+2c_0+3 .$$

As above this is a contradiction. If $c_1=e-3$, then

$$v = 2e^2+(e-1)e+c_0+1+D$$

and

$$2C+6e+3 = 2e^2+2c_0+3 .$$

As above this is a contradiction. Hence we obtain that $c_2=3$ or $c_2=2$. If $c_2=3$, then from (63) and (61) we obtain the following equations:

$$\begin{aligned} &(2e^2+c_1e+c_0)(d_2e^2+d_1e+d_0) \\ &= e^4+e^3+(e-2)e^2+(e-2)e; \\ &v = (4+d_2)e^2+(c_1+2+d_1)e+c_0+1+d_0; \\ &2C+6e+3-v \\ &= (2-d_2)e^2+(c_1+4-d_1)e+c_0+2-d_0 . \end{aligned} \tag{74}$$

Further from (70) we obtain that

$$\begin{aligned} &a\{(2-d_2)e^2+(_1+4-d_1)e+c_0+2-d_0\} \\ &= (4+d_2)e^2+(c_1+2+d_1)e+c_0+2+d_0 . \end{aligned} \tag{75}$$

Now from (74) we obtain that $d_2=0$. If $a\geqq 5$, from (75) we obtain that

$$\begin{aligned} &10e^2+5(c_1+4-d_1)e+5(c_0+2-d_0) \\ &\leqq 4e^2+(c_1+2+d_1)e+c_0+2+d_0 , \end{aligned}$$

which implies that

$$6e^2+(4c_1+18)e+(4c_0+8)\leqq 6d_1e+6d_0 \leqq 6e^2-6 .$$

This is a contradiction. If $a=4$, as above we obtain that

$$4e^2+(3c_1+14)e+(3c_0+6) = 5d_1e+5d_0. \tag{76}$$

On the other hand, from (74) we obtain that

$$\begin{aligned} &2d_1e^3+(2d_0+c_1d_1)e^2+(c_1d_0+c_0d_1)e+c_0d_0 \\ &= e^4+e^3+(e-2)e^2+(e-2)e , \end{aligned}$$

which implies that $2d_1\leqq e+1$. But then we obtain that

$$\begin{aligned} & 5d_1e+5d_0 \\ \leqq\ & 5d_1e+5(e-1) \\ =\ & 5(d_1+1)e-5 \\ \leqq\ & 5\{\tfrac{1}{2}(e+1)+1\}e-5 \\ =\ & 3e^2+\tfrac{1}{2}(15-e)e-5\,. \end{aligned}$$

Since $e\geqq 7$, this contradicts (76). If $a=3$, as above we obtain the following equation:

$$2e^2+(2c_1+10)e+2c_0+5=4d_1e+4d_0\,,$$

which is clearly a contradiction. Hence we obtain that $c_2=2$.

Now we have the following equations:

$$(77)\qquad \begin{aligned} & (e^2+c_1e+c_0)(d_2e^2+d_1e+d_0) \\ =\ & e^4+e^3+(e-2)e^2+(e-2)e; \end{aligned}$$

$$(78)\qquad v=(3+d_2)e^2+(c_1+2+d_1)e+c_0+1+d_0;$$

and

$$(79)\qquad 2C+6e+3-v=(1-d_2)e^2+(c_1+4-d_1)e+c_0+2-d_0\,.$$

Hence we obtain that either $d_2=1$ or 0.

There seems to be a rather big difference between the cases $d_2=1$ and $d_2=0$.

First we assume that $d_2=1$. Then from (77) it follows that

$$(80)\qquad \begin{aligned} & (c_1+d_1)e^3+(c_0+c_1d_1+d_0)e^2+(c_0d_1+c_1d_0)e+c_0d_0 \\ =\ & e^3+(e-2)e^2+(e-2)e. \end{aligned}$$

Hence we obtain that $c_1+d_1\leqq 1$.

Case (i) where $c_1=d_1=0$. If $c_0=d_0=e-1$, then $e=1$. Hence either $c_0\leqq e-2$ or $d_0\leqq e-2$. So we obtain that

$$(2e-3)e^2+(e-1)(e-2)\geqq e^3+(e-2)e^2+(e-2)e\,,$$

which is a contradiction.

