Manabe, S.
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ON THE HITTING PROPERTIES OF A CLASS OF ONE-DIMENSIONAL MARKOV PROCESSES

SHÔJIRÔ MANABE

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1. Introduction. Let $X=(X_t, P_x, x\in R^1)$ be a one-dimensional standard Markov process with generator A

$$(1.1)\qquad \begin{aligned} Au(x) &= a(x)u'(x)+\sigma(x)^2u''(x)/2 \\ &\quad +\int_{-\infty}^{\infty}\left\{u(x+y)-u(x)-\frac{y}{1+y^2}u'(x)\right\}n(x, dy)\,. \end{aligned}$$

In this article, we shall discuss how the sample paths of X approach a single point. Let σ_0 be the first hitting time of the sample path to the origin: $\sigma_0=\inf\{t>0; X_t=0\}$. Set $\Omega_1=\{\omega; \sigma_0(\omega)<+\infty\}$. Define Ωί, ΩΓ and $\Omega_1^\pm$ by

$$\Omega_1^+ = \{\omega\in\Omega_1;\ \exists\varepsilon>0,\ \forall t\in[\sigma_0(\omega)-\varepsilon\sigma_0(\omega)),\ X_t(\omega)>0\}\ ,$$
$$\Omega\Gamma = \{\omega\in\Omega_1;\ \exists\varepsilon>0,\ \forall t\in[\sigma_0(\omega)-\varepsilon,\sigma_0(\omega)),\ X_t(\omega)<0\}\ ,$$

and

$$\Omega_1^\pm = \{\omega\in\Omega_1;\ \exists t_n\uparrow\sigma_0(\omega)\quad s.t.\ X_{t_{2n-1}}(\omega)<0<X_{t_{2n}}(\omega)\}\ .$$

Our present problem is to decide whether $P_x(\Omega_1^+/\Omega_1)$, $P_x(\Omega_1^-/\Omega_1)$ and $P_x(\Omega_1^\pm/\Omega_1)$ are positive or not. When X is spatially homogeneous, the problem was treated by T.Takada [15] in case $\sigma\neq 0$ and by N.Ikeda-S.Watanabe [3] in case $\sigma=0$ who also applied their results to the study of two-dimensional diffusion processes. Their method is based on the estimate of the singularity of the Green function on the diagonal set. Recently P.W.Millar [10] solved a similar problem independently. Let T_x be the first exit time of the sample path of a spatially homogeneous process from $(-\infty, x]$ $(x>0)$. Millar gave, in terms of the exponent, a necessary and sufficient condition that $P_0(X_{T_x}=x)>0$.

Here we shall consider the class of spatially inhomogeneous Markov processes determined by A under certain regularity conditions on a, σ and Lévy measure $n(x, dy)$. We shall give some sufficient conditions that $P_x(\Omega_1^+/\Omega_1)=1$, $P_x(\Omega_1^-/\Omega_1)=1$ and $P_x(\Omega_1^\pm/\Omega_1)=1$. Our method consists in estimating the singularity of the Green function as in [3]. More precisely, under the regularity conditions that will be given in §2, put

$$(1.2) \qquad \psi(x, \xi) = ia(x)\xi - \sigma(x)^2\xi^2/2 + \int_{-\infty}^{\infty}\left(e^{i\xi}-1-\frac{i\xi w}{1+w^2}\right)n(x, dw).$$

Then A can be written in the form:

$$(1.3) \qquad Af(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi}\psi(x, \xi)\hat{f}(\xi)d\xi,$$

namely, we can regard A as a pseudo-differential operator having its symbol $\psi(x,\xi)$. On account of the theory of pseudo-differential operators, the equation $(\lambda-A)u=f$ admits a fundamental solution $g_\lambda(x, y)$, which is of the form

$$(1.4) \qquad g_\lambda(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(x-y)\xi}\frac{d\xi}{\lambda-\psi(x, \xi)} + g_{\lambda,1}(x,y),$$

with a continuously differentiable function $g_{\lambda,1}(x, y)$. Therefore the singularity of the Green function is the same as in the spatially homogeneous case and the results of [3], [15] concerning the manners of hitting remain true in the present case.

The organization of the present paper is as follows. In section 2, we state our theorems. In section 3, we mention some related facts from the theory of pseudo-differential operators. We construct the above mentioned fundamental solution in section 4 and estimate its singularity in section 5. In section 6, 7 and 8 we prove our theorems by making use of the estimates established in section 5.

2. Theorems. Denote by $\mathcal{B}(R^n)$ ($\dot{\mathcal{B}}(R^n)$) the space of $C^\infty(R^n)$-functions whose derivatives of any order are bounded (vanishing at infinity). Let $a(x)$ and $\sigma(x)$ be the bounded C^∞-functions with its first derivatives $a'(x)$ and $\sigma'(x)$ belonging to $\dot{\mathcal{B}}$.[1)] Let $\nu(x,y)$ be a nonnegative function of $\mathcal{B}(R^2)$. We assume that $\nu(x,y)$ satisfies following two conditions:

(ν.1) there exists a positive constant c_1 such that

$$\nu(x, y) > c_1 \text{ on } R^1 \times \{y;\ |y| \leqq 1\},$$

(ν.2) there exists a positive constant L such that

$$\nu(x,y) \text{ is independent of } x, \text{ if } |x| \geqq L.$$

Define $n(x, y)$ by

$$n(x,y) = \begin{cases} \dfrac{\nu(x, y)}{y^{1+\alpha_1}}, & y>0, \\ \dfrac{\nu(x, y)}{|y|^{1+\alpha_2}}, & y<0, \end{cases} \qquad \text{where } 0<\alpha_i<2,\ i=1, 2.$$

1) We denote $\mathcal{B}$ (or $\dot{\mathcal{B}}$) for $\mathcal{B}(R^1)$ (or $\dot{\mathcal{B}}(R^1)$) for simplicity.

Consider the following operator A on $\mathring{\mathcal{B}}$:

$$
(2.1)\qquad \begin{aligned} Au(x) &= a(x)u'(x)+\sigma(x)^2u''(x)/2 \\ &\quad +\int_{-\infty}^{\infty}\left\{u(x+y)-u(x)-\frac{y}{1+y^2}u'(x)\right\}n(x, y)dy . \end{aligned}
$$

It is known that there exists a unique standard Markov process $X=\{X_t, P_x, x\in R^1\}$ whose generator is the closure of A (in fact, X is a Hunt process, see Sato [11]). Let G_λ be the resolvent of X:

$$
G_\lambda f(x) = E_x\left[\int_0^\infty e^{-\lambda t}f(X_t)dt\right] = \int_{R^1} G_\lambda(x, dy)f(y), \quad f\in C_b(R^1) .^{1)}
$$

Then it can be shown that there exists a density $g_\lambda(x, y)$ of $G_\lambda(x, dy)$ with respect to Lebesgue measure dy (see §4).

Now our results are as follows.

Theorem 1. *If* (i) $\sigma(x)^2 \geqq \sigma^2 > 0$ *or* (ii) $\sigma(x)=0$ *and* $\max(\alpha_1, \alpha_2)>1$, *then* $P_x(\Omega_1)>0$ *for any* $x\neq 0$.

Theorem 2. (1) *If* $\sigma(x)^2 \geqq \sigma^2 > 0$, *then we have*

(i) $P_x(\Omega_1^+ \cup \Omega_1^- | \Omega_1)=1$ *for any* $x\neq 0$.

(ii) *More precisely, for any* $x\neq 0$,

$$
(2.2a)\quad E_x[e^{-\lambda\sigma_0}; \Omega_1^+] = \frac{\sigma(0)^2}{2g_\lambda(0, 0)}\left[g_\lambda(0, 0)\frac{\partial g_\lambda}{\partial y}(x, 0)-g_\lambda(x, 0)\frac{\partial g_\lambda}{\partial y}(0, 0+)\right],
$$

$$
(2.2b)\quad E_x[e^{-\lambda\sigma_0}; \Omega_1^-] = \frac{\sigma(0)^2}{2g_\lambda(0, 0)}\left[-g_\lambda(0, 0)\frac{\partial g_\lambda}{\partial y}(x, 0)+g_\lambda(x, 0)\frac{\partial g_\lambda}{\partial y}(0, 0-)\right].
$$

(2) *If* $\sigma(x)=0$ *and* $\alpha_1>\alpha_2$, $\alpha_1>1$ [*resp.* $\alpha_1<\alpha_2$, $\alpha_2>1$], *then* $P_x(\Omega_1^+|\Omega_1)=1$ [*resp.* $P_x(\Omega_1^-|\Omega_1)=1$] *for any* $x\neq 0$.

(3) *If* $\sigma(x)=0$ *and* $\alpha_1=\alpha_2>1$, *then* $P_x(\Omega_1^\pm|\Omega_1)=1$ *for any* $x\neq 0$.

3. Pseudo-differential operators. In this section, we shall collect a number of known facts needed for later section from the theory of pseudo-differential operators. We refer to Kumano-go [6] for details. We denote by $S^m_{\rho,\delta}$, $0\leqq\delta<\rho\leqq 1$, $-\infty<m<\infty$, the set of $C^\infty(R\times R)$-functions $p(x,\xi)$ such that

$$
(3.1)\qquad |D_x^\alpha\partial_\xi^\beta p(x,\xi)|\leqq C_{\alpha,\beta}(1+|\xi|)^{m+\delta\alpha-\rho\beta} \text{for any } x, \xi\in R^1 ,
$$

where $D_x^\alpha=(-i\partial/\partial x)^\alpha$ and $\partial_\xi^\beta=(\partial/\partial\xi)^\beta$. An element of $S^m_{\rho,\delta}$ is called a symbol. Set $S^\infty_{\rho,\delta}=\bigcup_m S^m_{\rho,\delta}$ and $S^{-\infty}_{\rho,\delta}=\bigcap_m S^m_{\rho,\delta}$. For a function $p(x,\xi)$ belonging to $S^m_{\rho,\delta}$, we define a pseudo-differential operator $P=p(x, D_x)$

1) $C_b(R^1)$ is the totality of bounded continuous functions on R^1.

by

$$Pu(x)=p(x,\ D_x)u(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\xi}p(x,\ \xi)\hat{u}(\xi)d\xi^{1)},\quad u\in\mathcal{S}\ .^{2)}$$

Let $S^m_{\rho,\delta}$ be the totality of $p(x, D_x)$, where $p(x, \xi)$ is a function of $S^m_{\rho,\delta}$. Set $S^\infty_{\rho,\delta}=\bigcup_m S^m_{\rho,\delta}$ and $S^{-\infty}_{\rho,\delta}=\bigcap_m S^m_{\rho,\delta}$. We call E belonging to $S^\infty_{\rho,\delta}$ a right (or left) parametrix of P, if $PE=I-K_1$ for some $K_1\in S^{-\infty}_{\rho,\delta}$ (or $EP=I-K_2$ for some $K_2\in S^{-\infty}_{\rho,\delta}$) A parametrix is a right and left parametrix. Following theorem gives a sufficient condition for the existence of parametrix.

Theorem 3.1. (Hörmander [2]). *Let $p(x, \xi)$ be a symbol belongs to $S^m_{\rho,\delta}$. If $p(x,\xi)$ satisfies the following conditions* (i), (ii):
(i) *For some $\delta_0>0$ and m_1,*

$$|p(x,\ \xi)|\geqq\delta_0(1+|\xi|)^{m_1}\qquad \textit{for any } x,\ \xi\in R^1\ .$$

(ii) *For any nonnegative integers α, β, there exists a constant $C_{\alpha\beta}>0$ such that*

$$|D^\beta_x\partial^\alpha_\xi p(x,\xi)/p(x,\xi)|\leqq C_{\alpha\beta}(1+|\xi|)^{-\rho\alpha+\delta\beta}\qquad \textit{for any } x,\ \xi\in R^1\ .$$

Then there exists a parametrix $E=e(x, D_x)$, where $e(x, \xi)\in S^{-m_1}_{\rho,\delta}$.

REMARK 3.1. $e(x, \xi)$ can be constructed as follows (Hörmander [2]). Let $e_j, j=0, 1, 2, \cdots$ be functions determined by the following relations

$$\text{(3.2)}\qquad \begin{aligned} &e_0(x,\ \xi)=1/p(x,\ \xi)\,,\\ &e_j(x,\ \xi)=\frac{-1}{p(x,\ \xi)}\sum_{\substack{|\alpha|+k=j\\ k<j}}\frac{1}{\alpha!}\partial^\alpha_\xi e_k(x,\ \xi)D^\alpha_x p(x,\xi)\qquad \text{for any } j\geqq 1\ . \end{aligned}$$

Let ϕ be a C^∞-function such that $\phi(\xi)=0$ if $|\xi|\leqq 1/2$, $=1$ if $|\xi|\geqq 1$. Choose a sequence $1<t_1<t_2<\cdot<t_n\uparrow\infty$ such that

$$\text{(3.3)}\qquad |\partial^\alpha_x\partial^\beta_\xi\{e_j(x,\ \xi)\phi(\xi/t_j)\}|\leqq 2^{-j}(1+|\xi|)^{-m-\rho\beta+\delta\alpha-j+1}\,,$$

for $|\xi|\geqq t_j$, $|\alpha+\beta|\leqq j$. Then $e(x, \xi)$ can be written in the form

$$\text{(3.4)}\qquad e(x,\ \xi)=e_0(x,\ \xi)+\sum_{j=1}^{\infty}e_j(x,\xi)\phi(\xi/t_j)$$

Let H_s, $-\infty<s<\infty$ be the Sobolev space with norm $\|u\|^2_s=\int_{-\infty}^{\infty}(1+|\xi|^2)^s|\hat{u}(\xi)|^2 d\xi$. We need the following sharp form of Gårding inequality.

Theorem 3.2. (Kumano-go [7]). *Let $p(x, \xi)$ be a function of $S^m_{1,0}$. If $p(x, \xi)$ satisfies the following inequality: For some $\delta>0$ and m_1 such that* $0\leqq m$

1) We define $\hat{u}(\xi)=\int_{-\infty}^{\infty}e^{-i\xi x}u(x)dx$ for $u\in\mathcal{S}$.

2) $\mathcal{S}$ is the space of rapidly decreasing functions.

$-m_1 \equiv \theta < 1$,

$$|p(x, \xi)| \geqq \delta(1+|\xi|)^{m_1} \quad \textit{for any } x, \xi \in R^1,$$

then

$$\delta^2 \|u\|^2_{s+m-\theta/2} \leqq \|p(x, D_x)u\|^2_s + C_s \|u\|^2_{s+m-1/2}, \qquad u \in \mathcal{S}.$$

We shall give the characterization of $S^{-\infty}_{\rho,\delta}$.

Theorem 3.3. (Kumano-go [7]). *Let $P=p(x,D_x)$ be an element of $S^{-\infty}_{\rho,\delta}$. If we define $K(x, w)$ by*

$$K(x, w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iw\xi} p(x, \xi) d\xi,$$

then $K(x, w)$ belongs to $\mathcal{B}(R\times R)$ and $K(x, w)$ satisfies the following: For any α, β, N, there exists a constant $C=C_{\alpha\beta N}$ such that

$$(3.5) \qquad |\partial_x^\alpha \partial_w^\beta K(x,w)| \leqq C(1+|w|^2)^{-N/2} \quad \textit{for any } x, w \in R^1.$$

$Pu(x)$ can be written in the form

$$(3.6) \qquad Pu(x) = \int_{-\infty}^{\infty} K(x, x-y)u(y)dy, \quad u \in \mathcal{S}.$$

Proof. Because $p(x, \xi) \in S^{-\infty}_{\rho,\delta}$, we have

$$(1+|w|^2)^l \partial_x^\alpha \partial_w^\beta K(x,w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iw\xi}(1+D_\xi^2)^l \{(i\xi)^\beta \partial_x^\alpha p(x, \xi)\} d\xi.$$

From this we get (3.5). By the Fubini theorem we have (3.6).

For later use we shall quote the index theorem of Kumano-go [8]. We call a function $p(x, \xi)$ of $S^m_{\rho,\delta}$ be slowly varying, if for any α, β,

$$|D_x^\alpha \partial_\xi^\beta p(x,\xi)| \leqq C_{\alpha\beta}(x)(1+|\xi|)^{l+\delta-\rho\beta} \qquad \text{for any } x, \xi \in R^1,$$

where $C_{\alpha\beta}(x)$ is a bounded function such that $C_{\alpha\beta}(x) \to 0$ as $|x| \to \infty$, if $\alpha \neq 0$.

