MULTIPLICATIVE OPERATIONS IN BP COHOMOLOGY

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Introduction. In the present work we study multiplicative operations in BP cohomology. In §1 we show that all multiplicative operations in BP* are automorphisms (Theorem 1.3). Thus they form the group Aut (BP). In §2 we define Adams operations in BP* by the formal group μBP of BP cohomology and study the basic properties of them. These operations are primarily defined for units in Zp and then extended to p-adic units. Thereby we discuss BP* by extending the ground ring Zp to the ring of p-adic integers Zp. To achieve this extension simply by tensoring with Zp we restrict our cohomologies to the category of finite CW-complexes. Correspondingly we consider all multiplicative operations in BP*( ) ⊗ Zp whenever it becomes necessary to do so. Adams operations could be defined also for non-units, but we are not interested in such a case in this paper. In §3 we prove that the center of Aut (BP) consists of all Adams operations (Theorem 3.1).

We regard the lecture note [2] as our basic reference and use the results contained there rather freely.

1. Multiplicative operations in BP*.

Let BP* denote the Brown-Peterson cohomology for a specified prime p. By a multiplicative operation in BP* we understand a stable, linear and degree-preserving cohomology operation

\[ \Theta_a: BP*( ) \to BP*( ) \]

which is multiplicative and Θ(1) = 1. The set of all multiplicative operations in BP* forms a semi-group by composition, which will be denoted by Mult (BP).

With respect to the standard complex orientation of BP* [1], [2], [7], we denote by eBP(L) the Euler class of a complex line bundle L and by μBP the associated formal group. Let Θ_a ∈ Mult (BP). Putting

\[ \Theta_a(e^{BP}(L)) = \sum_{i \geq 0} \theta_i(e^{BP}(L))^i \]

for an arbitrary line bundle L, by naturality we obtain a well-determined power
series
\[ \theta_a(T) = \sum_{i \geq 0} \theta_i T^i, \quad \theta_i \in BP^{2-2i}(T). \]
By naturality \( \theta_0 = 0 \) and by stability \( \theta_1 = 1 \). In particular \( \theta_a \) is invertible.

Put
\[ \phi_a(T) = \theta_a^{-1}(T). \]
Then
\[ (1.2) \quad \Theta_a(pt) \ast \mu_{BP} = \mu_a, \quad \mu_a = \mu_{BP}^{a \ast}. \]
Recall that \( \mu_{BP} \) is typical. Hence \( \mu_a \) is a typical formal group and \( \phi_a \) is a typical curve over \( \mu_{BP} \).

Conversely, given a typical curve \( \phi_a \) over \( \mu_{BP} \), by the universality of \( BP^* \), [2], Theorem 7.2, \( \phi_a \) determines uniquely a multiplicative operation \( \Theta_a \) in \( BP^* \) satisfying
\[ (1.3) \quad \Theta_a(e^{BP}(L)) = \phi_a^{-1}(e^{BP}(L)). \]
Thus, via (1.3) multiplicative operations \( \Theta_a \) in \( BP^* \) correspond bijectively with typical curves \( \phi_a \) over \( \mu_{BP} \) such that
\[ (1.4) \quad \phi_a(T) \equiv T \text{ mod deg 2 and } \dim \phi_a^{-1}(e^{BP}(L)) = 2 \]
for complex line bundles \( L \).

Recall that a typical curve \( \phi_a \) satisfying (1.4) can be expressed uniquely as a Cauchy series
\[ (1.5) \quad \phi_a(T) = \sum_{k \geq 0} a_k T^k, \quad a_0 = 1, \quad a_k \in BP^{2(1-\rho^k)}(pt), \]
where \( \mu = \mu_{BP} \) (cf., [2], [3]). Thus multiplicative operations \( \Theta_a \) correspond bijectively with sequences
\[ (1.6) \quad a = (a_1, a_2, \ldots, a_n, \ldots), \quad a_n \in BP^{2(1-\rho^n)}(pt), \]
via (1.3) and (1.5). The identity operation corresponds to the zero sequence \( 0 = (0, 0, \ldots) \).

First we remark

**Proposition 1.1.** Let \( \Theta_a \) and \( \Theta_b \) be multiplicative operations in \( BP^* \) such that
\[ \Theta_a(pt) = \Theta_b(pt). \]
Then \( a = b \) as sequences (1.6). Hence \( \Theta_a = \Theta_b \).