Case (ii) where $c_1=0$ and $d_1=1$. From (80) it follows that

$$(81)\qquad (c_0+d_0)e^2+c_0e+c_0d_0=(e-2)e^2+(e-2)e\,.$$

Put $c_0+d_0=e-l$ and $c_0d_0=em$. Then $l\geqq 2$ and $e-3\geqq m\geqq 0$. Then from (81) we obtain that

$$(e-l)e+c_0+m=(e-2)e+e-2.$$

Now we have that $2e>c_0+m+2=(l-1)e\geqq e$. Hence $l=2$, $c_0+m+2=e$ and

$c_0+d_0+2=e$. So $m=d_0$. Thus $d_0=0$ and $c_0=e-2$. Then from (78) and (70) we obtain that

(82) $\quad v=4e^2+4e-1\,,$

and that

(83) $\quad a=e+l.$

Now from (67), (82) and (83) we obtain that

$$k^2-(4e^2+4e-1)k+2(e+1)^3(2e-1)=0\ .$$

Since

$$(4e^2+4e-1)^2-8(e+1)^3(2e-1)=\ -8e^3-16e^2+9$$

is negative, this is a contradiction.

Case (iii) where $c_1=1$ and $d_1=0$. From (80) it follows that

(84) $\quad (c_0+d_0)e^2+d_0e+c_0d_0=(e-2)e^2+(e-2)e.$

Put $c_0+d_0=e-l$ and $c_0d_0=me$. Then $l\geqq 2$ and $e-3\geqq m\geqq 0$. Then from (84) it follows that

$$(e-l)e^2+d_0e+me=\ (e-2)e^2+(e-2)e.$$

Thus we obtain that

$$(e-l)e+d_0+m=\ (e-2)e+e-2.$$

Now we have that $2e>d_0+m+2=(l-1)e\geqq e$. Hence $l=2$, $c_0+d_0=e-2$ and $d_0+m=e-2$. So $m=c_0$. Thus $c_0=0$ and $d_0=e-2$. Then from (78) and (70) we obtain that

(85) $\quad v=4e^2+4e-1\,,$

and that

(86) $\quad a=e\ .$

Now from (67), (85) and (86) we obtain that

(87) $\quad k^2-(4e^2+4e-1)k+2e^3(2e+3)=0\ .$

Hence $8e^3+8e^2-8e+1$ is a square. Thus we are confronted with a Diophantine equation

(88) $\quad Y^2=8X^3+8X^2-8X+1\ .$

It is easy to solve (88) with the restriction $X \not\equiv 2 \pmod 3$.

Lemma 3. *The integral solutions of* (88) *with* $X \not\equiv 2 \pmod 3$ *are the following:* (i) $X=0$, $Y=\pm 1$ *and* (ii) $X=1$, $Y=\pm 3$.

Proof. We start with

$$(89) \qquad Y^2 = (2X-1)(4X^2+6X-1).$$

If X is negative, from (89) we obtain that $-6X \geqq 4X^2-1$, which implies that $X \geqq -1$. So we may assume that X is non-negative. For $X=0$ and 1 we obtain the solutions (i) and (ii) respectively. Hence we may assume that $X \geqq 3$. Since

$$4X^2+6X-1 = (2X-1)(2X+4)+3,$$

$2X-1$ and $4X^2+6X-1$ are relatively prime by assumption. Hence we may put

$$(90) \qquad 2X-1 = R^2 \quad \text{and} \quad 4X^2+6X-1 = S^2,$$

where R and S are positive integers. Then $S>2X$ and $S<2X+2$. Hence $S=2X+1$, which implies that $X=1$ against the assumption.

For $X \leqq 14$ we obtain the further solutions (iii) $X=-1$, $Y=\pm 3$;(iv) $X=2$, $Y=\pm 9$and (v) $X=14$, $Y=\pm 153$.

REMARK 1. We owe to Dr. Jeffrey Leon the following fact that the solution (v) is the only solution of (88) in the interval $7 \leqq X \leqq 25000$.

If $e=14$, then $v=839$. Further we may assume that $k=343$. But these parameters do not satisfy (3). Hence we may assume that $e \geqq 15$.

We may put

$$(91) \qquad 2e-1 = 3R^2,$$

where R is an integer bigger than 3. Now we show the non-existence of tight 4-designs with parameters (85), (86), (87) and (91).

Assume that $2e-1$ has a prime divisor p bigger than 3. Then from (87) we obtain that

$$(92) \qquad k \equiv 1 \pmod p.$$

Further from (16), (21), (86) and (92) we obtain that

$$\mu_2 \equiv 1-e^2 \pmod p,$$

and that

$$\mu_1 \equiv 1-e-e^2 \pmod p.$$

If $e^2\equiv 1$ *(mod p),* then $4e^2\equiv 4=1$ (mod p), which implies that $p=3$. If $e^2+e\equiv 1$ (mod p), then $4e^2+4e\equiv 4\equiv 3$ *(mod p)*, which is absurd. Hence we obtain that

(93) $\mu_2\mu_1\not\equiv 0\ (\mathrm{mod}\,p)$.

