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$BP_*(BP)$ AND TYPICAL FORMAL GROUPS

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1. Introduction. D. Quillen showed in [6] that the formal group law of complex cobordism is a universal formal group, hence for a commutative ring R there is a natural bijection between ring homomorphisms $MU_*\to R$ and formal groups over R, where MU_* is the coefficient ring of complex cobordism. Similarly, S. Araki [4] has shown that for a fixed prime p, the formal group law of Brown-Peterson cohomology is universal for typical group laws over commutative $Z_{(p)}$-algebras. Thus if R is a commutative $Z_{(p)}$-algebra, there is a natural bijection between ring homomorphisms $BP_*\to R$ and typical formal groups over R, where BP_* is the coefficient ring of Brown-Peterson cohomology.

In this note we shall show that $BP_*(BP)$ represents isomorphisms between typical formal groups over $Z_{(p)}$-algebras. This places $BP_*(BP)$ in a purely algebraic setting, as was done for $MU_*(MU)$in the Appendix to [5]. We show how the structure maps for $BP_*(BP)$ arise in this context, and use our point of view to derive the formulas of J.F. Adams [2, Theorem 16.1] for these structure maps.

All this works as well for $MU_*(MU)$,by omitting mention of *typical* formal groups; this gives a description of $MU_*(MU)$which is somewhat different from the one given in [5]. In the BP-case it is essential to use coordinates for curves over a typical formal group μ which depend on μ. But in the MU- case, it is optional whether one uses "moving coordinates" (as we do here) or "absolute coordinates" as in [5].

The ideas in this note grew out of musings over D. Ravenel's paper [7] on multiplicative operations in $BP^*(BP)$.

2. **Recollections** (Araki [3, §1] and [4]) For the most part we follow Araki's notation. All rings and algebras are to be commutative. By an *isomor-*

Abstract. It is shown that $BP_*(BP)$ represents the functor which assigns to a commutative $Z_{(p)}$- algebra R the set of isomorphisms between typical formal groups over R. The structure maps of the Hopf algebra $BP_*(BP)$ all arise naturally from this point of view, and one can easily derive the formulas of Adams [2, Theorem 16.1] for them.

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phism ϕ: $\mu\to\mu'$ between formal groups we mean a homomorphism satisfying $\phi(T)\equiv T \bmod \deg 2$ (Araki calls this a *strict* isomorphism).

For a formal group μ, let C_μ denote the additive group of *curves* over μ, i.e. power series $\gamma(T)$ with zero constant term and with addition $(\gamma_1+^{\mu}\gamma_2)(T) =\mu(\gamma_1(T),\gamma_2(T))$. The *identity* curve is $\gamma_0(T)=T$. A homomorphism φ: $\mu\to\mu'$ induces a homomorphism $\phi_\#$: $C_\mu\to C_{\mu'}$, by $\phi_\#(\gamma)=\phi\circ\gamma$. Similarly a ring homomorphism /: $R\to R'$ sends μ to a formal group $f_*\mu$ and induces a homomorphism f_*: $C_\mu\to C_{f_*\mu}$ on curves, by applying f to the coefficients of power series over R.

Fix a prime p, and let $Z_{(p)}$ denote the integers localized at p. Let R be a $Z_{(p)}$-algebra and μ a formal group over R. From [4, 2.5] we recall the Frobenius operators f_n on curves; these satisfy $f_n\phi_\#=\phi_\# f_n$ A curve γ over μ is called *typical* if $f_q\gamma=0$ for all primes $q\neq p$. The formal group μ is called *typical* if the identity curve γ_0 over μ is typical. Theorem 3.6 of [4] states that for a typical formal group μ over a $Z_{(p)}$-algebra R, a curve 7 over μ is typical if and only if it has a series expansion in C_μ of the form

$$\gamma(t) = \sum_{k=1}^{\infty}{}^{\mu} c_k T^{p^k}$$

with (uniquely determined) coefficients $c_k\in R$.

From Theorems 4.6 and 5.6 of [4], we see that the formal group μ_{BP} over BP_* of Brown-Peterson cohomology is typical and universal for typical formal groups over $Z_{(p)}$-algebras.

2. Isomorphisms between typical formal groups

Let R be a $Z_{(p)}$-algebra and consider triples (μ, φ, μ') where μ and μ' are typical formal groups over R and φ: $\mu'\to\mu$ is an isomorphism. We write $TI(R)$ for the set of these triples, and $TF(R)$ for the set of typical formal groups over R. We know that

$$TF(R) \cong \operatorname{Hom}(BP_*, R)$$

on the category of $Z_{(p)}$-algebras, and plan to show that

$$TI(R) \cong \operatorname{Hom}(BP_*(BP), R)$$

on this category.