Proof. By (1.2) we see that
\[ \mu_a = \mu_b. \]
Then, by the uniqueness of logarithm we see that
\[ \log_{\mu_B} = \log_{\mu} \]
or
\[ \log_{BP} \circ \phi_a = \log_{BP} \circ \phi_b. \]
thus \( \phi_a = \phi_b. \) q.e.d.

Let \( \Theta_a \subseteq \text{Mult}(BP) \). We have
\[ \Theta_a(pt) \circ \log_{BP} = \log_{BP} \circ \phi_a(T) \]
over \( BP^*(pt) \otimes \mathbb{Q} \). Putting
\[ \log_{BP}(T) = \sum_{k \geq 0} n_k T^{p^k}, \quad n_k = [CP_{p^k-1}]_{p^k}, \]
expanding both sides of the above formula as power series of \( T \) and comparing coefficients of \( T^{p^k} \) we get
\begin{equation}
(1.7) \quad \epsilon \sum_{j=0}^{k} n_j a_j \epsilon_{-j}, \quad k \geq 0.
\end{equation}
This is a recursive formula to describe \( \Theta_a(n_k) \), hence determines \( \Theta_a(pt) \). We discuss another formula to describe \( \Theta_a(pt) \).

Denote by \( f_p \) and \( f_p^a \) the Frobenius operators for the prime \( p \) on curves over \( \mu_{BP} \) and \( \mu_a \) respectively. Recall that, if we put
\begin{equation}
(1.8) \quad (f_p^a \gamma_0)(T) = f_p^a f_p\gamma_0(T), \quad \gamma_0(T) = T,
\end{equation}
then \( v_k \in BP^{2(p-1)p^k}(pt) \) and the sequence \( (v_1, v_2, \cdots, v_n, \cdots) \) forms a polynomial basis of \( BP^*(pt) \), [2].

Since \( \Theta_a(pt) \star \mu_{BP} = \mu_a \), we have
\[ \Theta_a(pt) \star \mu_{BP} = \mu_a, \]
\begin{equation}
(f_p^a \gamma_0)(T) = \sum_{k \geq 1} \Theta_a(v_k) T^{p^{k-1}}.
\end{equation}
Using the fact that \( \phi_a : \mu_B \cong \mu_{BP}, \) a strict isomorphism, we compute \( (\phi_a f_p^a \gamma_0)(T) \) in two ways as follows:
\[ (\phi_a f_p^a \gamma_0)(T) = (f_p^a \phi_a \gamma_0)(T) \]
\[ = (f_p^a \phi_a)(T) = \sum_{k \geq 0} f_p(a_k T^{p^k}) \]
\[ = (f_p \gamma_0)(T) \sum_{k \geq 1} [\mu_{BP}(a_k T^{p^{k-1}})]_{p^k} \]
\[ = \sum_{k \geq 1} v_k T^{p^k} + \sum_{k \geq 1} \sum_{l \geq 1} w_l a_k^{p^l} T^{p^{k+l-1}}, \]
by [2], Propositions 2.4, 2.5 and 2.9, on one hand, where
\[ [\mu_{BP}(T) = \sum_{l \geq 0} w_l T^{p^l}, \quad w_0 - p, \quad w_k \in BP^{p-1-p^k}(pt); \]
on the other hand
Thus we obtain

\[(1.9) \quad \sum_{k \geq 1} \sum_{l \geq 0} \Theta_a(v_k) T^{\rho^k} \equiv \sum_{k \geq 1} \sum_{l \geq 0} \Theta_a(v_k) T^{\rho^k + l} \mod I^2.\]

This is a recursive formula to describe \( \Theta_a(v_k) \).

Let \( I = BP^*(p^j) \), the kernel of the augmentation \( \xi : BP^*(pt) \rightarrow \mathbb{Z}/(p) \). By [2], §10, we see that "the left hand side of (1.9)"

\[\equiv \sum_{k \geq 1} \Theta_a(v_k) T^{\rho^k} \quad \text{mod } I^2\]

\[= \Theta_a(v_1) T + \Theta_a(v_2) T^p + \cdots \mod I^2,\]

and "the right hand side of (1.9)"

\[\equiv \sum_{k \geq 1} v_k T^{\rho^k} \quad \text{mod } I^2\]

\[= (v_1 + p a_1) T + (v_2 + p a_2) T^p + \cdots \mod I^2.\]

Hence (1.9) implies

\[(1.10) \quad \Theta_a(v_k) = v_k + p a_k \mod I^2\]

for all \( k \geq 1 \). In particular

\[\Theta_a(v_k) \equiv v_k \mod (p + I^2)\]

for \( k \geq 1 \). This shows that \( \{ \Theta_a(v_k), k \geq 1 \} \) forms a polynomial basis of \( BP^*(pt) \).