On the other hand, from (2) we obtain that

$$4\lambda = (k-3)(\mu_2\mu_1-2\lambda) = -2(\mu_2\mu_1-2\lambda)\ (\mathrm{mod}\ p)\ ,$$

which implies that

(94) $\mu_2\mu_1 = 0\ (\mathrm{mod}\ p)$.

Obviously (93) and (94) are in contradiction.

Hence we obtain that

$$2e-1 = 3^E\ ,$$

where, by (91), E is an odd integer bigger than 3. Now from (89) and (91) we obtain that

(95) $F^2 = 3^{2E-1}+3^E+2.3^{E-1}+1$

is a square, where F is a positive integer. From (95) we obtain that either

(96) $F\text{-}l = G3^{E-1}\ ,$

or

(97) $F+1 = G3^{E-1}\ ,$

where G is a positive integer. If (96) occurs, then

$$F^2-1 = G^2 3^{2E-2}+2G3^{E-1}\ .$$

Since G is clearly bigger than 1, we obtain that

$$F^2-1 \geqq 4.3^{2E-2}+4.3^{E-1} > 3^{2E-1}+3^{2E-2}\ .$$

Since $E\geqq 5$, this is a contradiction. If (97) occurs, then

(98) $F^2-1 = G^2 3^{2E-2}-2G3^{E-1}$.

If $G\geqq 3$, then

$$G^2 3^{2E-2}-2G3^{E-1} > 2.3^{2E-1}\ ,$$

which contradicts (95). Hence $G=2$. Then (95) and (98) imply that $E=1$. This is against assumption.

REMARK 2. We are informed that Dr. Koichi Yamamoto, meantime, has solved (88) confirming that (i)—(v) are the only solutions. His proof seems to be not entirely elementary.

6. Relation between v and k

From now on we assume that $d_2=0$. By (77), (78), (79) and (70) we can start with the following equations:

(99) $$(e^2+c_1e+c_0)(d_1e+d_0)=e^4+e^3+(e-2)e^2+(e-2)e;$$

(100) $$v=3e^2+(c_1+d_1+2)e+c_0+d_0+1;$$

(101) $$2C+6e+3-v=e^2+(c_1-d_1+4)e+(c_0-d_0+2),$$

where $C=2e^2+c_1e+c_0$;

(102) $$\begin{aligned} &a\{e^2+(c_1-d_1+4)e+c_0-d_0+2\}\\ &=3e^2+(c_1+d_1+2)e+c_0+d_0+2\\ &=v+1\,. \end{aligned}$$

From (99) it follows that

(103) $$\begin{aligned} &d_1e^3+(c_1d_1+d_0)e^2+(c_0d_1+c_1d_0)e+c_0d_0\\ &=e^4+e^3+(e-2)e^2+(e-2)e\,. \end{aligned}$$

First we show that

(104) $$c_1+d_1=e+j \qquad \text{for some } j \text{ with } e-2\geqq j>0\,.$$

To show this put $d_1=e-A$ and $d_0=e-B$. Then $A\geqq 1$ and $B\geqq 1$. From (103) it follows that

$$\begin{aligned} &(c_1-A+1)e^3+(c_0+c_1-B-c_1A)e^2+(c_0-c_1B-c_0A)e-c_0B\\ &=e^3+(e-2)e^2+(e-2)e\,. \end{aligned}$$

This implies that $c_1-A+1>1$. Hence $c_1+d_1>e$.

Secondly we show that

(105) $$d_1<e-1\,.$$

Assume that $d_1=e-1$. Then from (103) it follows that

(106) $$\begin{aligned} &c_1e^3+(c_0+d_0+1)e^2+(c_1d_0+2)e+c_0d_0\\ &=3e^3+c_1e^2+c_0e\,. \end{aligned}$$