Lemma 1. *Let R be a $Z_{(p)}$-algebra and φ: $\mu'\to\mu$ cm isomorphism ojformal groups over R. Then $\mu!$ is typical if and only if φ is a typical curve over μ.*

Proof. φ induces an isomorphism $\phi_\#$: $C_{\mu'}\to C_\mu$ commuting with the Frobenius operators, and $\phi_\#(\gamma_0)=\phi$; the result is now immediate. QED

Notice that ϕ: $\mu\to\mu$ an isomorphism implies that

$$\mu'(X, Y) = \phi^{-1}(\mu(\phi(X), \phi(Y))$$

or $\mu'=\mu^{\phi}$ in the notation of [4, 2.11]. Thus μ' is determined by μ and φ, and we may view $TI(R)$ as the pairs (μ, ϕ) where μ is a typical formal group over R and φ is a typical curve over μ with $\phi(T)= T$ mod deg 2.

From [2, Theorem 16.1], we know that $BP_*(BP)$ is a polynomial algebra

$$BP_*[t_1, t_2, \ldots] = BP_*\otimes Z_{(p)}[t_1, t_2, \ldots]\,.$$

We agree to put $t_0=1$.

Theorem 1. *There is a natural bijection $TI(R)\cong \mathrm{Hom}(BP_*(BP), R)$ on the category of $Z_{(p)}$-algebras.*

Proof. Let $(\mu, \varphi, \mu^{\phi})\in TI(R)$, so μ is a typical formal group over R and φ is a typical curve over μ of the form

$$\phi(T) = \sum_{k=0}^{\infty} {}^{\mu}c_k T^{p^k}$$

with $c_k\in R$ and $c_0=1$. To μ we can associate a homomorphism

$$f: BP_* \to R$$

with $f_*(\mu_{BP})=\mu$. And then to μ we associate a homomorphism

$$g\colon Z_{(p)}[t_1, t_2, \cdots] \to R$$

with $g(t_k)=c_k$ for all k. Together we obtain a homomorphism

$$f\otimes g\colon\ BP_*(BP) = BP_*\otimes Z_{(p)}[t_i] \to R$$

from which we can recover f and g and so also μ and φ. Since any homomorphism $BP_*\otimes Z_{(p)}[t_i]\to R$ has the form $f\otimes g$, the result is proved. QED

3. The structure maps. We shall now account for the structure maps of the Hopf algebra $BP_*(BP)$ [1, Lecture 3] and the formulas given for them by Adams in [2, Theorem 16.1]. We begin by defining natural maps:

η_L: $TI(R)\to TF(R)$, $(\mu_1, \varphi, \mu_2)\mapsto \mu_1$
η_R: $TI(R)\to TF(R)$, $(\mu_1, \varphi, \mu_2)\mapsto \mu_2$
ε : $TF(R)\to TI(R)$, $\mu\mapsto(\mu, \gamma_0, \mu)$
c : $TI(R)\to TI(R)$, $(\mu_1, \phi, \mu_2)\mapsto(\mu_2, \phi^{-1}, \mu_1)$
ψ : $TI^2(R)\to TI(R)$, where $TI^2(R)$ is defined by the pull-back diagram

$$\begin{array}{ccc} TI^2(R) & \xrightarrow{\pi_2} & TI(R) \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \eta_L} \\ TI(R) & \xrightarrow{\eta_R} & TF(R) \end{array}$$

and the map is

$$((\mu_1, \phi, \mu_2), (\mu_2, \phi', \mu_3)) \mapsto (\mu_1, \phi\phi', \mu_3)$$

On general grounds, these give rise to ring homomorphisms which we give the same names:

η_L, η_R: $BP_* \to BP_*(BP)$
ε : $BP_*(BP) \to BP_*$
c : $BP_*(BP) \to BP_*(BP)$
ψ: $BP_*(BP) \to BP_*(BP) \otimes_{BP_*} BP_*(BP)$

where the tensor product is formed by viewing the left copy of $BP_*(BP)$ as a BP_*-module via η_R, and the right copy as a BP_*-module via η_L. One sees immediately that

$$\varepsilon\eta_L = 1 ,$$
$$c\eta_L = \eta_R$$

and $\psi\eta_L$ is η_L followed by the inclusion of the left copy of $BP_*(BP)$ into the tensor product.