Thus we obtain

**Proposition 1.2.** For any \( \Theta_a \in \text{Mult}(BP) \)

\[\Theta_a(pt) : BP^*(pt) \cong BP^*(pt), \text{ an isomorphism}.\]

Let \( \Theta_a \) and \( \Theta_b \) be two multiplicative operations in \( BP^* \) with corresponding sequences \( a = (a_1, a_2, \cdots) \) and \( b = (b_1, b_2, \cdots) \). Putting

\[\Theta_c = \Theta_a \circ \Theta_b, \quad c = (c_1, c_2, \cdots),\]

we shall discuss the sequence \( c \). Put

\[\tilde{\phi}_b(T) = \Theta_a(pt) \circ \phi_b(T) = \sum_{k \geq 0} \Theta_a(b_k) T^{\rho^k}.\]

Then
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On the other hand

\[ \Theta_a(pt) \ast \mu_b = \Theta_a(pt) \ast (\phi_b^{-1} \circ \phi_a \times \phi_b) = \phi_b^{-1} \circ \phi_a \circ \phi_b = \mu_{BP}^{e_{a,b}}. \]

Thus, likewise in the proof of Proposition 1.1, we have

\[ (1.11) \quad \phi_c = \phi_a \circ \phi_b, \]

or equivalently

\[ (1.12) \quad \sum_{k \geq 0} \epsilon_k T^{p^k} = \phi_a \left( \sum_{k \geq 0} \Theta_a(b_k) T^{p^k} \right) = \sum_{i=0}^{n} a_i (b_k) \epsilon_i T^{p^k+i}. \]

This is a recursive formula to describe \( \epsilon_k \).

A multiplicative operation \( \Theta_a \) in \( BP^* \) is called an automorphism of \( BP^* \) if

\[ \Theta_a(X, A) : BP^*(X, A) \cong BP^*(X, A), \]

isomorphic for all finite CW-pair \( (X, A) \). Clearly a multiplicative operation \( \Theta_a \) is an automorphism of \( BP^* \) iff it has an inverse. The set of all automorphisms of \( BP^* \) forms a group, which will be denoted by \( \text{Aut}(BP) \).

**Theorem 1.3.** \( \text{Aut}(BP) = \text{Mult}(BP) \).

Proof. It is sufficient to prove that every multiplicative operation \( \Theta_a \) has a right inverse.

Let \( t=(t_1, t_2, \cdots) \) and \( s=(s_1, s_2, \cdots) \) be sequences of indeterminates with \( \dim t_k = \dim s_k = 2(1-p^k) \). Put

\[ (*) (1) \quad \sum_{k \geq 0} u_k T^{p^k} = \sum_{k \geq 0} \sum_{l \geq 0} t_k s^l T^{p^k+l}, \]

where \( s_0 = t_0 = u_0 = 1 \). Then over \( BP^*(pt)[t, s] \) we have

\[ \sum_{k \geq 0} u_k T^{p^k} \equiv T + u_1 T^p + u_2 T^{p^2} + \cdots \mod \hat{I}, \]

and

\[ \sum_{k \geq 0} \sum_{l \geq 0} t_k s T^{p^k+l} \equiv T + (s_1 + t_1) T^p + (s_2 + t_2) T^{p^2} + \cdots \mod \hat{I}, \]

where \( \hat{I} = (s, t) \), the ideal of \( BP^*(pt) \) generated by \( s_1, s_2, \cdots, t_1, t_2, \cdots \). Thus we can put

\[ (*) (2) \quad u_k = t_k + s_k + P_k(t_1, \cdots, t_{k-1}, s_1, \cdots, s_{k-1}), \quad k \geq 1. \]