Hence we obtain that $c_1\leqq 3$. By (104) $c_1\geqq 2$. If $c_1=3$, then from (106) it follows that

$$(c_0+d_0)e^2+(3d_0+2)e+c_0d_0= 2e^2+c_0e\ .$$

If $c_0\leqq 1$, then we obtain a contradiction that $e=5$. If $c_0=2$, then $d_0=0$. Then from (102) it follows that

$$a(2e+1) = e^2+e+1\ .$$

This implies that $a\equiv 1 \pmod e$. Then $a=1$. This is against assumption. If $c_1=2$, then from (106) it follows that

$$(c_0+d_0)e^2+(2d_0+2)e+c_0d_0= e^3+e^2+c_0e\ . \tag{107}$$

If $c_0=0$, then $d_0=e-1$. Then from (102) and (100) it follows that

$$3a = 2e+1$$

and that

$$v = 4e(e+1)\,.$$

Further from (67) it follows that

$$3k^2-8e(e+1)k+2e(e+1)(2e+1)^2= 0\ .$$

Since

$$64e^2(e+1)^2-24e(e+1)(2e+1)^2 = 8e(e+1)(-4e^2-4e-3)$$

is negative, this is a contradiction. Hence $c_0>0$. If $d_0=0$, then $c_0>e$. This contradiction shows that $c_0d_0>0$. Put $c_0d_0=Ae$ and $2d_0+2+A-c_0=Be$. Then $0<A<e-3$ and $0\leqq B\leqq 2$. Further we have that

$$c_0+d_0+B = e+1\ . \tag{108}$$

If $B=0$, then by (107) and (108) we obtain that

$$e^2-(4d_0+1)e+d_0(d_0-1)= 0\ . \tag{109}$$

Hence we consider the function $f(X)=X^2-(4d_0+1)X+d_0(d_0-1)$. Since $d_0>1$, $f(0)=d_0(d_0-1)>0$ and $f(d_0)=-2d_0^2-2d_0<0$. Further we obtain that $f(4d_0)=d_0^2-5d_0$. If $d_0=5$, then $e=20$ and $c_0=16$. Then from (102) we obtain that $a=11$. From (100) we obtain that $v=1682$. Further from (101) we obtain that $C=856$. Now from (59) and (66) we obtain that

$$k(v\text{-}k) = a(C+ev)\ . \tag{110}$$

With data above (110) implies that

$$k^2-1682k+379456 = 0\ .$$

But then k cannot be integral. If $d_0 \leqq 4$, (109) has no integral solution. Hence we obtain that $d_0 > 5$ and that $e > 4d_0$. But $f(4d_0+1) = d_0(d_0-1) > 0$. This is a contradiction. If $B=1$, we can follow the above argument to get a contradiction. If $B=2$, we obtain that $d_0 = e-1$. This is a contradiction.

Thirdly we observe that

(111) $c_1 \geqq +2 \geqq 3$.

In fact, $c_1 + d_1 = e + j$ with $j > 0$. Hence by (105) $c_1 + e - 2 \geqq e + j$.

Now we show that

(112) $e > 2a$.

Assume not. Then from (102) it follows that

$$\begin{aligned} &e\{e^2+(c_1+4-d_1)e+c_0+2-d_0\} \\ &\leqq 6e^2+(2c_1+4+2d_1)e+2c_0+4+d_0, \end{aligned}$$

which implies that

(113)
$$\begin{aligned} &e^3+c_1e^2+c_0e \\ &\leqq (d_1+2)e^2+(2c_1+2+2d_1+d_0)e+2c_0+4+d_0 \\ &\leqq (d_1+7)e^2+1 . \end{aligned}$$

This implies that

(114) $d_1+7 \geqq e+c_1+1$,

provided $c_0 > 0$. But if $c_0 = 0$, (113) becomes

$$\begin{aligned} &e^3+c_1e^2 \\ &\leqq (d_1+2)e^2+(2c_1+2+2d_1+d_0)e+4+d_0 \\ &\leqq (d_1+7)e^2-2e+3 . \end{aligned}$$

Hence we recover (114). (114) implies that

(115) $6 \geqq A + c_1$,

where $d_1 = e - A$. Since $c_1 \geqq 3$ by (111), this is a contradiction for $A \geqq 4$. If $A=3$, then $c_1 = 3 + j \geqq 4$. Hence (115) shows a contradiction. Then by (105) we obtain that $A=2$. Hence $c_1 = 2 + j$ with $j \leqq 2$. If $j=2$, then (113) becomes

$$\begin{aligned} &e^3+4e+c_0e \\ &\leqq e^3+2e^2+(d_0+6)e+2c_0+4+d_0 \\ &\leqq e^3+3e^2+6e+2c_0+3 . \end{aligned}$$