Theorem 2. *These homomorphisms are the structure maps for the Hopf algebra $BP_*(BP)$—η_L is the leftunit, η_R is the right unit, ε is the counit, c is the conjugation and ψ is the coproduct.*

Let $\log_{BP}(T) = \sum_{k=0}^{\infty} m_{p^k-1} T^{p^k}$ $(m_0 = 1)$ be the logarithm for BP over $BP_* \otimes Q$ [2, §16], so $\log_{BP}$: $\mu_{BP} \to G_a$ is an isomorphism to the additive group:

$$\log_{BP}(\mu_{BP}(X, Y)) = \log_{BP}(X) + \log_{BP}(Y) .$$

Theorem 3. *We have*

i) $\eta_R(m_{p^k-1}) = \sum_{i+j=k} m_{p^i-1}(t_j)^{p^i}$

ii) η_L *is the obvious inclusion of* BP_* *into* $BP_*(BP) = BP_*[t_1, t_2, \ldots]$

iii) $\varepsilon(t_i) = 0$ for $i > 0$

iv) *c satisfies* $\sum_{h+i+j=k} m_{p^h-1}(t_i)^{p^h}(ct_j)^{p^{h+i}} = m_{p^k-1}$

v) ψ *atisfies* $\sum_{i+j=k} m_{p^i-1}(\psi t_j)^{p^i} = \sum_{h+i+j=k} m_{p^h-1}(t_i)^{p^h} \otimes (t_j)^{p^{h+i}}$.

Theorem 2 follows from Theorem 3 and the identical formulas of Adams [2, Theorem 16.1], in view of the identities preceeding the statement of Theorem 2 which determine the restrictions of *6, c* and ψ to BP_*.

Proof of Theorem 3: ii) η_L is a homomorphism $BP_* \to BP_*(BP)$ so that if $f \otimes g$: $BP_*(BP) = BP_* \otimes Z_{(p)}[t_i] \to R$ represents (μ_1, ϕ, μ_2) then $(f \otimes g)\eta_L BP_* \to R$ represents μ_1. *I.e.* this means that

$$(f\otimes g)\eta_L = f,$$

and is clearly satisfied by the obvious inclusion of BP_* in $BP_*[t_1 t_2,...]$.

i) η_R is a homomorphism $BP_*\to BP_*(BP)$ so that if $f\otimes g$: $BP_*(BP)=BP_*\otimes Z_{(p)}[t_i]\to R$ represents (μ_1, ϕ, μ_2), then $(f\otimes g)\eta_R$: $BP_*\to R$ represents $\mu_2=\mu_1{}^\phi$.

Take $R=BP_*(BP)$, $\mu_1=\mu_{BP}$ (extended from BP^* to $BP_*[t_i]$) and

$$\phi(T) = \sum_i {}^{\mu_{BP}} t_i T^{p^i}.$$

Then $f\otimes g$ is the identity, so η_R represents the formal group $\mu_{BP}{}^\phi$ over $BP_*(BP)$: $\eta_{R*}(\mu_{BP})=\mu_{BP}{}^\phi$ Now over $BP_*\otimes Q$ we have an isomorphism $\log_{BP}$: $\mu_{BP}\to G_a$, hence also an isomorphism $\eta_{R*}(\log_{BP})$: $\eta_{R*}(\mu_{BP})\to G_a$. Noting that $\eta_{R*}(\mu_{BP})=\mu_{BP}{}^\phi=(G_a{}^{\log_{BP}})^\phi$ we conclude that

$$\eta_{R*}(\log_{BP}) = \log_{BP}\circ\phi.$$

Hence

$$\begin{aligned}\sum_k \eta_R(m_{p^k-1})T^{p^k} &= \log_{BP}(\sum_j{}^{\mu_{BP}} t_j T^{p^j}) \\ &= \sum_j \log_{BP}(t_j T^{p^j}) \\ &= \sum_{i,j} m_{p^i-1}(t_j)^{p^i} T^{p^{i+j}}\end{aligned}$$

which proves i).

iii) ε is a homomorphism $BP_*[t_i]\to BP_*$ such that if f: $BP_*\to R$ represents μ then $f\circ\varepsilon$: $BP_*[t_i]\to R$ represents (μ, γ_0, μ), where $\gamma_0(T)=T$. Hence $f\circ\varepsilon(t_i)=0$ for $i>0$, from which it is immediate that $\varepsilon(t_i)=0$ for $i>0$.

iv) c is a homomorphism $BP_*[t_i]\to BP_*[t_i]$ so that if $f\otimes g$: $BP_*[t_i]\to R$ represents (μ_1, ϕ, μ_2) then $(f\otimes g)\circ c$ represents $(\mu_2, \phi^{-1}, \mu_1)$.