Here \( P_k \) is a polynomial of \( t_1, \cdots, t_{k-1}, s_1, \cdots, s_{k-1} \) with \( \dim P_k = 2(1-p^k) \) and
We want to find a right inverse of $\Theta_a$. Putting

\[ (\ast 3) \quad \Theta_a \circ \Theta_b = id \]

with undecided sequence $b=(b_1, b_2, \ldots)$, we shall decide the sequence $b$. By (1.12), (\ast 1) and (\ast 2), we get

\[ (\ast 4) \quad a_k + \Theta_a(b_k) + P_k(a_1, \ldots, a_{k-1}, \Theta_a(b_1), \ldots, \Theta_a(b_{k-1})) = 0 \]

for all $k \geq 1$. Since the coefficients of $P_k$ depend neither on $(a_1, a_2, \ldots)$ nor on $(\Theta_a(b_1), \Theta_a(b_2), \ldots)$ we may use (\ast 4) as a recursive formula to obtain $\Theta_a(b_k)$, so we get $\Theta_a(b_k)$ as polynomials of $a_1, \ldots, a_k$ successively for $k \geq 1$. By Proposition 1.2 $\Theta_a(pt)$ is an isomorphism. Thus we get a sequence $(b_1, b_2, \ldots)$ so that it satisfies (\ast 4). Thereby $\Theta_b$ is obtained to satisfy (\ast 3). q.e.d.


Let $\mathbb{Z}_p$ be the ring of integers localized at the prime $p$ and $\mathbb{Z}_p$ its completion, i.e., the ring of $p$-adic integers. As is well known the endomorphism

\[ [\alpha]_{BP} \in \text{End} (\mu_{BP}) \]

is defined for each $\alpha \in \mathbb{Z}_p$ so that

\[ [\alpha]_{BP}(T) = \alpha T + \text{higher terms}. \]

It is convenient for us to extend these endomorphisms $[\alpha]_{BP}$ to $\alpha \in \mathbb{Z}_p$. For this purpose we extend the ground ring $\mathbb{Z}_p$ of $BP^*$ to $\mathbb{Z}_p$ by tensoring, i.e., we consider $BP^*(\_ \otimes \mathbb{Z}_p)$ whenever it is necessary to talk of $p$-adic integers.

Let $A=BP^*(pt) \otimes \mathbb{Z}_p$. Let $F$ and $G$ be formal groups over $A$. Let

\[ c : \text{Hom}_A(F, G) \to A \]

be the homomorphism sending $f$ to $a_1$ when $f(T)=a_1T + \text{higher terms}$. Since $A$ is an integral domain of characteristic zero, $c$ is injective as is well known (cf., [4], [5]).

Since $A$ is a direct sum of copies of $\mathbb{Z}_p$ (corresponding to each monomials of $v_i$’s) we give a direct limit topology to $A$. (Each direct summand is given the topology of $\mathbb{Z}_p$). Then, using the argument of Lubin [5], Lemma 2.1.1, we see that $c$ is an isomorphism onto a closed subgroup of $A$.

In case $F=G=\mu_{BP}$,

\[ \text{Im} \ c \cong \mathbb{Z}_p, \]

because $c([\alpha]_{BP})=\alpha$ for $\alpha \in \mathbb{Z}_p$. Hence

\[ \text{Im} \ c \cong \bar{\mathbb{Z}}_p = \mathbb{Z}_p. \]
Since $c$ is injective, for each $\alpha \in \mathbb{Z}$ there exists a unique
$$[\alpha]_{BP} \in \text{End}_A(\mu_{BP})$$
such that $c([\alpha]_{BP})=\alpha$. Thus the definition of $[\alpha]_{BP}$ is extended to $\mathbb{Z}_p$.

Since $c: \text{End}_A(\mu_{BP}) \to A$ is a ring homomorphism, for any $p$-adic integers $a$ and $\beta$ we have the following relations:

(2.1) $[\alpha]_{BP}(T)=\alpha T + \text{higher terms},$

(2.2) $[\alpha]_{BP}+[\beta]_{BP}=[\alpha+\beta]_{BP}, \quad \mu = \mu_{BP} ,$

(2.3) $[\alpha]_{BP} \cdot [\beta]_{BP} = [\alpha \beta]_{BP}.$

Let $\alpha \in \mathbb{Z}_{(p)}$ (or $\in \mathbb{Z}_p$) be a unit. Put
$$\Psi_{\alpha}(T) = [\alpha^{-1}]_{BP}(\alpha T).$$