Thus $e \leqq 6$. This is against assumption. Hence we obtain that $j=1$. Thus $c_1=3$ and $d_1=e-2$. Now (113) becomes

$$\begin{aligned} & e^3+3e^2+c_0e \\ \leqq\ & e^3+2e^2+(4+d_0)e+2c_0+4+d_0 \\ \leqq\ & e^3+2e^2+(7+d_0)e+1 . \end{aligned}$$

This implies that

(116) $\quad d_0+7 \geqq e+c_0 .$

If $c_0=6$, then $d_0=e-1$ by (116). Then by (103) e divides 6. This is against assumption. If $c_0=5$, then $d_0 \geqq e-2$. If $d_0=e-1$, then by (103) e divides 5. This is against assumption. If $d_0=e-2$, then by (103) $e=10$. Hence $d_1=8$, $c_1=3$, $d_0=8$ and $c_0=5$. Then from (102) it follows that $a=5$. From (100) and (101) we obtain that ɀ $=444$ and $C=235$. Then from (110) we obtain that

$$k^2-444k+23375=0 .$$

But then k cannot be integral. If $c_0=4$, then $d_0 \geqq e-3$. If $d_0=e-1$, then by (103) e divides 4. This is against assumption. If $d_0=e-2$, then by (103) $e=8$. Then $c_1=3$, $d_1=6$, $c_0=4$ and $d_0=6$. Then from (102) it follows that $a=4$. From (100) and (101) we obtain that $v=291$ and $C=156$. Then from (110) we obtain that

$$k^2 - 291k+9936=0 .$$

But then k cannot be integral. If $d_0=e-3$, then by (103) $e=12$. Thus $c_1=3$, $d_1=10$, $c_0=4$ and $d_0=9$. Then from (102) it follows that $105a=627$. This is a contradiction. If $c_0=3$, then $d_0 \geqq e-4$. If $d_0 \geqq e-2$, then by (103) $e \leqq 6$. This is against assumption. If $d_0=e-3$, then by (103) $e=9$. Thus $c_1=3$, rfι=7, $c_0=3$ and rfo—6. Then from (102) it follows that $40a=181$. This is a contradiction. If $d_0=e-4$, then by (103) $e=12$. Thus $c_1=3$, $d_1=10$, $c_0=3$ and rfo=8. Then from (102) it follows that $21a=125$. This is a contradiction. If $c_0=2$, then $d_0 \geqq e-5$. If $d_0 \geqq e-3$, then by (103) $e \leqq 6$. This is against assumption. If $d_0=e-4$, then by (103) $e=8$. Thus $c_1=3$, $d_1=6$, $c_0=2$ and $d_0=4$. Then from (102) it follows that $a=4$. From (100) and (101) we obtain that $v=287$ and $C=154$. Then from (110) we obtain that

$$k^2-287k+9800=0 .$$

But then k cannot be integral. If $d_0=e-5$, then by (103) $e=10$. Thus $c_1=3$, $d_1=8$, $c_0=2$ and $d_0=5$. Then from (102) it follows that $89a=439$. This is a contradiction. If $c_0=1$, then $d_0 \geqq e-6$. Then by (103) $e \leqq 6$. This is against assumption. If $c_0=0$, then from (103) we obtain that

$$(d_0-6)e+3d_0= (e-2)e+e-2.$$

This implies that

$$d_0(e+3) = e(e+3)+2(e-1)\ .$$

Hence $e+3$ divides $2(e-1)$. Thus $e+3=2e-2$, and $e=5$. This is against assumption. This completes the proof of (112).

By (100) we may put

$$k = k_2e^2+k_1e+k_0\,,$$

where $e-1\geqq k_i\geqq 0$ for $i=2,\ 1,\ 0$. Then from (110) it follows that

(117) $$a\{3e^3+(c_1+d_1+4)e^2+(c_1+c_0+d_0+1)e+c_0\}$$
$$= (k_2e^2+k_1e+k_0)\{(3-k_2)e^2+(c_1+d_1+2-k_1)e+c_0+d_0+1-k_0\}\ .$$

Now under the assumption that $k\leqq\frac{1}{2}v$ we show that

(118) $k_2=0$.