Take $f\otimes g$ to be the identity, so / represents μ_{BP} with scalars extended to $BP_*[t_i]$ and g represents

$$\phi(T) = \sum_i {}^{\mu_{BP}} t_i T^{p^i}.$$

Then $c=f'\otimes g$ where f' represents $\mu_{BP}{}^\phi$ (giving $c\eta_L=\eta_R$ as noted above) and g': $Z_{(p)}[t_i]\to BP_*[t_i]$ must be determined on the t_i's. Now g' represents ϕ^{-1}: $\mu_{BP}\to\mu_{BP}{}^\phi$; thus

$$\phi^{-1}(T) = \sum_i{}^{\mu_{BP}{}^\phi} c(t_i)T^{p^i}$$

as a curve over $\mu_{BP}{}^\phi$. Applying $\phi_\#$: $C_{\mu_{BP}{}^\phi}\to C_{\mu_{BP}}$ and noting that

$$\phi_\#(\phi^{-1}) = \phi\circ\phi^{-1} = \gamma_0,$$

we compute:

$$T = \sum_j{}^{\mu_{BP}} \phi_*(c(t_j))T^{p^j}$$
$$= \sum_{i,j}{}^{\mu_{BP}} t_i(c(t_j))^{p^i} T^{p^{i+j}} .$$

Finally, we apply $\log_{BP}$ and obtain the desired formula.

v) ψ is a homomorphism $BP_*(BP) \to BP_*(BP) \otimes_{BP_*} BP_*(BP)$ such that if

$$f \otimes g \text{ represents } (\mu_1, \phi, \mu_2)$$

and

$$f' \otimes g' \text{ represents } (\mu_2, \phi' , \mu_3)$$

then

$$[(f \otimes g) \otimes (f' \otimes g')] \circ \psi = f'' \otimes g''$$

represents $(\mu_1, \phi\phi', \mu_3)$. Note that

$$\mu_3 = {\mu_2}^{\phi'} = ({\mu_1}^{\phi})^{\phi'} = {\mu_1}^{\phi\phi'} .$$

We seek a universal example. Take $R = BP_*(BP) \otimes_{BP_*} BP_*(BP)$, which is a polynomial algebra over BP_* on generators $t_i \otimes 1$ and $1 \otimes t_j$ for i and $j > 0$. Take

$$\mu_1 = \mu^{BP} \text{ (extended to } R)$$
$$\phi = \sum_i{}^{\mu_{BP}}(t_i \otimes 1)T^{p^i}$$
$$\mu_2 = (\mu^{BP})^{\phi}$$
$$\phi' = \sum_j{}^{\mu_2}(1 \otimes t_j)T^{p^j}$$
$$\mu_3 = {\mu_2}^{\phi'} = {\mu_{BP}}^{\phi\phi'} .$$

One verifies easily that $f \otimes g$ is the inclusion of the left copy of $BP_*(BP)$ in R, while $f' \otimes g'$ is the inclusion of the right copy. Thus $(f \otimes g) \otimes (f' \otimes g')$ is the identity, and so in this situation $\psi = f'' \otimes g''$. We want to find $\psi(t_j)$. The series $\sum_j{}^{\mu_{BP}} \psi(t_j)T^{p^j}$ must agree with the composition

$$(\sum_j{}^{\mu_{BP}} t_i \otimes 1\, T^{p^i}) \circ (\sum_{i,j}{}^{(\mu_{BP})^{\phi}} 1 \otimes t_j\, T^{p^j}) ,$$

hence

$$\sum_j{}^{\mu_P} \psi(t_j)T^{p^i} = \sum_{i,j}{}^{\mu_{BP}} t_i \otimes (t_j)^{p^i} T^{p^{i+j}} .$$

The formula v) now follows by applying $\log_{BP}$. This completes the verifications. QED

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References

[1] J.F. Adams: *Lectures on generalized cohomology*, Lecture Notes in Mathematics, 99 (1969), Springer-Verlag, 1-138.

[2]————: Quillen's Work on Formal Groups and Complex Cobordism, University of Chicago, Lecture Notes, 1970.

[3] S. Araki: *A typical formal group in K-theory*, Proc. Japan Acad. 49 (1973), 477-482.

[4]————· Typical Formal Groups in Complex Cobordism and K-theory, Lectures in Mathematics, Kyoto University, 6, Kinokuniya Book-Store, 1973.

[5] P.S. Landweber: *Associated prime ideals and Hopf algebras*, J. Pure Appl. Algebra 3 (1973), 43-58.

[6] D. Quillen: *On the formal group laws of unoriented and complex cobordism theory*, Bull. Amer. Math. Soc. 75 (1969), 1293-1298.

[7] D.C. Ravenel: *Multiplicative operations in $BP_*(BP)$*, Columbia University, preprint, 1974.