Theorem 3.4. (Kumano-go [8]). *Let $p(x, \xi) \in S^m_{\rho,\delta}$, $m>0$ be slowly varying. Suppose that there exist positive constants C_0 and $0<\tau\leqq 1$ such that $(p(x, \xi)-\zeta)^{-1}$ exists on*

$$\Xi_\xi = \{\zeta \in C;\ \operatorname{dist}(\zeta, (-\infty, 0]) \leqq C_0(1+|\xi|)^{\tau m}\}$$

and the estimate of the form

$$(3.7) \qquad |(D_x^\alpha \partial_\xi^\beta p(x,\xi))(p(x,\xi)-\zeta)^{-1}| \leqq C_{\alpha\beta}(x)(1+|\xi|)^{\delta\alpha-\rho\beta}$$

holds uniformly on Ξ_ξ, where $C_{\alpha\beta}(x)$ is a bounded function such that $C_{\alpha\beta}(x) \to 0$ as $|x| \to \infty$, if $\alpha \neq 0$. Then $P=p(x, D_x)$ as the map from L^2 into itself with the *domain* $D(P)=\{u\in L^2; Pu\in L^2\}$ *has index* 0.

4. Construction of the fundamental solution. Throughout this section, we always assume that (a) $\alpha_1\geqq\alpha_2$ and (b) $\sigma(x)^2\geqq\sigma^2>0$ or $\alpha_1>1$.

Let A be the integro-differential operator defined by (2.1) and $\psi(x, \xi)$ be the function defined by the following equation

$$(4.1)\qquad \psi(x, \xi)=ia(x)\xi-\sigma(x)^2\xi^2/2+\int_{-\infty}^{\infty}\left[e^{i\xi w}-1-\frac{i\xi w}{1+w^2}\right]n(x,w)dw\,.$$

Then it is easy to see that A can be written in the form

$$(4.2)\qquad Au(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ix\xi}\psi(x,\xi)\hat{u}(\xi)d\xi,\qquad u\in\mathcal{S}\,.$$

In Lemma 4.1, we collect some properties of A as a pseudo-differential operator when the following condition (c) is satisfied.

(c) There exists a constant $L_1>1$ such that

$$\nu(x, y)=0\qquad\text{on }\ R\times\{y;\ |y|\geqq L_1\}\,.$$

Lemma 4.1. *Let the condition* (c) *be satisfied. Then we have*

(1°) *$\psi(x, \xi)$ belongs to $S^2_{1,0}$. If $\sigma(x)=0$, then ψ belongs to $S^{\alpha_1}_{1,0}$.*

(2°) *For any $\lambda>0$, the symbol of $\lambda-A$ satisfies the following: There exists a constant $\delta_0>0$ such that*

$$(4.3)\qquad |\lambda-\psi(x,\xi)|\geqq\delta_0(1+|\xi|)^{\alpha_1}\quad \textit{for any } x,\ \xi\in R^1\,.$$

(3°) *For any $\lambda>0$ and for any nonnegative integers α, β, there exists $C_{\alpha\beta}>0$ such that for any x, $\xi\in R^1$*

$$(4.4)\qquad |(D_x^\alpha\partial_\xi^\beta\{\lambda-\psi(x,\xi)\})(\lambda-\psi(x,\xi))^{-1}|\leqq C_{\alpha\beta}(1+|\xi|)^{-\beta+\delta\alpha}$$
$$\textit{where }\ \delta=2-\alpha_1\ \textit{ if }\ \alpha_1>1,\ =0\ \textit{ if }\ \sigma(x)^2\geqq\sigma^2>0\,.$$

(4°) *Choose a constant γ such that $1<\gamma<2$ if $\sigma(x)^2\geqq\sigma^2>0$, $1<\gamma<\alpha_1$ if $\alpha_1>1$ and fix it. Let A_0 be the pseudo-differential operator having $\psi_0(x,\xi)=(\lambda-\psi(x,\xi))(1+|\xi|^2)^{-\gamma/2}$ as to symbol. Then we have index$(A_0)=0$.*

Proof. Proof of (1°). Since a' and σ' belong to $\mathring{\mathcal{B}}$, it is obvious that $ia(x)\xi-\sigma(x)^2\xi^2/2$ belongs to $S^2_{1,0}$. Set

$$\phi(x,\xi)=\int_{-L_1}^{L_1}\left(e^{i\xi w}-1-\frac{i\xi w}{1+w^2}\right)n(x,w)dw\,.$$

Since ϕ can be written in the form

$$\phi(x,\xi)=\int_{-L_1}^{0}\left(e^{i\xi w}-1-\frac{i\xi w}{1+w^2}\right)\frac{\nu(x,w)}{|w|^{1+\alpha_2}}dw$$

$$+\int_0^{L_1}\left(e^{i\xi w}-1-\frac{i\xi w}{1+w^2}\right)\frac{\nu(x, w)}{w^{1+\alpha_1}}dw,$$

it is sufficient to consider $F(x, \xi)=\int_0^{L_1}\left(e^{i\xi w}-1-\frac{i\xi w}{1+w^2}\right)\frac{\nu(x,w)}{w^{1+\alpha}}dw$ and to prove that $F(x, w)\in S^1_{1,0}$ if $0<\alpha<1$, $F(x, w)\in S^\alpha_{1,0}$ if $1<\alpha<2$, $|\partial_x^m\partial_\xi^n F(x, w)|\leqq C_{m,n}(1+|\xi|)^{1-}(1+\log(|\xi|\vee 1))$ if $\alpha=1$.

First we consider the case of $0<\alpha<1$. In this case, $F(x,\xi)$ can be written in the form

$$F(x, \xi)=\int_0^{L_1}(e^{i\xi w}-1)\frac{\nu(x, w)}{w^{1+\alpha}}dw-i\xi\int_0^{L_1}\frac{\nu(x, w)}{w^\alpha(1+w^2)}dw. \tag{4.5}$$

Set $F_1(x, \xi)=\int_0^{L_1}(e^{i\xi w}-1)\frac{\nu(x, w)}{w^{1+\alpha}}dw$ and $F_2(x, \xi)=-i\xi\int_0^{L_1}\frac{\nu(x, w)}{w^\alpha(1+w^2)}dw$. It is clear that F_2 belongs to $S^1_{1,0}$. We show F_1 belongs to $S^\alpha_{1,0}$. Set $M_l=\sup_{x,w\in R^1, m+n\leqq l}|\partial_x^m\partial_w^n\nu(x,w)|$, $l=0, 1, 2, \cdots$. If $n=0$, then we have $\partial_x^m F_1(x, \xi)=\int_0^{L_1}(e^{i\xi w}-1)\frac{\partial_x^m\nu(x, w)}{w^{1+\alpha}}dw$. Therefore, for $|\xi|\leqq 1$, noting that $|e^{i\xi w}-1|\leqq|\xi w|\leqq|w|$, we get

$$|\partial_x^m F_1(x, \xi)|\leqq\int_0^{L_1}\frac{|\partial_x^m\nu(x, w)|}{w^\alpha}dw\leqq M_m\frac{L_1^{1-\alpha}}{1-\alpha}.$$

For the case $|\xi|>1$, by putting $|\xi|w=y$, we have

$$\begin{aligned}|\partial_x^m F_1(x, \xi)| &= |\xi|^\alpha\left|\int_0^{L_1|\xi|}\frac{e^{i\xi/|\xi|y}-1}{y^{1+\alpha}}\partial_x^m\nu\left(x, \frac{y}{|\xi|}\right)dy\right| \\ &\leqq|\xi|^\alpha\left\{\int_0^1\frac{M_m}{y^\alpha}dy+\int_1^{L_1|\xi|}\frac{2M_m}{y^{1+\alpha}}dy\right\}\leqq\frac{2-\alpha}{\alpha(1-\alpha)}M_m|\xi|^\alpha.\end{aligned}$$

Thus we have

$$|\partial_x^m F_1(x, \xi)|\leqq C_{m,0}(1+|\xi|)^\alpha. \tag{4.6}$$

If $n\geqq 1$, we have

$$\partial_x^m\partial_\xi^n F_1(x, \xi)=(i)^n\int_0^{L_1}\frac{e^{i\xi w}}{w^{\alpha-n+1}}\partial_x^m\nu(x, w)dw.$$

For $|\xi|\leqq 1$, we get

$$|\partial_x^m\partial_\xi^n F_1(x, \xi)|\leqq M_m\int_0^{L_1}\frac{dw}{w^{\alpha-n+1}}\leqq\frac{M_m L_1^{n-\alpha}}{n-\alpha}.$$

For $|\xi|>1$, we have

$$\partial_x^m \partial_\xi^n F_1(x, \xi) = (i)\ |\xi|^{\alpha-n}\int_0^{L_1|\xi|} e^{i\,\xi/|\xi|\,y}\, y^{n-\alpha-1}\partial_x^m \nu\left(x, \frac{y}{|\xi|}\right)dy$$
$$= (i)^n|\xi|^{\alpha-n}F_3(x, \xi)\,.$$

For the estimate of $F_3(x, \xi)$, set $F_3^{(1)}(x, \xi)=\int_0^1 e^{i\,\xi/|\xi|\,y}y^{n-\alpha-1}\partial_x^m \nu\left(x, \frac{}{|\xi|}\right)dy$ and $F_3^{(2)}(x, \xi)=\int_1^{L_1|\xi|} e^{i\,\xi/|\xi|\,y}y^{n-\alpha-1}\partial_x^m \nu\left(x, \frac{y}{|\xi|}\right)dy$. Then we obtain for $F_3^{(1)}$, $|F_3^{(1)}(x, \xi)| \leqq M_m\int_0^1 y^{n-\alpha-1}dy=\frac{M}{n-\alpha}$. For $F_3^{(2)}$, by integration by parts, we have

$$F_3^{(2)}(x, \xi)= \sum_{l=1}\left(-\frac{|\xi|}{i\xi}\right)^l e^{i\,\xi/|\xi|\,y}\frac{\partial^l}{\partial y^l}\left\{y^{n-\alpha-1}\partial_x^m \nu\left(x, \frac{y}{|\xi|}\right)\right\}\Bigg|_{y=1}$$
$$+\left(-\frac{|\xi|}{j\xi}\right)^n \sum \binom{n}{l}(n-\alpha-1)(n-\alpha-2)\cdots(l-\alpha)$$
$$\times\int_1^{L_1|\xi|} e^{i\,\xi/|\xi|\,y}y^{l-\alpha-1}(\partial_y^l\partial_x^m \nu)\left(x, \frac{y}{|\xi|}\right)\frac{dy}{|\xi|^l}\,.^{1)}$$

For the terms corresponding to $l\geqq 1$, we have

$$\left|\frac{1}{|\xi|^l}\int_1^{L_1|\xi|} e^{i\,\xi/|\xi|\,y}y^{l-\alpha-1}(\partial_y^l\partial_x^m \nu)\left(x, \frac{y}{|\xi|}\right)dy\right|$$
$$=\left|\frac{1}{|\xi|^\alpha}\int_{1/|\xi|}^{L_1} e^{i\xi w}w^{l-\alpha-1}(\partial_w^l\partial_x^m \nu)(x, w)dw\right|\leqq M_{l+m}\int_{1/|\xi|}^{L_1} w^{l-\alpha-1}dw$$
$$\leqq M_{l+m}\frac{L_1^{l-\alpha}}{l-\alpha}\,.$$

For the term corresponding to $l=0$,

$$\left|\int_1^{L_1|\xi|} e^{i\,\xi/|\xi|\,y}\frac{1}{y^{1+\alpha}}(\partial_x^m \nu)\left(x, \frac{y}{|\xi|}\right)dy\right| = M_m\int_1^{L_1|\xi|}\frac{dy}{y^{1+\alpha}}\leqq\frac{M_m}{\alpha}\,.$$

Thus F_3 is bounded and the inequality

$$|\partial_x^m \partial_\xi^n F_1(x, \xi)| \leqq C_{m,n}(1+|\xi|)^{\alpha-n}, \qquad n\geqq 1 \tag{4.7}$$

holds. It follows from (4.6) and (4.7) that F_1 belongs to $S_{1,0}^\alpha$. Combining the fact that F_2 belongs to $S_{1,0}^1$ and $\alpha<1$, we have $F\in S_{1,0}^1$.

Next let us consider the case of $1<\alpha<2$. Set

$$F_1(x, \xi) = \int_0^{L_1}(e^{i\xi w}-1-i\xi w)\frac{\nu(x, w)}{w^{1+\alpha}}dw \text{ and}$$

1) $(\partial_y{}^l\partial_x{}^m \nu)\left(x, \frac{y}{|\xi|}\right)$ means $\left.\frac{\partial^{l+m}\nu(x, z)}{\partial x^m \partial z^l}\right|_{z=\frac{y}{|\xi|}}$

$$F_2(x, w) = i\xi\int_0^{L_1} \frac{w^{2-\alpha}}{1+w^2} \nu(x, w)dw .$$

Then F can be written in the form $F(x, \xi) = F_1(x,\xi) + F_2(x,\xi)$. It is evident that F_2 belongs to $S^1_{1\,0}$. In order to prove (3.1) for F_1, it suffices to show (3.1) for the case $\beta=0$. In fact, since

$$\partial_\xi F_1(x, \xi) = i\int_0^{L_1} \frac{e^{i\xi w}-1}{w^{1+(\alpha-1)}} \nu(x, w)dw ,$$

it reduces to the case $0<\alpha<1$. If $\beta=0$, then we have for $|\xi| \geq 1$,

$$\begin{aligned}|\partial_x^m F_1(x, \xi)| &= |\xi|^\alpha \left|\int_0^{|\xi|L_1}\left(e^{i\xi/|\xi|w}-1-i\frac{\xi}{|\xi|}w\right)\frac{\partial_x^m \nu\left(x, \frac{w}{|\xi|}\right)}{w^{1+\alpha}}dw\right| \\ &\leqq |\xi|^\alpha\left\{\int_0^1 \frac{M_m}{w^{\alpha-1}}dw + \int_1^\infty \frac{2+w}{w^{1+\alpha}}M_m dw\right\} \\ &= \left(\frac{2}{\alpha}+\frac{1}{\alpha-1}+\frac{1}{2-\alpha}\right)M_m|\xi|^\alpha .\end{aligned}$$

For $|\xi|<1$, noting $|e^{i\xi w}-1-i\xi w| \leqq \frac{1}{2}|\xi w|^2 \leqq \frac{1}{2}|w|^2$, we have

$$|\partial_x^m F_1(x, \xi)| \leqq M_m\int_0^{L_1} w^{1-\alpha}dw = \frac{M_m}{2-\alpha}(L_1)^{2-\alpha} .$$

Thus F_1 belongs to $S^\alpha_{1,0}$. Combining this and the fact that $F_2 \in S^1_{1,0}$, we obtain that F belongs to $S^\alpha_{1,0}$.