First of all, from (117) and (112) it follows that

(119) $$6k_2e^4+2\{k_2(c_1+d_1+2)+3k_1\}\,e^3$$
$$+2\{k_2(c_0+d_0+1)+k_1(c_1+d_1+2)+3k_0\}\,e^2$$
$$+2\{k_1(c_0+d_0+1)+k_0(c_1+d_1+2)\}\,e+k_0(c_0+d_0+1)$$
$$<(3+2k_2^2)e^4+\{(c_1+d_1+4)+4k_1k_2\}\,e^3$$
$$+(c_0+d_0+1+4k_0k_2+2k_1^2)e^2+(c_0+4k_0k_1)e+k_0^2\ .$$

Now from (100) it follows that $k_2\leqq 2$. If $k_2=2$, then from (119) we obtain that

$$e^4+\{3(c_1+d_1)+4\}\,e^3+\{3(c_0+d_0+1)+2k_1(c_1+d_1+2)\}\,e^2$$
$$+2\{k_1(c_0+d_0+1)+k_0(c_1+d_1+2)\}\,e+k_0(c_0+d_0+1)$$
$$<2k_1e^3+(2k_0+2k_1^2)e^2+(c_0+4k_1k_0)e+k_0^2$$
$$\leqq 4e^4-7e^2+(c_0+2)e+1\ .$$

By (104) this is a contradiction. If $k_2=1$, then from (119) we obtain that

$$e^4+(e_1+d_1+2k_1)e^3+\{c_0+d_0+1+k_1(c_1+d_1+2)+2k_0\}\,e^2$$
$$+2\{k_1(c_0+d_0+1)+k_0(c_1+d_1+2)\}\,e+k_0(c_0+d_0+1)$$
$$<2k_1^2e^2+(c_0+4k_1k_0)e+k_0^2$$
$$\leqq 2e^4-5e^2+(c_0+2)e+1\ .$$

By (104) this is a contradiction. Thus $k_2=0$.

From (118) it follows that

(120) $4k<v$.

In fact, otherwise, from (100) we obtain that

$$3e^2+(c_1+d_1+2)e+c_0+d_0+1$$
$$\leqq 4k_1e+4k_0\leqq 4e^2-4 \ .$$

By (104) this is a contradiction.

7. Cases with small *a*

Now it is convenient to eliminate cases with small a. From (50) and (120) we obtain that

(121) $8a^2-9a+1>18\mu_1$.

On the other hand, let us consider the complementary design $(X, \mathcal{A}c)$. Let ν_2 and ν_1 be the intersection numbers of $(X, \mathcal{A}c)$, where $\nu_2>\nu_1$. Then easily we obtain that

(122) $v-2k=\nu_2-\mu_2=\nu_1-\mu_1$.

(122) implies that

(123) $\nu_2-\nu_1=\mu_2-\mu_1=a$,

and that

(124) $v-k-\nu_2=k-\mu_2=ea$.

Put $x_c=\nu_2+\nu_1-1$. Then from (123) and (124) it follows that

(125) $x_c=2v-2k-1-a(2e+1)=2v-4k+x$.

Furthermore corresponding to (1) we obtain that

(126) $x_c=\dfrac{2(v-k-1)(v-k-2)}{v-3}$.

From (1), (126) and (50) we obtain that

(127) $(a^2-1)(v-2)=xx_c$.

Then from (30), (125) and (127) we obtain that

(128) $(a^2-1)(v-2)=(2\mu_1-1+a)(2v-4ae-2\mu_1-3a-1)$.

(128) implies that

(129) $4\mu_1 > a^2 - 2a + 1$.

Now we have to eliminate the case where $10 \geqq a \geqq 3$. For each case we can argue exactly in the same way. So we demonstrate only for the case $a=3$. If $a=3$, then from (121) and (129) we obtain that $\mu_1=2$. Then $\mu_2=5$ and $x=6$. Then from (1) and (125) we obtain that

$$v = 6k - 3,$$

and that

$$k^2 - 21k + 50 = 0.$$

But then k cannot be integral. It is easy to proceed by hand up to $a=10$. Hence from now on we may assume that $a \geqq 11$.

8. Completion of the proof

First we scrutinize (103) a little more. By (103) we may put

(130) $c_0 d_0 = Ee$

(131) $c_0 d_1 + c_1 d_0 + E = Fe - 2$;

and

(132) $c_1 d_1 + d_0 + F = Ge - 1$

where E is a non-negative integer and F and G are positive integers. Then from (130)-(132) and (103) it follows that

(133) $d_1 + G = e + 2$.

Further from (133), (105) and (106) it follows that

(134) $c_1 + 2 = j + G \geqq j + 4$.

Secondly we show that

(135) $k_1 > a$.

In fact, by (21) $k_1 \geqq a$. Hence assume that $k_1 = a$. Then $k_0 = \mu_2 < e$. From (47), (100) and (104) we obtain that

$$4ye^2 < y(v-2) = kx = (ae + \mu_2)(2\mu_2 - a - 1) < 2(a+1)e^2,$$

which implies that

(136) $y < \frac{1}{2}(a+1)$.