Since
$$(f_\gamma \Psi_{\alpha})(T) = f_\gamma([\alpha^{-1}]_{BP}(\alpha T)) = [\alpha^{-1}]_{BP}([\alpha^\varphi]f_\gamma \gamma_\varphi(T)) = 0$$
for every $q > 1$ such that $(p,q)=1$ by [2], Propositions 2.3 and 2.9, where $\gamma_\varphi(T)=T$, we see that $\Psi_{\alpha}$ is a typical curve over $\mu_{BP}$. Moreover $\Psi_{\alpha}$ satisfies (1.4) as is easily seen. Thus there corresponds a multiplicative operation in $BP^*$ to $\Psi_{\alpha}$. We denote this multiplicative operation by $\Psi^*$ and call Adams operations in $BP^*$.

REMARK 1. Even for non-units $a$ Adams operations can be defined in the same way as above. But these operations are defined in $BP^*(\bigotimes \mathbb{Q})$ or $BP^*(\bigotimes \mathbb{Q}_p)$. And these cohomology theories are essentially ordinary cohomologies (corresponding to generalized Eilenberg-MacLane spectra), so we are not interested in these operations in the present work.

REMARK 2. Adams operations in complex cobordism are defined by Novikov [6]. When we regard $BP^*$ as a direct summand of $U^*(\bigotimes \mathbb{Q}_p)$, our Adams operations will be the restrictions of Novikov's Adams operations to $BP^*$.

Let $a$ be a unit of $\mathbb{Z}_{(p)}$ (or of $\mathbb{Z}_p$). Since
$$\Psi_{\alpha}([\alpha]_{BP}(T)) = [\alpha^{-1}]_{BP}[\alpha]_{BP}(T)=T,$$
we see that

(2.4) $\Psi^*(e_{BP}(L)) = \alpha^{-1}[\alpha]_{BP}(e_{BP}(L))$

for any complex line bundle $L$.

Since $\Psi^*(pt)_*\mu_{BP} = \mu_{BP}$, we see that
\[ \Psi^a(\phi_t) \log_{BP} \alpha = \log_{BP} \Psi^a \\cdot \]

Here

\[ (\log_{BP} \Psi^a)(T) = \log_{BP}[\alpha^{-1}]_{BP}(\alpha T) = \alpha^{-1} \cdot \log_{BP}(\alpha T) = \sum_{k \geq 0} \alpha^{b^k - 1} n_k T^{b^k} . \]

Thus

\[ \sum_{k \geq 0} \Psi^a(n_k) T^{b^k} = \sum_{k \geq 0} \alpha^{b^k - 1} n_k T^{b^k} , \]

or

(2.5) \[ \Psi f_0 = \alpha^{b^k - 1} n_k , \quad k \geq 1 , \]

after extending \( \Psi^a(\phi_t) \) to \( \Psi^a(\phi_t) \otimes 1_Q \).

**Proposition 2.1.** \( \Psi^a(\phi_t) BP^{-2s}(\phi_t) = c \cdot \text{id.} \)

Proof. \( (k_1, n_2, \ldots) \) is a polynomial basis of \( BP^*(\phi_t) \otimes Q \). Since \( \Psi^a \) is linear and multiplicative, for every polynomials \( x_s \) of \( n_k \)'s with \( \text{dim} x_s = -2s \) by (2.5) we see easily that

\[ \Psi^a(x_s) = \alpha^s x_s . \quad \text{q.e.d.} \]

**Corollary 2.2.** If we put

\[ \mu_{BP}(X, Y) = \sum_{i,j} a_{ij} X^i Y^j , \]

them

\[ \mu_{BP}^a(X, Y) = \sum_{i,j} \alpha^{i+j-1} a_{ij} X^i Y^j . \]

Next we prove

**Proposition 2.3.** \( \Psi^a \Psi^b = \Psi^{ab} = \Psi^b \Psi^a \).

Proof. Put

\[ [\alpha]_{BP}(T) = \sum_{\ell \geq 0} \alpha_{\ell} T^{\ell \cdot 3^{\ell + 1}} , \quad \alpha_{\ell} \in BP^{-2s-1 \cdot 3^\ell}(\phi_t) . \]