Finally consider the case $\alpha=1$. We have

$$\partial_x^m F(x, \xi) = \int_0^{L_1}\left\{e^{i\xi w}-1-\frac{i\xi w}{1+w^2}\right\}\frac{\partial_x^m \nu(x, w)}{w^2}dw , \tag{4.8a}$$

$$\partial_x^m \partial_\xi F(x, \xi) = i\int_0^{L_1}(e^{i\xi w}-1)\frac{\partial_x^m \nu(x, w)}{w}dw + i\int_0^{L_1}\frac{w\partial_x^m \nu(x, w)}{1+w^2}dw , \tag{4.8b}$$

$$\partial_x^m \partial_\xi^n F(x, \xi) = (i)^n\int_0^{L_1} e^{i\xi w}w^{n-2}\partial_x^m \nu(x, w)dw,\ n\geqq 2 . \tag{4.8c}$$

First we consider the case $n=0$. If $|\xi| \leq 1$, then we have

$$\begin{aligned}|\partial_x^m F(x, \xi)| &\leqq \left|\int_0^{L_1}(e^{i\xi w}-1-i\xi w)\frac{\partial_x^m \nu(x, w)}{w^2}dw\right| + |\xi|\left|\int_0^{L_1}\frac{\partial_x^m \nu(x, w)}{1+w^2}dw\right| \\ &\leqq 2M_m L_1 .\end{aligned}$$

If I ξI > 1, then we have

$$|\partial_x^m F(x,\xi)| = |\xi|\left|\int_0^{L_1|\xi|}\left\{e^{i\cdot sgn(\xi)y}-1-\frac{i\cdot sgn(\xi)y}{1+y^2/\xi^2}\right\}\frac{\partial_x^m v\left(x,\frac{y}{|\xi|}\right)}{y^2}dy\right|$$

$$\leqq |\xi|\left|\int_0^1\left\{e^{i\cdot sgn(\xi)y}-1-\frac{i\cdot sgn(\xi)y}{1+y^2/\xi^2}\right\}\frac{\partial_x^m v\left(x,\frac{y}{|\xi|}\right)}{y^2}dy\right|$$

$$+ |\xi|\left|\int_1^{L_1|\xi|}\left\{e^{i\cdot sgn(\xi)y}-1-\frac{i\cdot sgn(\xi)y}{1+y^2/\xi^2}\right\}\frac{\partial_x^m v\left(x,\frac{y}{|\xi|}\right)}{y^2}dy\right|$$

$$\equiv I_1|\xi|+I_2|\xi|.$$

For I_1, we have

$$I_1\leqq\left|\int_0^1\{e^{i\cdot sgn(\xi)y}-1-i\cdot sgn(\xi)y\}\frac{\partial_x^m v\left(x,\frac{y}{|\xi|}\right)}{y^2}dy\right|$$

$$+\left|\int_0^1 i\cdot sgn(\xi)y\left(1-\frac{1}{1+y^2/\xi^2}\right)\frac{\partial_x^m v\left(x,\frac{y}{|\xi|}\right)}{y^2}dy\right|$$

$$\leqq\int_0^1\left|\partial_x^m v\left(x,\frac{y}{|\xi|}\right)\right|dy+\int_0^1\frac{\left|\partial_x^m v\left(x,\frac{y}{|\xi|}\right)\right|}{y^2+\xi^2}dy\leqq 2M_m.$$

For I_2, we have

$$I_2\leqq\left|\int_1^{L_1|\xi|}\{e^{i\cdot sgn(\xi)y}-1\}\frac{\partial_x^m v\left(x,\frac{y}{|\xi|}\right)}{y^2}dy\right|$$

$$+\left|\int_1^{L_1|\xi|}\frac{-i\cdot sgn(\xi)y}{(1+y^2/\xi^2)y^2}\partial_x^m v\left(x,\frac{y}{|\xi|}\right)dy\right|$$

$$\leqq 2M_m\int_1^\infty\frac{dy}{y^2}+\int_1^{L_1|\xi|}\frac{M_m}{y(1+y^2/\xi^2)}dy = 2M_m+M_m\int_{1/|\xi|}^{L_1}\frac{dw}{w(1+w^2)}$$

$$\leqq M_m(2+\log L_1|\xi|).$$

Therefore we have

$$|\partial_x^m F(x,\xi)|\leqq C_{m,0}(1+|\xi|)(1+\log(|\xi|\vee 1)).^{1)} \tag{4.9}$$

Next we consider the case $n=1$. The second term of (4.8b) is bounded. Denote by / the first term of (4.8b). For $|\xi|>1$, we have

1) $a\vee b=\max(a,b), a\wedge b=\min(a,b)$.

$$|I| = \left| \int_0^{L_1|\xi|} \frac{e^{i\, sgn(\xi) y} - 1}{y} \partial_x^m v\left(x, \frac{y}{|\xi|}\right) dy \right| \leqq 2M_m \log L_1 |\xi| .$$

For $|\xi| \leqq 1$, we have

$$|I| \leqq \int_0^{L_1} |\xi| \, |\partial_x^m v(x, w)| \, dw \leqq M_m L_1 .$$

Therefore we get

$$(4.10) \qquad |\partial_x^m \partial_\xi F(x, \xi)| \leqq C_{m,1}(1 + \log(|\xi| \vee 1)) .$$

Finally we consider the case $n \geqq 2$. If $|\xi| \leqq 1$, then we obtain

$$|\partial_x^m \partial_\xi^n F(x, \xi)| \leqq M_m \int_0^{L_1} w^{n-2} dw = \frac{M_m}{n-1} L_1^{n-1} .$$

If $|\xi| > 1$, then we have

$$\partial_x^m \partial_\xi^n F(x, \xi) = \frac{(i)^n}{|\xi|^{n-1}} \int_0^{L_1|\xi|} e^{i \cdot sgn(\xi) y} y^{n-2} \partial_x^m v\left(x, \frac{y}{|\xi|}\right) dy$$

$$= \frac{(i)^n}{|\xi|^{n-1}} \left\{ \int_0^1 e^{i \cdot sgn(\xi) y} y^{n-2} \partial_x^m v\left(x, \frac{y}{|\xi|}\right) dy + \int_1^{L_1|\xi|} e^{i \cdot sgn(\xi) y} y^{n-2} \partial_x^m v\left(x, \frac{y}{|\xi|}\right) dy \right\} .$$

Set

$$J_1 = \int_0^1 e^{i \cdot sgn(\xi) y} y^{n-2} \partial_x^m v\left(x, \frac{y}{|\xi|}\right) dy \text{ and } J_2 = \int_1^{L_1|\xi|} e^{i \cdot sgn(\xi) y} y^{n-2} \partial_x^m v\left(x, \frac{y}{|\xi|}\right) dy .$$

Then we have for J_1,

$$|J_1| \leqq M_m/(n-1) .$$

For J_2, by integration by parts, we get

$$J_2 = \sum_{l=1}^{n-1} \left(-\frac{1}{i \cdot sgn(\xi)} \right)^l e^{i\, sgn(\xi) y} \frac{\partial^l}{\partial y^l} \left\{ y^{n-2} \partial_x^m v\left(x, \frac{y}{|\xi|}\right) \right\} \Big|_{y=1}$$

$$+ \left(-\frac{1}{i\, sgn(\xi)} \right)^{n-1} \sum_{l=1}^{n-1} \binom{n-1}{l} (n-2)(n-3) \cdots l$$

$$\times \int_1^{L_1|\xi|} e^{i \cdot sgn(\xi) y} y^{l-1} (\partial_y^l \partial_x^m v) \left(x, \frac{y}{|\xi|}\right) \frac{dy}{|\xi|^l} .$$

For the terms corresponding to $l \geqq 1$, we have

$$\left| \frac{1}{|\xi|^l} \int_1^{L_1|\xi|} e^{i \cdot sgn(\xi) y} y^{l-1} (\partial_y^l \partial_x^m v) \left(x, \frac{y}{|\xi|}\right) dy \right| \leqq M_{m+n-1} (L_1)^{n-1} .$$

For the term corresponding to $l=0$, we have

$$\left|\int_1^{L_1|\xi|} e^{i\cdot sgn(\xi)y}\frac{1}{y}(\partial_x^m v)\left(x, \frac{y}{|\xi|}\right)dy\right| \leqq M_m \log L_1|\xi| .$$

Therefore we get $|J_2| \leqq C(1+\log|\xi|)$ for $|\xi|>1$. Hence we have

$$(4.11) \qquad |\partial_x^m\partial_\xi^n F(x,\xi)| \leqq C_{m,n}(1+|\xi|)^{1-n}(1+\log(|\xi|\vee 1)).$$

It follows from (4.9), (4.10) and (4.11) that

$$|\partial_x^m\partial_\xi^n F(x,\xi)| \leqq C_{m,n}(1+|\xi|)^{1-n}(1+\log(|\xi|\vee 1)) .$$

Thus the proof of (1°) is completed.

Proof of (2°). Note that

$$\lambda-\psi(x, \xi) = \lambda+\frac{1}{2}\sigma(x)^2\xi^2+\int_{-\infty}^{\infty}(1-\cos\xi w)n(x,w)dw$$
$$-i\left\{a(x)\xi+\int_{-\infty}^{\infty}\left(\frac{\xi w}{1+w^2}-\sin\xi w\right)n(x, w)dw\right\} .$$

Set $c_2=\int_0^1 \frac{1-\cos y}{y^{1+\alpha_1}}dy$. Then usi[illegible] (1), we have for $|\xi|>1$

$$|\lambda-\psi(x, \xi)| \geqq \lambda+\frac{1}{2}\sigma(x)^2\xi^2+\int_{-\infty}^{\infty}(1-\cos\xi w)n(x,w)dw$$
$$\geqq \lambda+\frac{1}{2}\sigma(x)^2\xi^2+|\xi|^{\alpha_1}\int_0^1(1-\cos y)\frac{\nu\left(x, \frac{y}{|\xi|}\right)}{y^{1+\alpha_1}}dy$$
$$\geqq \lambda+\frac{1}{2}\sigma(x)^2\xi^2+c_1c_2|\xi|^{\alpha_1} .$$

Thus (2°) is proved.

Proof of (3°). Let $\alpha_1>1$. For $m\geqq 1$, we have by (1°) and (2°)

$$\left|\frac{\partial_\xi^n\partial_x^m(\lambda-\psi(x, \xi))}{\lambda-\psi(x, \xi)}\right| \leqq C_{m,n}(1+|\xi|)^{-n+(2-\alpha_1)} \leqq C_{m,n}(1+|\xi|)^{-n+(2-\alpha_1)m} .$$

For the case $m=0$ and $n\geqq 2$, we have

$$\left|\frac{\partial_\xi^n(\lambda-\psi(x, \xi))}{\lambda-\psi(x, \xi)}\right| \leqq C_{0,n}(1+|\xi|)^{-n} .$$

For $m=0$ and $n=1$, we have

$$\left|\frac{\partial_\xi(\lambda-\psi(x,\xi))}{\lambda-\psi(x,\xi)}\right|\leqq\frac{\sigma(x)^2|\xi|+C_3(1+|\xi|)^{\alpha_1-1}}{\lambda+\frac{1}{2}\sigma(x)^2\xi^2+C_4(1+|\xi|)^{\alpha_1}}\leqq C_{0,1}(1+|\xi|)^{-1}.$$

Therefore we get

$$\left|\frac{\partial_\xi^n\partial_x^m(\lambda-\psi(x,\xi))}{\lambda-\psi(x,\xi)}\right|\leqq C_{m,n}(x)\,(1+|\xi|)^{-n+(2-\alpha_1)m}.$$

In case $\sigma(x)^2\geqq\sigma^2>0$, (4.4) is clear. The proof of (3°) is completed.

Proof of (4°). It is sufficient to verify that the conditions in Theorem 3.4 are satisfied for A_0. First note that $\lambda-\psi(x,\xi)$ is slowly varying. This follows from the fact that $a'(x)$ and $\sigma'(x)$ belong to $\dot{\mathscr{B}}$ and $\partial_x^m\nu(x,y)$, $m\geqq1$ are zero for $|x|\geqq L$. Next we show the estimate (3.7) for $\psi_0(x,\xi)$. If $\eta=\psi_0(x,\xi)$, then we have

$$\mathrm{dist}(\eta,(-\infty,0])=|\eta|\geqq\mathrm{Re}\,\eta\geqq\delta_0(1+|\xi|^2)^{(\alpha_1-\gamma)/2}.$$

Therefore $\{(\lambda-\psi(x,\xi))(1+|\xi|^2)^{-\gamma/2}-\zeta\}^{-1}$ exists on Ξ_ξ if we choose $\tau<\alpha_1^{-1}(\alpha_1-\gamma)$ Then since $\mathrm{Re}(\lambda-\psi(x,\xi))>0$,

$$\begin{aligned}|(\lambda-\psi(x,\xi))\,(1+\xi^2)^{-\gamma/2}-\zeta|&\geqq|\mathrm{Re}(\lambda-\psi(x,\xi))\,(1+\xi^2)^{-\gamma/2}-\mathrm{Re}\,\zeta|\\&\geqq\mathrm{Re}(\lambda-\psi(x,\xi))-\mathrm{Re}\,\zeta\quad\text{if }\mathrm{Re}\,\zeta\geqq0,\\&\geqq\mathrm{Re}(\lambda-\psi(x,\xi))\quad\text{if }\mathrm{Re}\,\zeta\leqq0.\end{aligned}$$

In view of the proof of (2°), we have

$$|(\lambda-\psi(x,\xi))\,(1+\xi^2)^{-\gamma/2}-\zeta|\geqq C_3|(\lambda-\psi(x,\xi))\,(1+\xi^2)^{-\gamma/2}|\quad\text{on }\Xi_\xi.$$

It is easy to see that

$$|\partial_x^\alpha\partial_\xi^\beta(\lambda-\psi(x,\xi))\,(1+\xi^2)^{-\gamma/2}|\leqq C_{\alpha,\beta}|\partial_x^\alpha\partial_\xi^\beta(\lambda-\psi(x,\xi))|\,(1+\xi^2)^{-\gamma/2}.$$

Combining these two estimates and (4.4) it follows that the estimate (3.7) holds. Thus the conditions of Theorem 3.4 holds for $\psi_0(x,\xi)$. The proof of (4°) is completed.

Although the following remark is well known, we shall state here for later use.

REMARK 4.1. (Maximum principle). Let $u(\not\equiv0)$ be a function of $\dot{\mathscr{B}}$. If $u(x_0)=\sup_{x\in\dot{R}^1}u(x)$ for some x_0, then $Au(x_0)\leqq0$. Moreover there exists a point x_1 such that $u(x_1)=u(x_0)$ and $Au(x_1)<0$.

Now we shall show the existence of the Green operator of A and construct its kernel.

Lemma 4.2. *Let the condition (c) be satisfied and A be the operator defined by* (2.1). *Then we have*

(i) *For any* $f\in H_\infty$, *there exists a unique solution* $u\in\dot{\mathcal{B}}$ *of the equation*

$$(\lambda-A)u=f. \tag{4.12}$$

(ii) *For any* $f\in H_\infty$, *define* $G_\lambda f$ *as the unique solution of* (4.12) *Then* G_λ *is a continuous operator from* H_∞ *to* $\dot{\mathcal{B}}$.

Proof. First we note that A can be written in the form

$$Au(x)=\mathcal{F}^{-1}(\psi(x,\ )\mathcal{F}u(\cdot))(x),\ u\in\dot{\mathcal{B}}, \tag{4.13}$$

where $\mathcal{F}u$, $\mathcal{F}^{-1}u$ denote the Fourier transform and the Fourier inverse transform in distribution sense respectively. In fact, for any $u\in\dot{\mathcal{B}}$, there exists a sequence $\{u_n\}$ of $\mathcal{S}$ such that $u_n\to u$ in $\dot{\mathcal{B}}$. Therefore $\mathcal{F}u_n\to\mathcal{F}u$ in $\mathcal{S}'$. Because $\psi(x,\ )\in\mathcal{O}_M$[1], we have $\psi(x,\ )\mathcal{F}u_n\to\psi(x,\ )\mathcal{F}u$ in $\mathcal{S}'$. Define A_0 by

$$A_0u(x)=\mathcal{F}^{-1}((\lambda-\psi(x,\xi))(1+|\xi|^2)^{-\gamma/2}\mathcal{F}u(\xi))(x),\qquad u\in\dot{\mathcal{B}}\ (1<\gamma<m^{2)}).$$

Consider A_0 as a map of L^2 with domain $D(A_0)=\{u\in L^2; A_0u\in L^2\}$. First we show that A_0 is one-to-one. Let u be a function of L^2 such that $A_0u=0$. Set

$$g(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ix\xi}}{(1+|\xi|^2)^{\gamma/2}}d\xi \text{ and } v(x)=g*u(x).^{3)}$$

Since the symbol ψ_0 satisfies the condition of Theorem 3.2 for $m=2-\gamma$, $m_1=2-\gamma$ if $\sigma(x)^2\geqq\sigma^2>0$, $\alpha_1-\gamma$ if $\alpha_1>1$, we have $\|u\|_{s+m-\theta/2}\leqq C_s'\|u\|_{s+m-1/2}(\forall s)$. Therefore $u\in H_\infty$. Combining this and $g\in L^1$, we obtain $v\in\dot{\mathcal{B}}$. So we have

$$(\lambda-A)v(x)=\mathcal{F}^{-1}\{(\lambda-\psi(x,\ \cdot))\ \mathcal{F}v(\cdot)\}(x)=A_0v(x)=0.$$

By Remark 4.1, we have $v(x)=0$, hence $u(x)=0$. Therefore $\ker(A_0)=\{0\}$. Combining this and $\text{index}(A_0)=0$, we have $\text{coker}(A_0)=L^2/Im(A_0)=\{0\}$. Thus A_0 maps $D(A_0)$ onto $L^2(R^1)$. Let f be any function of H_∞. Then since f belongs to L^2, it follows that there exists a unique $v\in L^2$ such that $A_0v=f$. Set $u=g*v$. Then by Theorem 3.2, $u\in\dot{\mathcal{B}}$ and using the same argument as above, we have $(\lambda-A)u=f$. Uniqueness follows easily from Remark 4.1.