On the other hand, let d be the greatest common divisor of k and $v-2$. Then from (1) we obtain that

$$x+4=0 \quad (\bmod d)\,.$$

Now by (47) y is divisible by k/d. Hence in order to get a contradiction to (136), it suffices to show that

$$(137) \qquad \frac{k}{x+4} \geqq \tfrac{1}{2}(a+1)\,.$$

Since

$$2k-(a+1)(x+4)= 2a(e-\mu_2)+a^2-2a-3\,,$$

(137) holds.

As a consequence of (135) we obtain that

$$(138) \qquad e<\tfrac{1}{9}a^2\,.$$

In fact, (135) implies that $\mu_2>e$. Hence from (121) it follows that

$$2e<2\mu_2<\frac{8a^2+9a+1}{9}<a^2\,.$$

Thirdly we show that

$$(139) \qquad c_1 \leqq \frac{4e}{a}\,.$$

In fact, first assume that $ac_1>5e$, which implies that

$$a(d_1-c_1-5)<(a-5)e\,.$$

Hence by (102) we obtain that

$$(140) \qquad v+1 \geqq 5e^2.$$

If $c_1 \geqq d_1$, from (102) it follows that $ae^2<6e^2$. This contradiction shows that $c_1<d_1$. Now from (102) and (105) it follows that

$$v+1 \leqq 5e^2-e\,,$$

which contradicts (140). Thus $ac_1 \leqq 5e$. Next assume that $ac_1>4e$. Then by (100) and (102) we obtain that

$$ae^2+4ae+a(c_0+2)<(j+2)e+c_0+d_0+2+ad_1e+ad_0\,.$$

Hence by (105) we obtain that

$$6ae+a(c_0+2)<(j+2)e+c_0+2+a(d_0+1).$$

Since, by (105), (138), and (139)

$$j=c_1-(e-d_1)\leqq(5e/a)-2\leqq 3a-2,$$

we obtain that

$$3ae+a(c_0+2)<c_0+2+a(d_0+1),$$

which is a contradiction.

Now from (30), (125) and (127) we obtain that

(141) $(a^2-1)(v-2)=(2k-1-2ae-a)(2v-2k-1-2ae-a)$

From (141) we obtain that

(142) $4k^2=4a^2e^2+4a^2e+4ae+3a^2+2a-1 \pmod{v}.$

On the other hand, from (67) we obtain that

(143) $2k^2=6ae+3a-1 \pmod{v}.$

From (142) and (143) it follows that

(144) $4a^2e^2+(4a^2-8a)e+3a^2-4a+1=0 \pmod{v}.$

Further from (100) and (104) we obtain that

$$a^2v=4a^2e^2+(j+2)a^2e+(c_0+d_0+1)a^2.$$

Hence together with (144) we may put

(145) $Yv=\{(j-2)a^2+8ae+(c_0+d_0-2)a^2+4a-1,$

where Y is a rational integer. Since $v\equiv-l \pmod{a}$ by (102), from (145) we obtain that $Y\equiv 1 \pmod{a}$, and hence that

(146) $(aZ+1)v=\{(j-2)a^2+8ae+(c_0+d_0-2)a^2+4a-1,$

where Z is a rational integer.

We show that $Z=0$ in (146). First we show that Z is non-negative. In fact, assume that Z is negative. Then $j\leqq 2$. If $j=2$, then $c_0+d_0\leqq 1$. But since $e>2a$ by (112), $8ae-2a^2>0$. This is absurd. Hence $j=1$. Then from (146) we obtain that

$$(a+1)(4e^2+3e+c_0+d_0+1)+8ae+(c_0+d_0)a^2+4a$$
$$\leqq a^2e+2a^2+1.$$

Since $e > 2a$ by (112), this is absurd. Next we show that $Z \leqq 1$. In fact, otherwise, from (146) we obtain that

$$\begin{aligned}&\{(j-2)a^2+8a\}\, e+(c_0+d_0-2)a^2+4a-1\\ \geq\ &(2a+1)\{4e^2+(j+2)e+c_0+d_0+1\}.\end{aligned}$$

Since $e>2a$ by (112), from the above inequality we obtain that $ja>8e$. Hence by (134) we obtain that $ac_1>8e$. This contradicts (139). Now assume that $Z=1$. Then as above we obtain that $ja>4e$. Hence as above we obtain that $ac_1>4e$. This contradicts (139). Thus $Z=0$. Hence we obtain that