For any complex line bundle \( L \) we have

\[ \Psi^b(\Psi^a(e_{BP}(L))) = \Psi^b(\alpha^{-1}[\alpha]_{BP}(e_{BP}(L))) = \alpha^{-1} \Psi^b(\sum_{\ell \geq 0} \alpha_{\ell} e_{BP}(L)) \]

\[ = \alpha^{-1} \beta \Psi^b(e_{BP}(L)) \]

by Proposition 2.1

\[ = \alpha^{-1} \beta^{-1} \sum_{\ell \geq 0} \alpha_{\ell} (\beta \Psi^b(e_{BP}(L)))^{\ell \cdot 3^{\ell + 1}} \]

by (2.4)

\[ = (\alpha \beta)^{-1} [\alpha]_{BP}(e_{BP}(L)) \]

by (2.3)

\[ = \Psi^{ab}(e_{BP}(L)) . \]
Therefore, by the universality of $BP^*$, [2], Theorem 7.2, we conclude the Proposition.

Let $a$ and $\beta$ be $p$-adic units. By Propositions 1.1 and 2.1 we see that

\[(2.6) \quad \Psi^a = \Psi^\beta \iff \alpha^{p^{-1}} = \beta^{p^{-1}}.\]

Let $U(Z_p)$ be the multiplicative group of $p$-adic units and $U_i(Z_p)$ be its subgroup consisting of $p$-adic integers $a$ such that

$$\alpha \equiv 1 \mod p.$$ 

As is well known

$$U_i(Z_p) = \{\alpha^{p^{-1}} ; \alpha \in U(Z_p)\}.$$ 

By Proposition 2.3 all Adams operations (for $p$-adic units) form a multiplicative subgroup of Aut($BP$). We denote this subgroup by $Ad(BP)$. Then, (2.6) implies that

**Proposition 2.4.** $Ad(BP) \cong U_i(Z_p)$.

And also

**Proposition 2.5.** $\Psi^\lambda = 1 \iff \lambda^{p^{-1}} = 1$.

Next we discuss the relations of Adams operations with Quillen operations (of Landweber-Novikov type). We recall the definition of Quillen operations, [2], [7]. Let $t = (t_1, t_2, \ldots)$ be a sequence of indeterminates such that $\dim t_k = 2(1 - p^k)$ and

$$\phi_t(T) = \sum_{\ell \geq 0} r(\mu_{BP} \otimes T^\ell) \cdot t_0 = 1,$$

a typical curve over $\mu_{BP}$ by extending the ground ring of $\mu_{BP}$ to $BP^*(pt)[t]$. Then

$$r_t : BP^*(\ ) \to BP^*(\ ) [t]$$

is the multiplicative operation such that

$$r_t(e^{BP}(L)) = \phi_t^{-1}(e^{BP}(L))$$

for any complex line bundle $L$. Putting

$$r_t(x) = \sum_E r_E(x)t^E, \quad x \in BP^*(X, A),$$

where $E = (e_1, e_2, \ldots)$ runs over all sequences of non-negative integers such that all $e_k$ but a finite are zero, we get linear stable operations

$$r_E : BP^*(\ ) \to BP^{* + \ell(E)}(\ )$$
of degree $2|E|$, where $|E| = \sum_i e_i (p^i - 1)$.

Now for a $p$-adic unit $\alpha$ we have

$$ (2.7) \quad r, \circ \psi \left( e_{BP}(L) \right) = r, \left( \psi_{\alpha^{-1}}(e_{BP}(L)) \right) $$
$$ = (r, (pt) \circ \psi_{\alpha})^{-1}(r, (e_{BP}(L))) $$
$$ = (\phi_{\alpha} \circ r, (pt) \circ \psi_{\alpha})^{-1}(e_{BP}(L)). $$

And

$$ (r, (pt) \circ \psi_{\alpha}) (T) = r, (pt) \circ ([\alpha^{-1}]_{BP}(\alpha T)) = [\alpha^{-1}]_{\mu'}(\alpha T), $$

where $\mu' = \mu_{BP^*}$. Thus

$$ (2.8) \quad \langle \phi_{\alpha} \circ r, (pt) \circ \psi_{\alpha} \rangle (T) = \phi_{\mu}([\alpha^{-1}]_{\mu'}(\alpha T)) $$
$$ = [\alpha^{-1}]_{BP}(\phi_{\alpha}(\alpha T)) = [\alpha^{-1}]_{BP}(\sum_{k \geq 0} \alpha^k \phi_{\alpha} t_k T^p k). $$