Finally we shall prove the continuity of G_λ. By the closed graph theorem, it is sufficient to show the following:

$$f, f_n\in H_\infty, f_n\to f \text{ in } H_\infty,\ G_\lambda f_n\to v \text{ in } \dot{\mathcal{B}},\ \text{then } v=G_\lambda f.$$

Since it follows from (2.1) that A maps $\dot{\mathcal{B}}$ into $\dot{\mathcal{B}}$ continuously, we have $f_n=(\lambda-A)G_\lambda f_n\to(\lambda-A)v$ in $\dot{\mathcal{B}}$. On the other hand, $f_n\to f$ in H_∞, we have

1) $\mathcal{O}_M$ is the totality of C^∞-functions such that each derivative is dominated by some polynomial.

2) $m=2$ if $\sigma(x)^2\geqq\sigma^2>0$, $=\alpha_1$ if $\alpha_1>1$.

3) $g*u(x)=\int_{-\infty}^{\infty}g(x-y)u(y)dy$.

$f=(\lambda-A)v$ in $\mathring{\mathcal{B}}$. The proof is completed.

By Lemma 4.1, there exists a parametrix $Q=q(x, D_x)$ of $\lambda-A$, where $q\in S^{-\alpha_1}_{1,2-\alpha_1}$. Define K by the following equation:

$$K=I-(\lambda-A)Q\,. \tag{4.14}$$

Then K maps $H_{-\infty}$ to H_{∞} continuously. Hence $G_\lambda K=G_\lambda-Q$ maps $H_{-\infty}$ to $\mathcal{B}$ continuously. This implies that $G_\lambda-Q$ maps $\mathcal{E}'$ to $\mathcal{E}$ continuously. By Schwartz' kernel theorem (Schwartz [13]), there exists a C^∞-function $k(x, y)$ such that

$$G_\lambda f(x)-Qf(x)=\int_{-\infty}^{\infty} k(x, y)f(y)dy, \quad f\in\mathcal{D}\,.^{1)} \tag{4.15}$$

Set

$$g_{\lambda,0}(x, y)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(x-y)\xi}q_0(x, \xi)d\xi\,, \tag{4.16}$$

$$g_{\lambda,1}(x, y)=\sum_{j=1}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(x-y)\xi}q_j(x, \xi)\phi\left(\frac{\xi}{t_j}\right)d\xi+k(x,y) \tag{4.17}$$

and

$$g_\lambda(x,y)=g_{\lambda,0}(x, y)+g_{\lambda,1}(x, y) \tag{4.18}$$

Since $q_k\in S^{-\alpha_1-k}_{1,2-\alpha}$, $k=0, 1, 2, \cdots$, $\int_{-\infty}^{\infty} e^{i(x-y)\xi}q_0(x, \xi)d\xi$ and $\int_{-\infty}^{\infty} e^{i(x-y)\xi}q_j(x, \xi)\phi\left(\frac{\xi}{t_j}\right)d\xi$ are well-defined. By (3.2) of Remark 3.1, the right hand side of (4.17) converges uniformly in x, y.

Lemma 4.3. *Under the same condition in Lemma 4.2,*
(i) *G_λ has the kernel representation:*

$$G_\lambda f(x)=\int_{-\infty}^{\infty} g_\lambda(x, y)f(y)dy, \quad f\in\mathcal{D}\,, \tag{4.19}$$

where g_λ is the function defined by (4.18).
(ii) (a) *$g_\lambda(x,y)$ is a nonnegative function,* (b) *$g_{\lambda,0}(x,y)$ is a continuous function and C^∞-function except on the diagonal set,* (c) *$g_{\lambda,1}(x, y)$ is a continuously differentiable function and C^∞-function except on the diagonal set.*

Proof. First note that

$$G_\lambda f(x)= Qf(x)+\int_{-\infty}^{\infty} k(x,y)f(y)dy, \qquad f\in\mathcal{D}\,. \tag{4.20}$$

On the other hand, $Qf(x)$ can be written in the form

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1) $\mathcal{D}$ is the totality of C^∞-functions on R^1 of compact support.

$$Qf(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ix\xi}q(x,\xi)\hat{f}(\xi)d\xi,\qquad f\in\mathcal{S}.$$

Using the fact that $q_k\in S^{-\alpha_1-k}_{1,2-\alpha_1}$, $k=0, 1, 2, \cdots$ and (3.2), we have

$$Qf(x)=\int_{-\infty}^{\infty}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i(x-y)\xi}q_0(x,\xi)d\xi+\sum_{j=1}\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i(x-y)\xi}q_j(x,\xi)\phi\left(\frac{\xi}{t_j}\right)d\xi\right\}f(y)dy.$$

From this and (4.20), we obtain (4.19). The proof of (i) is completed. Next we show (ii). It is clear that $g_{\lambda,0}$ and $g_{\lambda,1}$ are continuous in x, y. So g_λ is continuous in x, y. By Remark 4.1, we have $G_\lambda f\geqq 0$ if $f\geqq 0$. Therefore g_λ is nonnegative. Although the proof of the fact that $g_{\lambda,0}$ and $g_{\lambda,1}$ belong to $C^\infty(R\times R-\Delta)$ is found in [2], for completeness we present the proof. By integration by parts, we have

$$g_{\lambda,0}(x,y)=\frac{(i)^{k+l}}{2\pi(x-y)^{k+l}}\int_{-\infty}^{\infty}e^{i(x-y)\xi}\frac{\partial^{k+l}}{\partial\xi^{k+l}}q_0(x,\xi)d\xi.$$

Since $q_0\in S^{-\alpha_1}_{1,2-\alpha_1}$, we have

$$\begin{aligned}&\left|\frac{\partial^{k+l}}{\partial x^k\partial y^l}\left\{e^{i(x-y)\xi}\frac{\partial^{k+l}}{\partial\xi^{k+l}}q_0(x,\xi)\right\}\right|\\&=\left|(-i\xi)^l\sum_{r=0}^{k}\binom{k}{r}(i\xi)^r e^{i(x-y)\xi}\frac{\partial^{k-r+k+l}q_0(x,\xi)}{\partial x^{k-r}\partial\xi^{k+l}}\right|\\&\leqq C_5|\xi|^l\sum_{r=0}^{k}\binom{k}{r}|\xi|^r(1+|\xi|)^{-\alpha_1-(k+l)+\delta(k-r)}\\&\leqq C_6(1+|\xi|)^{-\alpha_1}.\end{aligned}$$

Therefore we have

$$\begin{aligned}\frac{\partial^{k+l}g_{\lambda,0}}{\partial x^k\partial y^l}(x,y)&=\frac{1}{2\pi}\sum_{r_1=0}^{k}\sum_{r_2=0}^{l}\binom{k}{r_1}\binom{l}{r_2}\left\{\frac{\partial^{r_1+r_2}}{\partial x^{r_1}\partial y^{r_2}}\frac{1}{(y-x)^{k+l}}\right\}\\&\quad\times\int_{-\infty}^{\infty}\frac{\partial^{(k-r_1)+(l-r_2)}}{\partial x^{k-r_1}\partial y^{l-r_2}}\left\{e^{i(y-x)\xi}\frac{\partial^{k+l}q_0}{\partial\xi^{k+l}}(x,\xi)\right\}d\xi.\end{aligned}$$

Thus we have $g_{\lambda,0}\in C^\infty(R\times R-\Delta)$. By the same way we have $g_{\lambda,1}\in C^\infty(R\times R-\Delta)$ $\cap\, C^1(R\times R)$. The proof of (ii) is completed.

5. The estimates of the singularity of $\frac{\partial g_\lambda}{\partial y}$.

In this section we estimate the singularity of $\partial g_\lambda/\partial y$. By Lemma 4.3 it is sufficient to consider $g_{\lambda,0}(x,y)$ only.

Lemma 5.1. *Let* $g_{\lambda,0}(x,y)$ *be the function defined by* (4.16). *Then we have*

$$(5.1)\qquad \frac{\partial g_{\lambda,0}}{\partial y}(x, y) = -\frac{1}{\pi}\int_0^\infty \frac{\xi[\lambda-\psi_R(x, \xi)]\sin(y-x)\xi}{[\lambda-\psi_R(x, \xi)]^2+\psi_I(x, \xi)^2}d\xi$$
$$+\frac{1}{\pi}\int_0^\infty \frac{\xi\psi_I(x, \xi)\cos(y-x)\xi}{[\lambda-\psi_R(x, \xi)]^2+\psi_I(x, \xi)^2}d\xi, \qquad \text{for } x\neq y,$$

where ψ_R and ψ_I are the real part and the imaginary part of ψ respectively.

Proof. Since ψ_R and ψ_I are even and odd functions of ξ respectively, we have

$$g_{\lambda,0}(x, y) = \frac{1}{\pi}\int_0^\infty \frac{[\lambda-\psi_R(x, \xi)]\cos(y-x)\xi}{[\lambda-\psi_R(x, \xi)]^2+\psi_I(x, \xi)^2}d\xi$$
$$+\frac{1}{\pi}\int_0^\infty \frac{\psi_I(x, \xi)\sin(y-x)\xi}{[\lambda-\psi_R(x, \xi)]^2+\psi_I(x, \xi)^2}d\xi .$$

Therefore it suffices to prove that the following integrals

$$I_1 = \int_0^\infty \frac{\xi\{\lambda-\psi_R(x, \xi)\}\sin(y-x)\xi}{[\lambda-\psi_R(x, \xi)]^2+\psi_I(x, \xi)^2}d\xi$$

and

$$I_2 = \int_0^\infty \frac{\xi\psi_I(x, \xi)\cos(y-x)\xi}{[\lambda-\psi_R(x, \xi)]^2+\psi_I(x, \xi)^2}d\xi$$

are uniformly convergent for $|x-y|\geqq\delta>0$. First we show this for I_1. Set

$$F_1(\xi) = \int_0^\xi\left\{\frac{\partial}{\partial\eta}\left(\frac{\eta[\lambda-\psi_R(x, \eta)]}{[\lambda-\psi_R(x, \eta)]^2+\psi_I(x, \eta)^2}\right)\right\}\vee 0\, d\eta$$

and

$$F_2(\xi) = -\int_0^\xi\left\{\frac{\partial}{\partial\eta}\left(\frac{\eta[\lambda-\psi_R(x, \eta)]}{[\lambda-\psi_R(x, \eta)]^2+\psi_I(x, \eta)^2}\right)\right\}\wedge 0\, d\eta .$$

Then F_1 and F_2 are monotone increasing and

$$\frac{\xi[\lambda-\psi_R(x, \xi)]}{[\lambda-\psi_R(x, \xi)]^2+\psi_I(x, \xi)^2} = F_1(\xi)-F_2(\xi) .$$

Note that the set of pseudo-differential operators forms an algebra, that is if $p_i(x, D_x)$ belongs to $S^{m_i}_{\rho,\delta}$, $i=1, 2$, then $p_1(x, D_x)p_2(x, D_x)$ belongs to $S^{m_1+m_2}_{\rho,\delta}$ (see [6]). Using this, we have $\dfrac{\eta[\lambda-\psi_R(x,\eta)]}{[\lambda-\psi_R(x, \eta)]^2+\psi_I(x, \eta)^2}$ belongs to $S^{-m+1}_{1,0}$ for some $m>1$. Hence we get

$$\left|\frac{\partial}{\partial\eta}\left\{\frac{\eta[\lambda-\psi_R(x, \eta)]}{[\lambda-\psi_R(x, \eta)]^2+\psi_I(x, \eta)^2}\right\}\right|\leqq\frac{c_4}{\eta^m} \text{ for } |\eta|\geqq M ,$$

where M is a constant independent of x. According to this and $\lim_{|\xi|\to\infty}\frac{\xi[\lambda-\psi_R]}{[\lambda-\psi_R]^2+\psi_I^2}=0$ uniformly in x, we have $\lim_{\xi\to\infty}F_1(\xi)=\lim_{\xi\to\infty}F_2(\xi)=a$. Set $F_3(\xi)=F_1(\xi)-a$ and $F_4(\xi)=F_2(\xi)-a$. Then F_3 and F_4 converge to 0 uniformly in x and

$$\frac{\xi[\lambda-\psi_R(x,\xi)]}{[\lambda-\psi_R(x,\xi)]^2+\psi_I(x,\xi)^2}=F_3(\xi)-F_4(\xi)\,.$$

Therefore by the second mean value theorem, for $|x-y|\geqq\delta>0$,

$$\begin{aligned}&\left|\int_{N_1}^{N_2}\frac{\xi[\lambda-\psi_R(x,\xi)]\sin(y-x)\xi}{[\lambda-\psi_R(x,\xi)]^2+\psi_I(x,\xi)^2}d\xi\right|\\ &\leqq\left|\int_{N_1}^{N_2}F_3(\xi)\sin(y-x)\xi d\xi\right|+\left|\int_{N_1}^{N_2}F_4(\xi)\sin(y-x)\xi d\xi\right|\\ &\leqq\frac{2}{\delta}\{F_3(N_1)+F_3(N_2)+F_4(N_1)+F_4(N_2)\}\to 0,\ \text{as } N_1,\ N_2\to\infty\,.\end{aligned}$$

The proof for I_2 is the same as above. Q.E.D.

Lemma 5.2. (i) *If* $\sigma(x)^2\geqq\sigma^2>0$ *on* R^1 *or* (ii) $\sigma(x)^2\geqq\sigma^2>0$ *on a neighborhood of the origin and* $\alpha_1>1$, *then*

$$\lim_{x\uparrow 0}\frac{\partial g_{\lambda,0}}{\partial y}(x,0)=\lim_{y\downarrow 0}\frac{\partial g_{\lambda,0}}{\partial y}(0,y)=-\frac{1}{\sigma(0)^2}+C\,, \tag{5.2a}$$

$$\lim_{x\downarrow 0}\frac{\partial g_{\lambda,0}}{\partial y}(x,0)=\lim_{y\uparrow 0}\frac{\partial g_{\lambda,0}}{\partial y}(0,y)=\frac{1}{\sigma(0)^2}+C\,, \tag{5.2b}$$

where
$$C=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\xi\left[-a(0)\xi+\int_{-\infty}^{\infty}\left(\frac{\xi u}{1+u^2}-\xi u\right)n(0,u)du\right]}{[\lambda-\psi_R(0,\xi)]^2+\psi_I(0,\xi)^2}d\xi+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\xi\left[\int_{-\infty}^{\infty}(\xi u-\sin\xi u)n(0,u)du\right]}{[\lambda-\psi_R(0,\xi)]^2+\psi_I(0,\xi)^2}d\xi\,.$$

Proof. By Lemma 5.1, we have

$$\begin{aligned}\frac{\partial g_{\lambda,0}}{\partial y}(x,0)&=\frac{1}{\pi}\int_0^{\infty}\frac{\xi[\lambda-\psi_R(x,\xi)]\sin x\xi}{[\lambda-\psi_R(x,\xi)]^2+\psi_I(x,\xi)^2}d\xi+\frac{1}{\pi}\int_0^{\infty}\frac{\xi\psi_I(x,\xi)\cos x\xi}{[\lambda-\psi_R(x,\xi)]^2+\psi_I(x,\xi)^2}d\xi\\ &\equiv I_1+I_2\,.\end{aligned}$$

First we show $\lim_{x\downarrow 0}I_1=\frac{1}{\sigma(0)^2}$.

Set

$$G(x,\eta)=\left[\lambda x^2+\frac{\sigma(x)^2}{2}\eta^2+x^2\int_{-\infty}^{\infty}\left(1-\cos\frac{\eta u}{x}\right)n(x,u)du\right]^2$$

and

$$H(x,\eta)=\left[-a(x)x\eta+x^2\int_{-\infty}^{\infty}\left\{\frac{\eta u}{x(1+u^2)}-\sin\frac{\eta u}{x}\right\}n(x,u)du\right]^2.$$

Then we have

$$I_1=\frac{1}{\pi}\int_0^{\infty}\frac{\eta\sin\eta\left[\lambda x^2+\frac{\sigma(x)^2}{2}\eta^2+x^2\int_{-\infty}^{\infty}\left(1-\cos\frac{\eta u}{x}\right)n(x,u)du\right]d\eta}{G(X,\eta)+H(x,\eta)}.$$

Since

$$\left|\frac{\partial}{\partial\eta}\left\{\frac{\eta\left[\lambda x^2+\frac{\sigma(x)^2}{2}\eta^2+x^2\int_{-\infty}^{\infty}\left(1-\cos\frac{\eta u}{x}\right)n(x,u)du\right]}{G(x,\eta)+H(x,\eta)}\right\}\right|\leqq\frac{C}{\eta^2}\qquad |\eta|\geqq M,$$

by the same argument as in the proof of Lemma 5.1, we can write

$$\frac{\eta\left[\lambda x^2+\frac{\sigma(x)^2}{2}\eta^2+x^2\int_{-\infty}^{\infty}\left(1-\cos\frac{\eta u}{x}\right)n(x,u)du\right]}{G(x,\eta)+H(x,\eta)}=F_1(\eta)-F_2(\eta)$$

(F_1, F_2 have the same property as F_3, F_4 in the proof of Lemma 5.1).