(147) $\quad v = \{(j-2)a^2+8a\}e+(c_0+d_0-2)a^2+4a-1.$

From (100), (104) and (147) we obtain that

(148) $$\begin{aligned}&j(a^2-1)e+(c_0+d_0+1)(a^2-1)\\ =\ &4(e-1)e+2(a-1)(a-3)e+(3a-1)(a-1).\end{aligned}$$

From (138) and (148) we obtain that

(149) $$j+\frac{c_0+d_0+1}{e} = \frac{4(e-1)}{a^2-1}+\frac{2(a-3)}{a+1}+\frac{3a-1}{(a+1)e} < 4.$$

By (104), (149) implies that

(150) $1 \leqq j \leqq 3$.

By (148) we may put

(151) $\quad 4(e-1)e = (a-1)(He+I)$,

where H and / are rational integers such that $H>0$ and that $e>I\geqq 0$. Then from (148) we obtain that

(152) $\quad (c_0+d_0+1)(a+1) = (H-8)e+I-4+(2-j)e(a+1)+3(a+1)$.

By (152) we may put

(153) $\quad (H-8)e+I-4 = J(a+1)$,

where J is a rational integer.

From (152) and (153) we obtain that

(154) $\quad c_0+d_0 = J+(2-j)e+2$.

From (100), (104) and (154) we obtain that

(155) $\quad v = 4e^2+4e+J+3$.

Now from (102) it follows that

$$a(c_0-d_0+2)=c_0+d_0+2 \qquad (\text{mod } e).$$

So we may put

(156) $(a-1)(c_0+2)-(a+1)d_0 = Ke,$

where K is a rational integer.

From (102), (133), (134) and (156) we obtain that

(157) $a(2c_1+4-j)+K= 4e+j+2.$

From (102) and (104) it follows that

(158) $v = (2c_1+4-j)ae+ac_0-ad_0+2a-1.$

Hence from (147) and (158) we obtain that

(159) $(2c_1-ja+2a-j-4)e= (a-1)(c_0-2)+(a+1)d_0$

From (156) and (159) we obtain that

(160) $(2c_1-ja+2a-j-4+K)e= 2(a-1)c_0.$

From (157) and (160) we obtain that

(161) $(2e+c_1-c_1a-a-1)e= (a-1)c_0.$

Similarly we obtain that

(162) $(c_1a+c_1+3a-2e-ja-j-3)e= (a+1)d_0-2(a-1).$

From (161) it follows that

(163) $a = \dfrac{C-e}{C-2e^2+e}.$

Further from (154) and (162) it follows that

(164) $J=C-2e^2+\dfrac{ae-4e^2-5e-4}{a+1}.$

Hence from (155), (163) and (164) we obtain that

(165) $v = \dfrac{C^2+2Ce+C-2e^2-2e}{C-e^2}.$

Now solve (70) in C and use (147), (149) and (154) to obtain

(166) $(a-1)C = \{2ae-(a+1)\}e.$

Going back to (1) and (2) once again and using the definition of a and e, we obtain that

(167) $$k(k-1)(v-k)(v-k-1)=a^2e(e+1)(v-2)(v-3).$$

From (50) and (167) we obtain that

(168) $$\frac{4(k-2)(v-k-2)}{k(v-k)}=\frac{(a^2-1)(v-3)}{a^2e(e+1)},$$

Subtracting 4 from both sides of (168) and using (147) and (154), we obtain that

(169) $$\frac{2v-4}{k(v-k)}=\frac{2ae+a-1}{a^2e(e+1)},$$

From (110) and (169) we obtain that

(170) $$(2ae+a-1)C+4ae(e+1)=a(e+1)v.$$

From (165) and (170) we obtain that

(171) $$(ae-1)C^2-(2ae^3+a-ae^2-ae-e^2)C-2ae(e+1)(2e^2-e-1)=0.$$

Substitute C in (171) by (166). Then we obtain that

(172) $$4a^2e^4-(4a^3+8a)e^3+(7a^2+4a-1)e^2+(a^3-7a^2+5a-1)e$$
$$+(3a^3-4a^2+a)=0.$$

(172) allows the following factorization:

$$(e-a)\{4a^2e^3-8ae^2-(a^2-4a+1)e-(3a^2-4a+1)\}=0.$$

Since $e>a$, we obtain that

$$4a^2e^3-8ae^2-(a^2-4a+1)e-(3a^2-4a+1)=0.$$

Since a and e are not so small and hence the first term of the left hand side is too large comparing with other terms, this is a contradiction.

This completes the proof.

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