Let

$$ \sigma_{\alpha}: \mathbb{Z}_{(p)}[t] \to \mathbb{Z}_{(p)}[t] $$

be an algebra homomorphism such that

$$ \sigma_{\alpha}(t_k) = \alpha^{k^{-1}} t_k, \quad k \geq 1, $$

and define an operation

$$ \widetilde{\Psi}^{\alpha^{*}}: BP^*( ) [i] \to BP^*( ) [i] $$

by $\widetilde{\Psi}^{\alpha} = \Psi^{\alpha} \otimes \sigma_{\alpha}$. Then

$$ (2.9) \quad (\Psi^{\alpha} \circ r, ) (e_{BP}(L)) = \widetilde{\Psi}^{\alpha}(\phi_{\alpha}^{-1}(e_{BP}(L))) $$
$$ = (\widetilde{\Psi}^{\alpha}(pt) \circ \phi_{\alpha})^{-1}(\widetilde{\Psi}^{\alpha}(e_{BP}(L))) $$
$$ = (\Psi^{\alpha} \circ \widetilde{\Psi}^{\alpha}(pt) \circ \phi_{\alpha})^{-1}(e_{BP}(L)). $$

Remark that

$$ \widetilde{\Psi}^{\alpha}(pt) \circ \mu_{BP} = \mu_{BP^*}^{\psi_{\alpha}}. $$

Thus

$$ (\widetilde{\Psi}^{\alpha}(pt) \circ \phi_{\alpha}) (T) = \sum_{k \geq 0} \alpha^k \phi_{\alpha} t_k T^p k, $$

where $\mu'' = \mu_{BP^*}\psi_{\alpha}$. And

$$ (2.10) \quad (\phi_{\alpha} \circ \widetilde{\Psi}^{\alpha}(pt) \circ \phi_{\alpha}) (T) = \phi_{\alpha}(\sum_{k \geq 0} \alpha^k \phi_{\alpha} t_k T^p k) $$
$$ = \sum_{k \geq 0} \phi_{\alpha} (\alpha^k \phi_{\alpha} t_k T^p k) $$
$$ = \sum_{k \geq 0} [\alpha^{-1}]_{BP}(\alpha^k t_k T^p k) $$
$$ = [\alpha^{-1}]_{BP}(\sum_{k \geq 0} \alpha^k t_k T^p k). $$
Thus by (2.8) and (2.10) we see that
\[ \phi_t \circ r_t(pt) \circ \psi_a = \psi_a \circ \overline{\psi}_a(pt) \circ \phi_t, \]
then, by (2.7) (2.9) and the universality of $BP^*$ we obtain

**Proposition 2.6.** For any unit of $\mathbb{Z}_p$ there holds the commutativity
\[ r_t \circ \psi_a = \overline{\psi}_a \circ r_t. \]

**Corollary 2.7.** Let $E=(e_1, e_2, \ldots)$ be a sequence of non-negative integers of which all but a finite terms are zero. There holds the commutativity
\[ r_E \circ \psi_a = \alpha^{|E|} \psi_a \circ r_E. \]

**Corollary 2.8.** For any linear stable cohomology operation
\[ \Xi_s: BP^*(\ ) \to BP^*+2s(\ ) \]
of degree $2s$ there holds the commutativity
\[ \Xi_s \circ \psi_a = \alpha^s \psi_a \circ \Xi_s. \]

Remark that every stable cohomology operation in $BP^*$ can be expressed as linear combinations of Quillen operations $r_E$ over $BP^*(pt)$. Then Corollary 2.8 follows from Proposition 2.1 and Corollary 2.7.

**Corollary 2.9.** Adams operations in $BP^*$ commute with all multiplicative operations.

**REMARK.** Properties of Adams operations in complex cobordism which correspond to Propositions 2.1, 2.2, 2.3, 2.7 and 2.8 are obtained in Novikov [7] by different arguments.