Using the second mean value theorem, for any $\varepsilon>0$, there exists a constant $N>0$ such that

$$\left|\frac{1}{\pi}\int_N^{\infty}\frac{\eta\sin\eta\left[\lambda x^2+\frac{1}{2}\sigma(x)^2\eta^2+x^2\int_{-\infty}^{\infty}\left(1-\cos\frac{\eta u}{x}\right)n(x,u)du\right]d\eta}{G(x,\eta)+H(x,\eta)}\right|<\varepsilon$$

for any x. Since the integrand is bounded in $[0, N]$,

$$\lim_{x\downarrow 0}\frac{1}{\pi}\int_0^{N}\frac{\eta\sin\eta\left[\lambda x^2+\frac{1}{2}\sigma(x)^2\eta^2+x^2\int_{-\infty}^{\infty}\left(1-\cos\frac{\eta u}{x}\right)n(x,u)du\right]d\eta}{G(x,\eta)+H(x,\eta)}$$
$$=\frac{1}{\pi}\int_0^{N}\frac{2}{\sigma(0)^2}\frac{\sin\eta}{\eta}d\eta.$$

Consequently we obtain $\lim_{x\downarrow 0}I_1=\frac{1}{\sigma(0)^2}$. In case $x<0$, we have

$$I_1=-\frac{1}{\pi}\int_0^{\infty}\frac{\eta\sin\eta\left[\lambda x^2+\frac{1}{2}\sigma(x)^2\eta^2+x^2\int_{-\infty}^{\infty}\left(1-\cos\frac{\eta u}{x}\right)n(x,u)du\right]d\eta}{G(x,\eta)+H(x,\eta)},$$

so $\lim_{x\uparrow 0} I_1 = \frac{1}{\sigma(0)^2}$. For I_2, we devide I_2 into two terms

$$I_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\xi\left[-a(x)\xi + \int_{-\infty}^{\infty}\left(\frac{\xi u}{1+u^2} - \xi u\right) n(x, u)du\right]\cos x\xi}{[\lambda - \psi_R(x, \xi)]^2 + \psi_I(x, \xi)^2} d\xi$$
$$+ \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\xi\left[\int_{-\infty}^{\infty}(\xi u - \sin \xi u) n(x, u)du\right]\cos x\xi}{[\lambda - \psi_R(x, \xi)]^2 + \psi_I(x, \xi)^2} d\xi .$$

Using Lebesgue's convergence theorem,

$$\lim_{x\to 0} I_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\xi\left[-a(0)\xi + \int_{-\infty}^{\infty}\left(\frac{\xi u}{1+u^2} - \xi u\right) n(0, u)\, du\right]}{[\lambda - \psi_R(0, \xi)]^2 + \psi_I(0, \xi)^2} d\xi$$
$$+ \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\xi\left[\int_{-\infty}^{\infty}(\xi u - \sin \xi u) n(0, u)du\right]}{[\lambda - \psi_R(0, \xi)]^2 + \psi_I(0, \xi)^2} d\xi = C .$$

The same estimate shows that

$$\lim_{y\downarrow 0} \frac{\partial g_{\lambda,0}}{dy}(0, y) = -\frac{1}{\sigma(0)^2} + C, \lim_{y\uparrow 0} \frac{\partial g_{\lambda,0}}{\partial y}(0, y) = \frac{1}{\sigma(0)^2} + C .$$

The proof of Lemma 5.2 is completed.

Next we shall consider the case $\sigma(x) = 0$. We shall devide into the following three cases:

(I) $1 < \alpha_2 \leqq \alpha_1 < 2$, (II) $0 < \alpha_2 < 1 < \alpha_1 < 2$ and (III) $1 = \alpha_2 < \alpha_1 < 2$.

We write ψ in the following form according to the above cases.

Case (I). $\psi(x, \xi) = ic_1(x)\xi + \int_{-\infty}^{\infty} (e^{i\xi u} - 1 - i\xi u) n(x, u) du ,$

where $c_1(x) = a(x) + \int_{-\infty}^{\infty} \frac{u^3 n(x, u)}{1+u^2} du .$

Case (II). $\psi(x, \xi) = ic_2(x)\xi + \int_{-\infty}^{0} (e^{i\xi u} - 1) \frac{\nu(x, u)}{|u|^{1+\alpha_2}} du$

$$+ \int_0^{\infty} (e^{i\xi u} - 1 - i\xi u) \frac{\nu(x, u)}{u^{1+\alpha_1}} du ,$$

where $c_2(x) = a(x) + \int_{-\infty}^{0} \frac{\nu(x, u)}{1+u^2} \frac{du}{|u|^{\alpha_2}} + \int_0^{\infty} \frac{u^{2-\alpha_1}}{1+u^2} \nu(x, u) du$

Case (III). $\psi(x, \xi) = ic_3(x)\xi + \int_{-\infty}^{0} \left(e^{i\xi u} - 1 - \frac{i\xi u}{1+u^2}\right) \frac{\nu(x, u)}{|u|^2} du$

$$+\int_0^\infty (e^{i\xi u}-1-i\xi u)\frac{\nu(x,u)}{u^{1+\alpha_1}}du\,,$$

where $c_3(x) = a(x)+\int_0^\infty \frac{u^3 n(x,u)}{1+u^2}du$.

Set $a_1(x,\xi)=\int_0^\infty (1-\cos v)\frac{\nu(x,v/\xi)}{v^{1+\alpha_1}}dv$, $a_2(x,\xi)=\int_{-\infty}^0 (1-\cos v)\frac{\nu(x,v/\xi)}{v^{1+\alpha_2}}dv$,

$$b_1(x,\xi)=\int_0^\infty (\sin v - v)\frac{\nu(x,v/\xi)}{v^{1+\alpha_1}}dv$$

and

$$b_2(x,\xi)=\begin{cases}\int_{-\infty}^0 (\sin v - v)\frac{\nu(x,v/\xi)}{|v|^{1+\alpha_2}}dv, & \text{if } \alpha_2>1\,,\\ \int_{-\infty}^0 \sin v \frac{\nu(x v/\xi)}{|v|^{1+\alpha_2}}dv, & \text{if } \alpha_2<1\,,\\ \int_{-\infty}\left(\sin v - \frac{v}{1+v^2/\xi^2}\right)\frac{\nu(x,v/\xi)}{v^2}dv\frac{1}{\log\xi}, & \text{if } \alpha_2=1\,,\end{cases}$$

Then $\psi(x,\xi)$ can be written in the form

Case (I). $(1<\alpha_2\leqq\alpha_1<2)$.

$$\psi(x,\xi) = -(a_1(x,\xi)\xi^{\alpha_1}+a_2(x,\xi)\xi^{\alpha_2}) + i(b_1(x,\xi)\xi^{\alpha_1}+b_2(x,\xi)\xi^{\alpha_2}+c_1(x)\xi)\,. \tag{5.3}$$

Case (II). $(0<\alpha_2<1<\alpha_1<2)$.

$$\psi(x,\xi)= -(a_1(x,\xi)\xi^{\alpha_1}+a_2(x,\xi)\xi^{\alpha_2}) + i(b_1(x,\xi)\xi^{\alpha_1}+b_2(x,\xi)\xi^{\alpha_2}+c_2(x)\xi)\,. \tag{5.4}$$

Case (III). $(1 = \alpha_2<\alpha_1<2)$.

$$\psi(x,\xi) = -(a_1(x,\xi)\xi^{\alpha_1}+a_2(x,\xi)\xi^{\alpha_2}) + i(b_1(x,\xi)\xi^{\alpha_1}+b_2(x,\xi)\xi\log\xi+c_3(x)\xi)\,. \tag{5.5}$$

Set

$$a_1(x) = \nu(x,0)\int_0^\infty \frac{1-\cos v}{v^{1+\alpha_1}}dv,\ a_2(x) = \nu(x,0)\int_{-\infty}^0 \frac{1-\cos v}{|v|^{1+\alpha_2}}dv\,,$$

$$b_1(x) = \nu(x,0)\int_0^\infty \frac{\sin v - v}{v^{1+\alpha_1}}dv$$

and

$$b_2(x)=\begin{cases}\nu(x,0)\int_{-\infty}^{0}\frac{\sin v-v}{|v|^{1+\alpha_2}}dv, & \text{if } \alpha_2>1\,,\\ \nu(x,0)\int_{-\infty}^{0}\frac{\sin v}{|v|^{1+\alpha_2}}dv, & \text{if } \alpha_2<1\,,\\ \nu(x,0), & \text{if } \alpha_2=1\,.\end{cases}$$

Lemma 5.3. *Let $a_i(x,\xi)$, $a_i(x)$, $b_i(x,\xi)$ and $b_i(x)$, $i=1,2$ be as above. Then we have the following estimates*

$$a_1(x,\xi)-a_1(x)=O(\xi^{-1})\,, \tag{5.6}$$

$$a_2(x,\xi)-a_2(x)=\begin{cases}O(\xi^{-1}), & \text{if } \alpha_2>1\,,\\ O(\xi^{-\alpha_2}), & \text{if } \alpha_2<1\,,\\ O(\xi^{-1}\log\xi), & \text{if } \alpha_2=1\,,\end{cases} \tag{5.7}$$

$$b_1(x,\xi)-b_1(x)=O(\xi^{1-\alpha_1}) \tag{5.8}$$

and

$$b_2(x,\xi)-b_2(x)=\begin{cases}O(\xi^{1-\alpha_2}), & \text{if } \alpha_2>1\,,\\ O(\xi^{-\alpha_2}), & \text{if } \alpha_2<1\,,\\ O((\log\xi)^{-1}), & \text{if } \alpha_2=1\,,\end{cases} \tag{5.9}$$

uniformly in x, as $\xi\to\infty$.

Proof. In the following, we use the notation $\frac{\partial\nu}{\partial y}(x, u/\xi)$ for $\left.\frac{\partial\nu(x,y)}{\partial y}\right|_{y=u/\xi}$. Throughout the proof we assume $\xi>1$. For $a_1(x,\xi)-a_1(x)$, we have

$$\begin{aligned}|a_1(x,\xi)-a_1(x)|&\leqq\left|\int_0^1\frac{1-\cos u}{u^{1+\alpha_1}}(\nu(x,u/\xi)-\nu(x,0))du\right|\\ &\quad+\left|\int_1^\infty\frac{1-\cos u}{u^{1+\alpha_1}}(\nu(x,u/\xi)-\nu(x,0))du\right|\\ &\leqq M_1\xi^{-1}\left\{\int_0^1\frac{1-\cos u}{u^{\alpha_1}}du+\int_1^\infty\frac{du}{u^{\alpha_1}}\right\}\leqq M_1\xi^{-1}\left(\frac{1}{2(3-\alpha_1)}+\frac{1}{\alpha_1-1}\right),\end{aligned}$$

where M_1 is the same constant as in the proof of Lemma 4.1. Hence we have (5.6). For $b_1(x,\xi)-b_1(x)$, we have

$$\begin{aligned}|b_1(x,\xi)-b_1(x)|&\leqq\left|\int_0^1\frac{\sin v-v}{v^{1+\alpha_1}}(\nu(x,v/\xi)-\nu(x,0))dv\right|\\ &\quad+\left|\int_1^{\xi L_1}\frac{\sin v-v}{v^{1+\alpha_1}}(\nu(x,v/\xi)-\nu(x,0))dv\right|\\ &\quad+\left|\int_{\xi L_1}^\infty\frac{\sin v-v}{v^{1+\alpha_1}}\nu(x,0)dv\right|\end{aligned}$$

$$\leqq M_1\left\{\xi^{-1}\int_0^1 \frac{v-\sin v}{v^{\alpha_1}}dv+\xi^{1-\alpha_1}\int_0^{L_1}\frac{dv}{v^{\alpha_1-1}}\right\}+M_1\int_{\xi L_1}^{\infty}\frac{1+v}{v^{1+\alpha_1}}dv$$

$$\leqq M_1\left\{\frac{1}{2(3-\alpha_1)\xi}+\frac{L_1^{2-\alpha_1}}{2-\alpha_1}\frac{1}{\xi^{\alpha_1-1}}+\frac{2}{L_1^{\alpha_1-1}(\alpha_1-2)}\frac{1}{\xi^{\alpha_1-1}}\right\}.$$

Thus we have (5.8). By the same way, we have (5.7) for $\alpha_2>1$. If $\alpha_2<1$, then we have

$$a_2(x, \xi)-a_2(x)=\int_{-1}^{0}\frac{1-\cos u}{|u|^{1+\alpha_2}}[\nu(x, u/\xi)-\nu(x, 0)]du$$
$$+\int_{-\xi L_1}^{-1}\frac{1-\cos u}{|u|^{1+\alpha_2}}[\nu(x, u/\xi)-\nu(x, 0)]du-\int_{-\infty}^{-\xi L_1}\frac{1-\cos u}{|u|^{1+\alpha_2}}\nu(x, 0)du$$
$$=I_1+I_2+I_3\ .$$

For I_1, we have

$$|I_1|=\left|\int_{-1}^{0}\frac{1-\cos u}{|u|^{\alpha_2}}\frac{1}{\xi}\frac{\partial \nu}{\partial y}(x, u/\xi)du\right|=M_1/\xi\int_{-1}^{0}\frac{1-\cos u}{|u|^{\alpha_2}}du=\frac{M_1}{2(3-\alpha_2)}\frac{1}{\xi}.$$

For I_2,

$$|I_2|=\left|\frac{1}{\xi^{\alpha_2}}\int_{-L_1}^{-1/\xi}\frac{1-\cos \xi v}{|v|^{1+\alpha_2}}[\nu(x, v)-\nu(x, 0)]dv\right|$$
$$=\left|\frac{1}{\xi^{\alpha_2}}\int_{-L_1}^{-1/\xi}\frac{1-\cos \xi v}{|v|^{\alpha_2}}\frac{\partial \nu}{\partial y}(x, v')dv\right|\leqq\frac{2M_1}{\xi^{\alpha_2}}\int_{-L_1}^{0}\frac{dv}{|v|^{\alpha_2}}=\frac{2M_1L_1^{1-\alpha_2}}{\alpha_2-1}\frac{1}{\xi^{\alpha_2}},$$

where v' is a mean value such that $v<v'<0$.

For I_3, we get

$$|I_3|\leqq\frac{2M_0}{\alpha_2L_1^{\alpha_2}}\frac{1}{\xi^{\alpha_2}}.$$

Therefore $|a_2(x, \xi)-a_2(x)|\leqq c_5\frac{1}{\xi^{\alpha_2}}$,

where c_5 depends only on L_1, α_2 and M_i, $i=1, 2$.

In case $\alpha_2=1$,

$$|a_2(x, \xi)-a_2(x)|=\left|\int_0^{\xi L_1}\frac{1-\cos y}{y^2}\nu(x, -y/\xi)dy-\nu(x, 0)\int_0^{\infty}\frac{1-\cos y}{y^2}dy\right|$$
$$=\left|\int_0^{\xi L_1}\frac{1-\cos y}{y^2}\{\nu(x, -y/\xi)-\nu(x, 0)\}dy-\nu(x, 0)\int_{\xi L_1}^{\infty}\frac{1-\cos y}{y^2}dy\right|$$

$$\leqq \int_0^1 \frac{1-\cos y}{y^2} |\nu(x, -y/\xi)-\nu(x, 0)| dy + \int_1^{\xi L_1} \frac{1-\cos y}{y^2} |\nu(x, -y/\xi)-\nu(x, 0)| dy$$
$$+\nu(x, 0)\int_{\xi L_1}^{\infty} \frac{1-\cos y}{y^2} dy$$
$$\equiv I_1+I_2+I_3 .$$

For I_1,

$$I_1 \leqq M_1/\xi \int_0^1 \frac{1-\cos y}{y^2} dy \leqq M_1/2\xi .$$

For I_2,

$$I_2 \leqq \frac{M_1}{\xi} \log L_1 \xi \int_1^{\xi L_1} \frac{1-\cos y}{y \log \xi L_1} dy \leqq \frac{2M_1 \log \xi L_1}{\xi} .$$

For I_3, we have

$$I_3 \leqq 2M_0/L_1\xi .$$

Therefore

$$|a_2(x, \xi)-a_2(x)| \leqq c_6 \frac{\log \xi}{\xi} , \tag{5.10}$$

where c_6 depends only on L_1, M_0 and M_1. Thus (5.7) is proved. Finally for $b_2(x, \xi)-b_2(x)$, *if* $\alpha_2>1$ then the same estimate as that of $b_1(x, \xi)-b_1(x)$ holds.

If $\alpha_2<1$, then we have

$$b_2(x, \xi)-b_2(x) = \int_{-1}^{0} \frac{\sin v}{|v|^{1+\alpha_2}} [\nu(x, v/\xi)-\nu(x, 0)]dv$$
$$+\int_{-\xi L_1}^{-1} \frac{\sin v}{|v|^{1+\alpha_2}} [\nu(x, v/\xi)-\nu(x, 0)]dv$$
$$+\int_{-\infty}^{-L_1\xi} \frac{\sin v}{|v|^{1+\alpha_2}} [\nu(x, v/\xi)-\nu(x, 0)]dv$$
$$\equiv I_1+I_2+I_3 .$$

For I_1, we have

$$|I_1| = \frac{M_1}{\xi}\int_{-1}^{0} \frac{dv}{|v|^{\alpha_2}} = \frac{M_1}{1-\alpha_2} \frac{1}{\xi} .$$

For I_2,

$$|I_2| = \left|\frac{1}{\xi^{\alpha_2}}\int_{-L_1}^{-1/\xi}\frac{\sin \xi w}{|w|^{1+\alpha_2}}[\nu(x, w)-\nu(x, 0)]dw\right| \leqq \frac{M_1}{\xi^{\alpha_2}}\int_{-L_1}^{0}\frac{dw}{|w|^{\alpha_2}} \leqq \frac{M_1 L_1^{1-\alpha_2}}{1-\alpha_2}\frac{1}{\xi^{\alpha_2}}.$$

For I_3,

$$|I_3| \leqq 2M_0\int_{-\infty}^{-\xi L_1}\frac{dv}{|v|^{1+\alpha_2}} = \frac{2M_0}{\alpha_2 L_1^{\alpha_2}}\frac{1}{\xi^{\alpha_2}}.$$

Thus we obtain

$$|b_2(x, \xi)-b_2(x)| \leqq c_7 \xi^{-\alpha_2}, \tag{5.11}$$

where c_7 does not depend on $x, \xi \in R^1$.

Finally consider the case $\alpha_2 = 1$.

$$\begin{aligned}
&b_2(x, \xi)-b_2(x)\\
&= \frac{1}{\log \xi}\left\{\int_0^1 \frac{y-\sin y}{y^2}\nu(x, -y/\xi)dy + \int_0^1\left(\frac{y}{1+y^2/\xi^2}-y\right)\frac{\nu(x, -y/\xi)}{y^2}dy\right.\\
&\quad \left. -\int_1^{\xi L_1}\frac{\sin y}{y^2}\nu(x, -y/\xi)dy\right\} + \left\{\frac{1}{\log \xi}\int_1^{\xi L_1}\frac{\nu(x, -y/\xi)}{y(1+y^2/\xi^2)}dy - \nu(x,0)\right\}\\
&\equiv I_1 + I_2 .
\end{aligned}$$

For I_1, we obtain

$$|I_1| \leqq \frac{1}{\log \xi}\left(\frac{M_0}{6}+\frac{M_0}{2\xi^2}+M_0\right).$$

For I_2, we get

$$\begin{aligned}
|I_2| &= \left|\frac{1}{\log \xi L_1}\int_1^{\xi L_1}\left\{\frac{\nu(x, -y/\xi)}{y(1+y^2/\xi^2)} - \frac{\nu(x, 0)}{y}\right\}dy \times \frac{\log \xi L_1}{\log \xi}\right|\\
&\leqq \frac{1}{\log \xi L_1}\int_1^{\xi L_1}\frac{|\nu(x, -y/\xi)-\nu(x, 0)|}{y(1+y^2/\xi^2)} + \frac{\nu(x, 0)}{\log \xi L_1}\int_1^{\xi L_1}\frac{y dy}{\xi^2+y^2}\\
&\leqq \frac{M_1}{\xi \log \xi L_1}\int_1^{\xi L_1}dy + \frac{M_0}{\log \xi L_1}\log\frac{\xi^2(1+L_1^2)}{\xi^2+1} \leq \frac{L_1 M_1 + 1}{\log \xi L_1}.
\end{aligned}$$

So we have

$$|b_2(x, \xi)-b_2(x)| \leqq c_8(\log \xi)^{-1}, \tag{5.12}$$

where c_8 does not depend on $x, \xi \in R^1$. By (5.11) and (5.12), we obtain (5.9), which completes the proof of Lemma 5.3.

Now we can estimate the singularity of $\frac{\hat{g}_{\lambda,0}}{dy}$. Set

(5.13) $$I(x,y)=-\frac{1}{\pi}\int_0^\infty\frac{\xi[\lambda-\psi_R(x,\xi)]\sin(y-x)\xi}{[\lambda-\psi_R(x,\xi)]^2+\psi_I(x,\xi)^2}d\xi$$

and

(5.14) $$J(x,y)=\frac{1}{\pi}\int_0^\infty\frac{\xi\psi_I(x,\xi)\cos(y-x)\xi}{[\lambda-\psi_R(x,\xi)]^2+\psi_I(x,\xi)^2}d\xi\,.$$

In the rest of this paper, we set

$$\Phi_R(x,\xi)=-(a_1(x)\xi^{\alpha_1}+a_2(x)\xi^{\alpha_2})$$

and

$$\Phi_I(x,\xi)=\begin{cases}(b_1(x)\xi^{\alpha_1}+b_2(x)\xi^{\alpha_2}+c_1(x)\xi), & \text{if } \alpha_2>1\,,\\ (b_1(x)\xi^{\alpha_1}+b_2(x)\xi^{\alpha_2}+c_2(x)\xi), & \text{if } \alpha_2<1\,,\\ (b_1(x)\xi^{\alpha_1}+b_2(x)\xi\log\xi+c_3(x)\xi), & \text{if } \alpha_2=1\end{cases}$$

Define $h_1(x)$ by

$$h_1(x)=\frac{\pi}{2\Gamma(\alpha_1+1)[a_1(x)^2+b_1(x)^2]}\,.$$

Then we can show the following lemma.

Lemma 5.4. *Assume that* $\alpha_1>\alpha_2$ *Let n be the largest positive integer such that* $\alpha_1-2+n(\alpha_1-\alpha_2)\leqq 0$ *Then we have*
(i) *//* $\alpha_2\neq 1$, *then*

(5.15) $$\begin{aligned}I(x,y)=&-\frac{h_1(x)}{\pi}\Gamma(2-\alpha_1)\operatorname{sgn}(y-x)|y-x|^{\alpha_1-2}\\&+\sum_{k=1}\sum_{j=0}^{k}\left\{D^{(j)}_{\alpha_2,k+1}(x)\int_0^\infty\xi^{j(1-\alpha_2)+1-\alpha_1-k(\alpha_1-\alpha_2)}\sin\xi d\xi\right\}\\&\times\operatorname{sgn}(y-x)|y-x|^{\alpha_1-2+k(\alpha_1-\alpha_2)+j(\alpha_2-1)}\\&+v_5(x,y)|y-x|^{2\alpha_1-3}+v_6(x,y)\,,\end{aligned}$$

(5.16) $$\begin{aligned}J(x,y)=&-\frac{h_1(x)}{\pi}\Gamma(2-\alpha_1)|y-x|^{\alpha_1-2}\\&+\sum_{k=1}^{n}\sum_{j=0}^{k}\left\{E^{(j)}_{\alpha_2,k+1}(x)\int_0^\infty\xi^{j(1-\alpha_2)+1-\alpha_1-k(\alpha_1-\alpha_2)}\cos\xi d\xi\right\}\\&\times|y-x|^{\alpha_1-2+k(\alpha_1-\alpha_2)+j(\alpha_2-1)}+v_7(x,y)|y-x|^{2\alpha_1-3}\\&+v_8(x,y)\,,\end{aligned}$$

where $\{D^{(j)}_{\alpha_2,k+1}\}_{0\leqq j\leqq k}$, , *are bounded continuous functions.*
(ii) *If* $\alpha_2=1$, *then*

(5.17)
$$I(x,y)=-\frac{h_1(x)}{\pi}\Gamma(2-\alpha_1)\operatorname{sgn}(y-x)|y-x|^{\alpha_1-2}$$
$$+\sum_{k=1}^{n}\sum_{r=0}^{k}\left\{\sum_{j=r}^{k}D_{1,k+1}^{(j)}(x)\int_0^\infty(\log\xi)^{j-r}\xi^{1-\alpha_1-k(\alpha_1-\alpha_2)}\sin\xi d\xi\right\}$$
$$\times\left(\log\frac{1}{|y-x|}\right)^r\operatorname{sgn}(y-x)|y-x|^{\alpha_1-2+k(\alpha_1-\alpha_2)}$$
$$+v_5(x,y)|y-x|^{2\alpha_1-3}+v_6(x,y),$$

(5.18)
$$J(x,y)=-\frac{h_1(x)}{\pi}\Gamma(2-\alpha_1)|y-x|^{\alpha_1-2}$$
$$+\sum_{k=1}\sum_{r=0}^{k}\left\{\sum_{j=r}^{k}E_{1,k+1}^{(j)}(x)\int_0^\infty(\log\xi)^{j-r}\xi^{1-\alpha_1-k(\alpha_1-\alpha_2)}\cos\xi d\xi\right\}$$
$$\times\log\left(\frac{1}{|y-x|}\right)^r|y-x|^{\alpha_1-2+k(\alpha_1-\alpha_2)}$$
$$+v_7(x,y)|y-x|^{2\alpha_1-3}+v_8(x,y),$$

where $\{D_{1,k+1}^{(j)}(x)\}_{0\leq j\leq k,1\leq k\leq n}$, $\{E_{1,k+1}^{(j)}(x)\}_{0\leq j\leq k,1\leq k\leq n}$ *and* v_j, $5\leqq j\leqq 8$ *are bounded continuous functions. In case* $\alpha_1-2+n(\alpha_1-\alpha_2)=0$, $|y-x|^{\alpha_1-2+n(\alpha_1-\alpha_2)}$ *should be replaced by* $\log\left(\frac{1}{|y-x|}\vee 1\right)$.

Proof. (i) First we show (5.15). Denote by $c(x)$ for $c_1(x)$ or $c_2(x)$. Then $I(x,y)$ can be written as follows:

$$I(x,y)=-\frac{1}{\pi}\int_0^\infty\frac{\lambda\xi\sin(y-x)\xi d\xi}{[\lambda+a_1(x,\xi)\xi^{\alpha_1}+a_2(x,\xi)\xi^{\alpha_2}]^2+[b_1(x,\xi)\xi^{\alpha_1}+b_2(x,\xi)\xi^{\alpha_2}+c(x)\xi]^2}$$
$$-\frac{1}{\pi}\int_0^\infty\frac{\sin(y-x)\xi d\xi}{[\lambda+a_1(x,\xi)\xi^{\alpha_1}+a_2(x,\xi)\xi^{\alpha_2}]^2+[b_1(x,\xi)\xi^{\alpha_1}+b_2(x,\xi)\xi^{\alpha_2}+c(x)\xi]^2}.$$

The first term is a bounded continuous function. Denote by $I_1(x,y)$ the second term. Then we have

$$I_1(x,y)=-\frac{1}{\pi}\int_0^\infty\frac{[a_1(x)\xi^{1+\alpha_1}+a_2(x)\xi^{1+\alpha_2}]\sin(y-x)\xi d\xi}{[\lambda-\Phi_R(x,\xi)]^2+\Phi_I(x,\xi)^2}$$
$$-\frac{1}{\pi}\int_0^\infty\left\{\frac{a_1(x,\xi)\xi^{1+\alpha_1}+a_2(x,\xi)\xi^{1+\alpha_2}}{(\lambda-\psi_R(x,\xi))^2+\psi_I(x,\xi)^2}-\frac{a_1(x)\xi^{1+\alpha_1}+a_2(x)\xi^{1+\alpha_2}}{(\lambda-\Phi_R(x,\xi))^2+\Phi_I(x,\xi)^2}\right\}$$
$$\times\sin(y-x)\xi d\xi$$
$$\equiv I_2(x,y)+I_3(\chi,y)$$

For the estimate of $I_2(x,y)$, we employ the method of Ikeda-S.Watanabe [3 p. 165]. Set $\tilde{b}_2(x,\xi)=b_2(x)+c(x)\xi^{1-\alpha_2}$. Define

$$A_1(x, \xi) = a_1(x),\ B_1(x, \xi) = a_2(x),\ D_{\alpha_2,k}(x, \xi) = \frac{A_k(x,\xi)}{a_1(x)^2+b_1(x)^2},$$

$$A_{k+1}(x, \xi) = B_k(x, \xi)-\frac{2(a_1(x)a_2(x)+b_2(x)\tilde{b}_2(x, \xi))}{a_1(x)^2+b_1(x)^2}A_k(x, \xi)$$

and

$$B_{k+1}(x, \xi) = -\frac{a_2(x)^2+\tilde{b}_2(x, \xi)^2}{a_1(x)^2+b_1(x)^2}A_k(x, \xi),\ 1\leqq k\leqq n+1\,.$$

Then we have the following formula:

$$(5.19)\quad \int_0^\infty \sin(y-x)\xi\frac{A_k(x, \xi)\xi^{1+\alpha_1-(k-1)(\alpha_1-\alpha_2)}+B_k(x, \xi)\xi^{1+\alpha_1-k(\alpha_1-\alpha_2)}}{(\lambda+a_1(x)\xi^{\alpha_1}+a_2(x)\xi^{\alpha_2})^2+(b_1(x)\xi^{\alpha_1}+b_2(x)\xi^{\alpha_2}+c(x)\xi)^2}d\xi$$
$$= \int_0^\infty D_{\alpha_2,k}(x, \xi)\sin(y-x)\xi\cdot\xi^{1-\alpha_1-(k-1)(\alpha_1-\alpha_2)}d\xi$$
$$+\int_0^\infty \sin(y-x)\xi\frac{A_{k+1}(x, \xi)\xi^{1+\alpha_1-k(\alpha_1-\alpha_2)}+B_{k+1}(x, \xi)\xi^{1+\alpha_1-(k+1)(\alpha_1-\alpha_2)}}{(\lambda+a_1(x)\xi^{\alpha_1}+a_2(x)\xi^{\alpha_2})^2+(b_1(x)\xi^{\alpha_1}+b_2(x)\xi^{\alpha_2}+c(x)\xi)^2}d\xi$$
$$+e_k(x, y)\,,$$

where $e_k(x, y)$ is a bounded continuous function. From this formula, it is easy to see that $D_{\alpha_2,k}(x, \xi)$ is a polynomial of degree k in $\xi^{1-\alpha_2}$. Therefore $D_{\alpha_2,k}(x,\xi)$ can be written in the form

$$D_{\alpha_2,1}(x, \xi) = \frac{a_1(x)}{a_1(x)^2+b_1(x)^2}\,,$$
$$D_{\alpha_2,k+1}(x, \xi) = D^{(0)}_{\alpha_2,k+1}(x)(\xi^{1-\alpha_2})^k+\cdots+D^{(k)}_{\alpha_2,k+1}(x)\,,\qquad 1\leqq k\leqq n\,,$$

where the coefficients $D^{(j)}_{\alpha_2,k+1}(x)$, $0\leqq j\leqq k$, are bounded continuous functions. Hence we have

$$I_2(x, y) = \sum_{k=1}\sum_{j=0} D^{(j)}_{\alpha_2,k}(x)\int_0^\infty \xi^{1-\alpha_1-(k-1)(\alpha_1-\alpha_2)}(\xi^{1-\alpha_2})^j\sin(y-x)\xi d\xi+v_8(x,y)$$
$$=\sum_{k=1}^{n+1}\sum_{j=0}^{k} D^{(j)}_{\alpha_2,k}(x)|y-x|^{\alpha_1-2+(k-1)(\alpha_1-\alpha_2)+j(\alpha_2-1)}$$
$$\times\operatorname{sgn}(y-x)\int_0^\infty \xi^{1-\alpha_1-(k-1)(\alpha_1-\alpha_2)+j(1-\alpha_2)}\sin\xi d\xi+v_8(x,y)\,,$$

where v_8 is a bounded continuous function. Noting that $D^{(1)}_{\alpha_2,1}(x)\equiv 0$ and $D^{(0)}_{\alpha_2,1}(x)=\dfrac{a_1(x)}{a_1(x)^2+b_1(x)^2}$, we obtain

$$I_2(x, y) = -\frac{h_1(x)}{\pi}\Gamma(2-\alpha_1)\operatorname{sgn}(y-x)|y-x|^{\alpha_1-2}$$
$$+\sum_{k=2}^{n+1}\sum_{j=0}^{k} D^{(j)}_{\alpha_2,k}(x)|y-x|^{\alpha_1-2+(k-1)(\alpha_1-\alpha_2)+j(\alpha_2-1)}$$

$$\times \operatorname{sgn}(y-x) \int_0^\infty \xi^{1-\alpha_1-(k-1)(\alpha_1-\alpha_2)+j(1-\alpha_2)} \sin \xi d\xi + v_8(x,y) .$$