3. The center of $Aut(BP)$.

For any $b \in BP^{2(1-p^k)}(pt)$ we define a sequence
\[ (b, k) = (0, \ldots, 0, b, 0, \ldots) \]
with $b$ as the $k$-th term and with all other terms zero. By (1.9) we obtain
\[ \sum_{i=1}^k \Theta_{b, k}(\psi_i)T^{p^i-1} + \mu \sum_{i=1}^k bT^{p^i+1-1} = \sum_{i=1}^k \psi_i T^{p^i-1} + \mu \sum_{i=1}^k b^i T^{p^i+1-1}. \]

In particular
\[ \sum_{i=1}^k \Theta_{b, k}(\psi_i)T^{p^i-1} \equiv \sum_{i=1}^k \psi_i T^{p^i-1} + \mu pbT^{p^k-1} \text{ mod } \deg p^{k-1} + 1. \]
Recursively on /, \(1 \leq l < k\), and deleting the same terms successively we see that

\[
\Theta_{(b, k)}(v_l) = v_l, \quad 1 \leq l < k,
\]

and

\[
\Theta_{(b, k)}(v_k) = v_k + pb.
\]

These imply that

\[
\Theta_{(b, k)}(x) = x \quad \text{for any } x \in BP^{-2x}(pt), s < p^k - 1,
\]

and

\[
\Theta_{(b, k)}(y) = y + pcb \quad \text{for } y \in BP^{2x(1-p^k)}(pt)
\]

when \(y = cv_k \text{ mod decomposables, } c \in \mathbb{Z}_p\).

Let \(\Theta_a\) be in the center of \(\text{Aut}(BP)\). Then

\[
\Theta_{(v_k, b)} \circ \Theta_a = \Theta_a \circ \Theta_{(v_k, b)}
\]

for all \(k \geq 1\). And by (1.12) we have

\[
\sum_{i \in \mathbb{Z}_b} \Theta_{(v_k, b)}(a_i) T^{p^i} + \sum_{i \in \mathbb{Z}_b} a_i \cdot \Theta_{(v_k, b)}(a_i) T^{p^i}
\]

\[
= \sum_{i \in \mathbb{Z}_b} a_i T^{p^i} + \sum_{i \in \mathbb{Z}_b} \Theta_a(v_k) T^{p^i}.
\]

In particular

\[
\Theta_{(v_k, b)}(a_k) T^{p^k} + u v_k T^{p^k} = \Theta_a(v_k) T^{p^k} \quad \text{mod deg } p^k + 1.
\]

Thus

\[
\Theta_{(v_k, b)}(a_k) + v_k = a_k + \Theta_a(v_k).
\]

Put

\[
a_k \equiv \lambda_k v_k \quad \text{mod decomposables, } \lambda_k \in \mathbb{Z}_p.
\]

Then by (3.4) and (3.5) we obtain

\[
\Theta_a(v_k) = (1 + p \lambda_k) v_k, \quad k \geq 1.
\]

Next, putting

\[
v_k' = v_k + v_1^{(p^k - 1)/p - 1}
\]

for \(k > 1\), by commutativity

\[
\Theta_{(v_k', b)} \circ \Theta_a = \Theta_a \circ \Theta_{(v_k', b)}
\]
and by the same argument as (3.5) we obtain

\[ \Theta_{\varphi_k'}(a_k + v'_k) = a_k + \Theta_{\varphi_k'}(v'_k). \]

Applying (3.4) and (3.7) to (3.8) we obtain

\[ (1 + p\lambda)\varphi_1^{(p^k-1)/p^{k-1}} = (1 + p\lambda)\varphi_1^{(p^k-1)/p^{k-1}}. \]

thus

\[ 1 + p\lambda_k = (1 + p\lambda_1)^{(p^k-1)/p^{k-1}}. \]

Let \( \lambda \) be a \( p \)-adic unit such that

\[ \lambda^{p-1} = 1 + p\lambda_1. \]

Then (3.9) implies that

\[ 1 + p\lambda_k = \lambda^{p^k-1} \]

for all \( k \geq 1 \). Thus, by (3.7), (3.10) and Proposition 2.1 we see that

\[ \Theta^a \BP^*(pt) = \Psi^a \BP^*(pt). \]

Then by Proposition 1.1

\[ \Theta^a = \Psi^a. \]

In other words every multiplicative operation which is in the center of \( \text{Aut}(\BP) \) is a suitable Adams operation. Let \( Z(\text{Aut}(\BP)) \) denote the center of \( \text{Aut}(\BP) \). The above result and Corollary 2.9 imply

**Theorem 3.1.** \( \text{Ad}(\BP) = Z(\text{Aut}(\BP)). \)

**Corollary 3.2.** \( Z(\text{Aut}(\BP)) = U_r(Z_p). \)

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**References**