For $I_3(x, y)$, we have

$$(5.20)\quad I_3(x, y) = -\frac{1}{\pi}\int_0^\infty \frac{(a_1(x, \xi)-a_1(x))\xi^{1+\alpha_1}+(a_2(x, \xi)-a_2(x))\xi^{1+\alpha_2}}{(\lambda-\psi_R(x, \xi))^2+\psi_I(x, \xi)^2} \sin(y-x)\xi d\xi$$

$$+\frac{1}{\pi}\int_0^\infty \left\{\frac{1}{(\lambda-\psi_R(x, \xi))^2+\psi_I(x, \xi)^2} - \frac{1}{(\lambda-\Phi_R(x, \xi))^2+\Phi_I(x, \xi)^2}\right\}$$

$$\times \xi\Phi_R(x, \xi) \sin(y-x)\xi d\xi$$

$$\equiv I_4(x, y)+I_5(x, y).$$

Using Lemma 5.3, it is easy to see that $I_4(x, y)$ is a bounded continuous function. For I_5, set

$$\begin{aligned}
H_1(x, \xi) &= (a_1(x, \xi)-a_1(x))\xi^{\alpha_1}+(a_2(x, \xi)-a_2(x))\xi^{\alpha_2} , \\
(5.21)\quad H_2(x, \xi) &= 2\lambda-(a_1(x,\xi)+a_1(x))\xi^{\alpha_1}-(a_2(x, \xi)+a_2(x))\xi^{\alpha_2} , \\
K_1(x, \xi) &= (b_1(x, \xi)-b_1(x))\xi^{\alpha_1}+(b_2(x, \xi)-b_2(x))\xi^{\alpha_2} , \\
K_2(x, \xi) &= (b_1(x, \xi)+b_1(x))\xi^{\alpha_1}+(b_2(x, \xi)+b_2(x))\xi^{\alpha_2}+2c(x)\xi
\end{aligned}$$

and

$$\Xi(x, y; \xi) = \frac{\xi\Phi_R(x, \xi) \sin(y-x)\xi}{\{(\lambda-\psi_R(x, \xi))^2+\psi_I(x, \xi)^2\}\ \{(\lambda-\Phi_R(x,\xi))^2+\Phi_I(x, \xi)^2\}} .$$

Then we have

$$I_5(x, y) = \frac{1}{\pi}\int_0^\infty \frac{H_1(x, \xi)H_2(x, \xi)}{\Xi(x, y; \xi)} d\xi + \frac{1}{\pi}\int_0^\infty \frac{K_1(x, \xi)K_2(x, \xi)}{\Xi(x, y; \xi)} d\xi$$

$$\equiv I_6+I_7.$$

By Lemma 5.3, it is easy to see that I_6 is bounded continuous. For the estimate of I_7, set

$$a_1\left(x, \frac{\eta}{y-x}\right)-a_1(x) = \bar{a}_1(x, y; \eta)\frac{|y-x|}{\eta} ,$$

$$b_1\left(x, \frac{\eta}{y-x}\right)-b_1(x) = \bar{b}_1(x, y; \eta)\frac{|y-x|^{\alpha_1-1}}{\eta^{\alpha_1-1}} ,$$

$$a_2\left(x, \frac{\eta}{y-x}\right)-a_2(x) = \begin{cases} \bar{a}_2(x, y; \eta)\dfrac{|y-x|}{\eta} , & \text{if } \alpha_2>1 , \\ \bar{a}_2(x, y; \eta)\dfrac{|y-x|}{\eta^{\alpha_2}} , & \text{if } \alpha_2<1 , \end{cases}$$

and

$$b_2\left(x, \frac{\eta}{y-x}\right)-b_2(x)=\begin{cases}\bar{b}_2(x, y; \eta)\dfrac{|y-x|^{\alpha_2-1}}{\eta^{\alpha_2-1}}, & \text{if } \alpha_2>1, \\ \bar{b}_2(x, y; \eta)\dfrac{|y-x|^{\alpha_2}}{\eta^{\alpha_2}}, & \text{if } \alpha_2<1.\end{cases}$$

Then we have

$$I_7(x, y) = -\frac{\operatorname{sgn}(y-x)}{\pi}|y-x|^{2\alpha_1-3}\int_0^\infty R(x,y \quad \eta) \sin \eta d\eta ,$$

where

$$R(x, y; \eta) = \frac{R_1(x, y; \eta)R_2(x,y, \eta)R_3(x, y, \eta)}{(S_1(x,y; \eta)^2+S_2(x, y; \eta)^2)(T_1(x, y \ \eta)^2+T_2(x, y; \eta)^2)},$$

$$R_1(x, y; \eta) = a_1(x)\eta^{1+\alpha_1}+a_2(x)\eta^{1+\alpha_2}(y-x)^{\alpha_1-\alpha_2},$$

$$R_2(x, y; \eta) = \bar{b}_1(x, y; \eta)\eta+\bar{b}_2(x, y; \eta)\eta^{\beta}(y-x)^{\gamma},$$

$$R_3(x, y; \eta) = \left(b\left(x, \frac{\eta}{y-x}\right)+b_1(x)\right)\eta^{\alpha_1}+\left(b_2\left(x, \frac{\eta}{y-x}\right)+b_2(x)\right)\eta^{\alpha_2}(y-x)^{\alpha_1-\alpha_2}$$
$$+2c(x)\eta(y-x)^{\alpha_1-1},$$

$$S_1(x, y; \eta) = \lambda(y-x)^{\alpha_1}+a_1\left(x, \frac{\eta}{y-x}\right)\eta^{\alpha_1}+a_2\left(x, \frac{\eta}{y-x}\right)\eta^{\alpha_2}(y-x)^{\alpha_1-\alpha_2},$$

$$S_2(x, y; \eta) = b_1\left(x, \frac{\eta}{y-x}\right)\eta^{\alpha_1}+b_2\left(x, \frac{\eta}{y-x}\right)\eta^{\alpha_2}(y-x)^{\alpha_1-\alpha_2}+c(x)\eta(y-x)^{\alpha_1-1},$$

$$T_1(x, y; \eta) = \lambda(y-x^{\alpha_1}+a_1(x)\eta^{\alpha_1}+a_2(x)\eta^{\alpha_2}(x-x)^{\alpha_1-\alpha_2}$$

and

$$T_2(x, y; \eta) = b_1(x)\eta^{\alpha_1}+b_2(x)\eta^{\alpha_2}(y-x)^{\alpha_1-\alpha_2}+c(x)\eta(y-x)^{\alpha_1-1}.$$

In the above expressions, we set $\beta=\alpha_1-1+(\alpha_1-\alpha_2)$ if $\alpha_2>1$, $=2(\alpha_1-1)$ if $\alpha_2<1$ and $\gamma=\alpha_2\vee 1$. Applying the second mean value theorem, we obtain the integral $\int_0^\infty R(x, y; \eta)d\eta$ is uniformly convergent in x, y. Hence $\int_0^\infty R(x, y; \eta)d\eta$ is bounded continuous. Thus we have (5.15). By the same argument as that of $I(x,y)$, we obtain (5.16).

Next we show (ii). $I(x,y)$ can be written as follows:

$$I(x, y) = -\frac{1}{\pi}\int_0^\infty \frac{(a_1(x, \xi)\xi^{1+\alpha_1}+a_2(x, \xi)\xi^{1+\alpha_2})}{(\lambda-\psi_R(x, \xi))^2+\psi_I(x, \xi)^2}\sin(y-x)\xi d\xi+v_9(x, y),$$

where v_9 is bounded continuous. Denote by $I_1(x,y)$ the first term of the right hand side of the above expression. Then we get

$$I_1(x, y) = -\frac{1}{\pi}\int_0^\infty \frac{(a_1(x)\xi^{1+\alpha_1}+a_2(x)\xi^{1+\alpha_2})\sin(y-x)\xi}{(\lambda-\Phi_R(x, \xi))^2+\Phi_I(x, \xi)^2}d\xi$$

$$-\frac{1}{\pi}\int_0^\infty \left\{\frac{a_1(x,\xi)\xi^{1+\alpha_1}+a_2(x,\xi)\xi^{1+\alpha_2}}{(\lambda-\psi_R(x,\xi))^2+\psi_I(x,\xi)^2}-\frac{a_1(x)\xi^{1+\alpha_1}+a_2(x)\xi^{1+\alpha_2}}{(\lambda-\Phi_R(x,\xi))^2+\Phi_I(x,\xi)^2}\right\}$$
$$\times \sin(y-x)\xi d\xi$$
$$\equiv I_2(x,y)+I_3(x,y).$$

For the estimate of $I_2(x,y)$, we set $\tilde{b}_2(x,\xi)=b_2(x)\log\xi+c(x)$, $A(x,\xi)=a_1(x)$, $B_1(x,\xi)=a_2(x)$, $D_{1,k}(x,\xi)=\dfrac{A_k(x,\xi)}{a_1(x)^2+b_1(x)^2}$,

$$A_{k+1}(x,\xi)=B_k(x,\xi)-\frac{2(a_1(x)a_2(x)+b_1(x)\tilde{b}_2(x,\xi))}{a_1(x)^2+b_1(x)^2}A_k(x,\xi)$$

and

$$B_{k+1}(x,\xi)=-\frac{a_2(x)^2+\tilde{b}_2(x,\xi)^2}{a_1(x)^2+b_1(x)^2}A_k(x,\xi), \qquad 1\leqq k\leqq n+1.$$

Then by (5.19), we get

$$\int_0^\infty \sin(y-x)\xi\frac{A_k(x,\xi)\xi^{1+\alpha_1-(k-1)(\alpha_1-\alpha_2)}+B_k(x,\xi)\xi^{1+\alpha_1-k(\alpha_1-\alpha_2)}}{(\lambda+a_1(x)\xi^{\alpha_1}+a_2(x)\xi^{\alpha_2})^2+(b_1(x)\xi^{\alpha_1}+b_2(x)\xi\log\xi+c(x)\xi)^2}d\xi$$
$$=\int_0^\infty D_{1,k}(x,\xi)\sin(y-x)\xi\cdot\xi^{1-\alpha_1-(k-1)(\alpha_1-\alpha_2)}d\xi$$
$$+\int_0^\infty \sin(y-x)\xi\frac{A_{k+1}(x,\xi)\xi^{1+\alpha_1-k(\alpha_1-\alpha_2)}+B_{k+1}(x,\xi)\xi^{1+\alpha_1-(k+1)(\alpha_1-\alpha_2)}}{(\lambda+a_1(x)\xi^{\alpha_1}+a_2(x)\xi^{\alpha_2})^2+(b_1(x)\xi^{\alpha_1}+b_2(x)\xi\log\xi+c(x)\xi)^2}d\xi$$
$$+e_k(x,y),$$

where e_k is bounded continuous. From this formula, it is clear that $D_{1,k}(x,\xi)$ is a polynomial of degree k in $\log\xi$. Hence $D_{1,k}(x,\xi)$ can be written in the form

$$D_{1,1}(x,\xi)=\frac{a_1(x)}{a_1(x)^2+\tilde{b}_1(x)^2},$$
$$D_{1,k+1}(x,\xi)=D_{1,k+1}^{(0)}(x)(\log\xi)^k+\cdots+D_{1,k+1}^{(k)}(x), \qquad 1\leqq k\leqq n,$$

where the coefficients $D_{1,k+1}^{(j)}$, $0\leqq j\leqq k$ are bounded continuous functions. Therefore we have

$$I_2(x,y)=\sum_{k=1}^{n+1}\sum_{j=0}^{k}D_{1,k}^{(j)}(x)\int_0^\infty \xi^{1-\alpha_1-(k-1)(\alpha_1-\alpha_2)}(\log\xi)^j\sin(y-x)\xi d\xi+v_8(x,y)$$
$$=\sum_{k=1}^{n+1}\sum_{j=0}D_{1,k}^{(j)}(x)\operatorname{sgn}(y-x)|y-x|^{\alpha_1-2+(k-1)(\alpha_1-\alpha_2)}\left(\log\frac{1}{|y-x|}\right)^r$$
$$\times\sum_{r=0}\binom{j}{r}\int_0^\infty(\log\xi)^{j-r}\xi^{1-\alpha_1-(k-1)(\alpha_1-\alpha_2)}\sin\xi d\xi+v_8(xy),$$

where $v_8(x,y)$ is a bounded continuous function.

Noting that $D_{1,1}^{(1)}(x)=0$, $D_{1,1}^{(0)}(x)=\frac{a_1(x)}{a_1(x)^2+b_1(x)^2}$, we obtain

$$I_2(x, y) = -\frac{h_1(x)}{\pi}\Gamma(2-\alpha_1)\mathrm{sgn}(y-x)|y-x|^{\alpha_1-2}$$
$$+\sum_{k=2}\sum_{r=0}\left\{\sum_{j=r}^{k}\binom{j}{r}D_{1,k}^{(j)}(x)\int_0^\infty(\log\xi)^{j-r}\xi^{1-\alpha_1-(k-1)(\alpha_1-\alpha_2)}\sin\xi d\xi\right\}$$
$$\times\left(\log\frac{1}{|y-x|}\right)^r\mathrm{sgn}(y-x)|y-x|^{\alpha_1-2+(k-1)(\alpha_1-\alpha_2)}+v_8(x,y)\,.$$

For I_3, we devide $I_3(x, y)$ into two terms: $I_3(x, y)=I_4(x,y)+I_5(x, y)$, where I_4, I_5 are defined by (5.20). By Lemma 5.3, it is easy to see that $I_4(x,y)$ is a bounded continuous function. For I_5, define $H_1(x, \xi)$, $H_2(x, \xi)$, $K_1(x, \xi)$ and $\Xi(x, y; \xi)$ by (5.21). We define $K_2(x, \xi)$ by

$$K_2(x, \xi) = (b_1(x, \xi)+b_1(x))\xi^{\alpha_1}+(b_2(x, \xi)+b_2(x))\xi\log\xi+2c(x)\xi\,.$$

Then we get

$$I_5(x, y)=\frac{1}{\pi}\int_0^\infty\frac{H_1(x, \xi)H_2(x, \xi)}{\Xi(x, y; \xi)}d\xi+\frac{1}{\pi}\int_0^\infty\frac{K_1(x, \xi)K_2(x, \xi)}{\Xi(x, y; \xi)}d\xi$$
$$\equiv I_6+I_7.$$

Using Lemma 5.3, we can show that I_6 is bounded continuous. For I_7, set

$$a_1\left(x, \frac{\eta}{y-x}\right)-a_1(x)=\tilde{a}_1(x,y;\eta)\frac{|y-x|}{\eta}\,,$$
$$b_1\left(x, \frac{\eta}{y-x}\right)-b_1(x)=\tilde{b}_1(x,y;\eta)\frac{|y-x|^{\alpha_1-1}}{\eta^{\alpha_1-1}}\,,$$
$$a_2\left(x, \frac{\eta}{y-x}\right)-a_2(x)=\tilde{a}_2(x,y;\eta)\frac{|y-x|}{\eta}\log\frac{\eta}{|y-x|}$$

and

$$b_2\left(x, \frac{\eta}{y-x}\right)-b_2(x)=\tilde{b}_2(x,y;\eta)\frac{1}{\log\frac{\eta}{|y-x|}}\,.$$

Substituting this into I_7 and applying the second mean value theorem to I_7, we obtain

$$I_7(x, y) = -\frac{\mathrm{sgn}(y-x)}{\pi}|y-x|^{2\alpha_1-3}v_{10}(x, y)\,,$$

where v_{10} is bounded continuous. Thus we have (5.17). By the same way, we obtain (5.18). This completes the proof.

Lemma 5.5. *If $\sigma(x)\equiv 0$ and $\alpha_1>\alpha_2$, $\alpha_1>1$, then $g_\lambda(x,y)$ satisfies one of the following conditions:*

(A) $\lim_{y\uparrow x}\frac{\partial g_\lambda}{\partial y}(x,y)=\frac{\partial g_\lambda}{\partial y}(x,x-)$ *exists finitely and* $\lim_{y\downarrow x}\frac{\partial g_\lambda}{\partial y}(x,y)=-\infty$.

$\lim_{x\downarrow y}\frac{\partial g_\lambda}{\partial y}(x,y)=\frac{\partial g_\lambda}{\partial y}(y+,y)$ *exists finitely and* $\lim_{x\uparrow y}\frac{\partial g_\lambda}{\partial y}(x,y)=-\infty$.

(B) *There exist nonnegative constants c_1, c_2, α, β, r and ε_0 such that $c_1>1$, $c_2>1$, $0<\alpha<\beta<1$ and*

$$\varepsilon^\beta/c_1<g_\lambda(x,x)-g_\lambda(x,x-\varepsilon)<c_1\varepsilon^\beta,$$
$$\varepsilon^\alpha\Big(\log\frac{1}{\varepsilon}\Big)^r\Big/c_2<g_\lambda(x,x)-g_\lambda(x,x+\varepsilon)<c_2\varepsilon^\alpha\Big(\log\frac{1}{\varepsilon}\Big)^r,$$

for any $\varepsilon\in(0,\varepsilon_0]$ and x.

Proof. By virtue of Lemma 5.4, we have

$$\frac{\partial g_\lambda}{\partial y}(x,y)=\begin{cases}O(|y-x|^{\alpha_1-2}), & \text{as } y\downarrow x,\\ o(|y-x|^{\alpha_1-2}), & \text{as } y\uparrow x.\end{cases}$$

Therefore only two cases occur: (i) $\lim_{y\uparrow x}\frac{\partial g_\lambda}{\partial y}(x,y)$ and $\lim_{x\downarrow y}\frac{\partial g_\lambda}{\partial y}(x,y)$ exist finitely or (ii) $\lim_{y\uparrow x}\frac{\partial g_\lambda}{\partial y}(x,y)$ and $\lim_{x\downarrow y}\frac{\partial g_\lambda}{\partial y}(x,y)$ do not exist. If the inequality $g_{\lambda,0}(x,x)-g_{\lambda,0}(x,y)\geqq 0$ holds (we show this soon later), then since $\alpha_1<2$ we have $\lim\frac{\partial g_\lambda}{\partial y}(x,y)=\lim\frac{\partial g_\lambda}{\partial y}(x,y)=-\infty$. Therefore if (i) occurs, it is nothing but the case (A). Let (ii) occur. Then $\frac{\partial g_\lambda}{\partial y}(x,y)=O\Big(|x-y|^{\alpha_3}\Big(\log\frac{1}{|x-y|}\Big)^r\Big)$ for some α_3: $\alpha_1-2<\alpha_3<0$ and $r\geqq 0$. Therefore we have

$$g_\lambda(x,x)-g_\lambda(x,x+\varepsilon)=\int_{x+\varepsilon}^x\frac{\partial g_\lambda}{\partial y}(x,y)dy=O(\varepsilon^{\alpha_1-1})$$

and

$$g_\lambda(x,x)-g_\lambda(x,x-\varepsilon)=\int_{x-\varepsilon}^x\frac{\partial g_\lambda}{\partial y}(x,y)dy=O\Big(\varepsilon^{\alpha_3+1}\Big(\log\frac{1}{\varepsilon}\Big)^r\Big).$$

Since $0<\alpha_1-1<\alpha_3+1<1$, it follows that the case (B) holds. Therefore it sufficies to prove that $g_{\lambda,0}(x,x)-g_{\lambda,0}(x,y)\geqq 0$ for all x, y. Let x be any point in R^1 and fix it. Set

$$g_\lambda^{(x)}(z)=\frac{1}{2\pi}\int_{-\infty}^\infty e^{-iz\xi}\frac{d\xi}{\lambda-\psi(x,\xi)}.$$

Let $\{X_t^{(x)}, P_y^{(x)}, y \in R^1\}$ be a Lévy process whose exponent $\psi^{(x)}(\xi)$ is $\psi(x, \xi)$. Then $g_\lambda^{(x)}(z)$ is the density of the Green measure of Lévy process $X^{(x)}$ with respect to Lebesgue measure. Let $G_\lambda^{(x)}$ be the corresponding Green operator and $\tilde{x}$ be any point such that $\tilde{x} \neq 0$. Let B be an interval containing the origin and $B \cap \{\tilde{x}\} = \phi$. Then for any Borel set $A(\subset B)$,

$$\int_A g_\lambda^{(x)}(y-\tilde{x})dy = \int_A E_{\tilde{x}}^{(x)}[e^{-\lambda\sigma_B} g_\lambda^{(x)}(y - X_{\sigma_B}^{(x)})]dy\,.$$

Since $g_\lambda^{(x)}$ is continuous, we have

$$g_\lambda^{(x)}(y-\tilde{x}) = E_{\tilde{x}}^{(x)}[e^{-\lambda\sigma_B} g_\lambda^{(x)}(y - X_{\sigma_B}^{(x)})] \qquad \text{for } y \in B\,.$$

Since the exponent $\psi^{(x)}$ of the Levy process $X^{(x)}$satisfies the following inequality

$$\int_{-\infty}^{\infty} \{\mathrm{Re}(\lambda - \psi^{(x)}(\xi))\} \,|\lambda - \psi^{(x)}(\xi)|^{-2} d\xi < +\infty \qquad (\lambda > 0)\,,$$

by virtue of Theorem 2 of [5] we have $P_{\tilde{x}}^{(x)}(\sigma_0 < \infty) > 0$. Letting $B \downarrow \{0\}$, by quasi-left continuity of Lévy process and by the above fact, we obtain

$$g_\lambda^{(x)}(-\tilde{x}) = E_{\tilde{x}}^{(x)}[e^{-\lambda\sigma_0}] g_\lambda^{(x)}(0) < g_\lambda^{(x)}(0)\,.$$

Noting that $g_\lambda^{(x)}(z) = g_{\lambda,0}(x, x+z)$, we have

$$g_{\lambda,0}(x, x-\tilde{x}) \leqq g_{\lambda,0}(x, x) \qquad \text{for any } x \text{ and } \tilde{x} \neq 0\,.$$

Thus Lemma 5.5 is proved.

Lemma 5.6. *If $\sigma(x) = 0$ and $\alpha_1 = \alpha_2 > 1$, then $g_\lambda(x, y)$ satisfies the following:*

$$\text{(5.22)} \qquad \lim_{\varepsilon \downarrow 0} \frac{g_\lambda(x, y) - g_\lambda(x, y \pm \varepsilon)}{g_\lambda(x, x) - g_\lambda(x, x \pm \varepsilon)} = 0\,, \qquad \textit{for any } y \neq x\,,$$

$$\text{(5.23)} \qquad -\infty < \varliminf_{\varepsilon \downarrow 0} \frac{g_\lambda(-\varepsilon, -\varepsilon) - g_\lambda(-\varepsilon, 0)}{g_\lambda(0, 0) - g_\lambda(0, -\varepsilon)} \leqq \varlimsup_{\varepsilon \downarrow 0} \frac{g_\lambda(-\varepsilon, -\varepsilon) - g_\lambda(-\varepsilon, 0)}{g_\lambda(0, 0) - g_\lambda(0, -\varepsilon)} < +\infty\,.$$

Proof. Set $\tilde{a}_1(x, \xi) = a_1(x, \xi) + a_2(x, \xi)$, $\tilde{a}_2(x, \xi) = b_1(x, \xi) + b_2(x, \xi)$, $\tilde{a}_1(x) = a_1(x) + a_2(x)$ and $\tilde{a}_2(x) = b_1(x) + b_2(x)$. Then $\frac{\partial g_{\lambda,0}}{\partial y}$ can be expressed in the form

$$\begin{aligned}
\frac{\partial g_{\lambda,0}}{\partial y}(x, y) = & \frac{1}{\pi}\int_0^\infty \frac{\lambda\xi \sin(y-x)\xi d\xi}{[\lambda - \psi_R(x, \xi)]^2 + \psi_I(x, \xi)^2} - \frac{1}{\pi}\int_0^\infty \frac{\tilde{a}_1(x)\xi^{1+\alpha_1} \sin(y-x)\xi d\xi}{[\lambda - \Phi_R(x, \xi)]^2 + \Phi_I(x, \xi)^2} \\
& - \frac{1}{\pi}\int_0^\infty \left\{ \frac{\tilde{a}_1(x, \xi)\xi^{1+\alpha_1}}{[\lambda - \psi_R(x, \xi)]^2 + \psi_I(x, \xi)^2} - \frac{\tilde{a}_1(x)\xi^{1+\alpha_1}}{[\lambda - \Phi_R(x, \xi)]^2 + \Phi_I(x, \xi)^2} \right\} \\
& \times \sin(y-x)\xi d\xi + \frac{1}{\pi}\int_0^\infty \frac{c_1(x)\xi^2 + \tilde{a}_2(x, \xi)\xi^{1+\alpha_1}}{[\lambda - \psi_R(x, \xi)]^2 + \psi_I(x, \xi)^2} \cos(y-x)\xi d\xi
\end{aligned}$$

$$\equiv I_1+I_2+I_3+I_4.$$

It is easy to see that I_1 is bounded **continuous.** Using Lemma 5.3, we have $I_3=O(|y-x|^{2\alpha_1-3})$. For I_4, noting that $\tilde{a}_2(x)=0$, we have $I_4=O(|y-x|^{2\alpha_1-3})$. Next we consider I_2. Let n be the largest positive integer such that $\alpha_1-2+(n-1)(2-2\alpha_1)\leqq 0$. Then we have

$$\int_0^\infty \frac{A_k(x)\xi^{1+\alpha_1+(k-1)(2-2\alpha_1)}\sin(y-x)\xi}{(\lambda+\tilde{a}_1(x)\xi^{\alpha_1})^2+c_1(x)^2\xi^2}d\xi = \int_0^\infty B_k(x)\xi^{1-\alpha_1+(k-1)(2-2\alpha_1)}\sin(y-x)\xi d\xi$$
$$+\int_0^\infty \frac{A_{k+1}(x)\xi^{1+\alpha_1+k(2-2\alpha_1)}\sin(y-x)\xi}{(\lambda+\tilde{a}_1(x)\xi^{\alpha_1})^2+c_1(x)^2\xi^2}d\xi+v_{11}(x,y),$$

where $A_1(x)=\tilde{a}_1(x)$, $B_k(x)=A_k(x)/\tilde{a}_1(x)^2$, $A_{k+1}(x)=-B_k(x)c_1(x)^2$, $1\leqq k\leqq n+1$, and $v_{11}(x,y)$ is a bounded continuous function.

From this formula we have

$$I_2(x,y) = -\frac{h_1(x)}{\pi}\operatorname{sgn}(y-x)|y-x|^{\alpha_1-2}+\sum_{j=2}^{n+1}h_j(x)\operatorname{sgn}(y-x)|y-x|^{1-2+(j-1)(2-2\alpha_1)}$$
$$+v(x,y),$$

where $h_1(x)=\frac{\;}{2\Gamma(\alpha_1-1)\cos\frac{\pi\alpha_1}{2}\cdot\tilde{a}_1(x)}$ h_j, $2\leqq j\leqq n+1$, and v are bounded continuous functions. (5.23) **follows from this** estimate. Since $g_\lambda\in C^\infty(R\times R-\Delta)$ and $g_\lambda(x,x)-g_\lambda(x,x\pm\varepsilon)=O(\varepsilon^{\alpha_1-2})$, we have (5.22). This completes the proof.

Now we shall show that Lemma 5.2, 5.5 and 5.6 hold without the condition *(c)*. Let A be the operator defined by (2.1) with the conditions (a) and (b). Let $\chi(y)$ be a C^∞-function of compact support such that $0\leqq\chi\leqq 1$ and $\chi(y)=1$ for $|y|\leqq L_1$, $=0$ for $|y|\geqq L_1+1$. Set

$$n_1(x,y) = n(x,y)\chi(y), \quad n_2(x,y) = n(x,y)(1-\chi(y)) \quad \text{and}$$
$$a_1(x) = a(x)-\int_{-\infty}^\infty \frac{yn_2(x,y)}{1+y^2}dy.$$

A can be written as follows.

$$Au(x) = \left\{a_1(x)u'(x)+\sigma(x)^2u''(x)/2+\int_{-\infty}^\infty\left[u(x+y)-u(x)-\frac{yu'(x)}{1+y^2}\right]n_1(x,y)dy\right\}$$

(5.24) $$+\int_{-\infty}^\infty [u(x+y)-u(x)]n_2(x,y)dy\equiv A^{(1)}u(x)+A^{(2)}u(x).$$

Since $A^{(1)}$ satisfies the condition (c), there exists the density $g_\lambda^{(1)}$ of the Green operator $G_\lambda^{(1)}$ of $A^{(1)}$. The next lemma is needed to show the regularity of $g_{\lambda,1}$

in the Theorem 5.1.

Lemma 5.7. *Let $k(x,y)$ be the function defined by* (4.15) *and K be the operator defined by* (4.14). *Then $k(x, y)$ can be written in the form.*

(5.25) $$k(x, y) = \int_{-\infty}^{\infty} g_\lambda^{(1)}(x, z)h(z, y)dz ,$$

where h is the kernel of K ($h(x,y)$ corresponds to $k(x, x-y)$ in Theorem 3.3).

Proof. Note that $k(x, y)$ is the kernel of $G_\lambda K$. On the other hand, we have for $f \in \mathcal{D}$,

$$G_\lambda Kf(x) = \int_{-\infty}^{\infty} g_\lambda^{(1)}(x, z)(Kf)(z)dz = \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} g_\lambda^{(1)}(x, z)h(z, y)dz\right\}f(y)\, dy .$$

Therefore we obtain (5.25), which completes the proof.

Theorem 5.1. *Let A be the operator defined by* (2.1) *with the conditions* (a) *and* (b). *Then we have*

(i) *For any $f \in H_\infty$, there exists a unique $u \in \mathring{\mathcal{B}}$ such that*

$$(\lambda - A)u = f .$$

(ii) *Let G_λ be the inverse of $(\lambda - A)$. Then G_λ has the kernel representation*

$$G_\lambda f(x) = \int_{-\infty}^{\infty} g_\lambda(x, y)f(y)dy,$$

where $g_\lambda(x, y)$ is of the form

$$g_\lambda(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i(x-y)\xi}}{\lambda - \psi(x, \xi)}d\xi + g_{\lambda,1}(x, y) , \qquad g_{\lambda,1} \in C^1(R \times R) .$$

Proof. Let χ, n_1, n_2 and a_1 be as in the paragraph before Lemma 5.7. Write A as in (5.24). It is easy to see that the proof of (i) is the same as the proof of [12, p. 537]. From (5.24), we have the following integral equation for the density $g_\lambda(x, y)$ of Green measure:

(5.26) $$g_\lambda(x,y) = g_\lambda^{(1)}(x,y) + \int_{-\infty}^{\infty} g_\lambda^{(1)}(x, z)A_z^{(2)}g_\lambda(z,y)dz.$$

The equation (5.26) can be solved by successive approximation and $g_\lambda(x,y)$ is given in the form

$$g_\lambda(x, y) = g_\lambda^{(1)}(x,y) + \sum_{k=2}^{\infty} g_\lambda^{(k)}(x, y) ,$$

where $g_\lambda^{(k)}(x, y) = \int_{-\infty}^{\infty} g_\lambda^{(1)}(x, z)A_z^{(2)}g_\lambda^{(k-1)}(z, y)dz$, $k=2, 3, \cdots$. Using the estimates for $g_\lambda^{(1)}$ in §5, Lemma 5.4, 5.6 and 5.7, we obtain $\sum_{k=2}^{\infty} g_\lambda^{(k)}(x,y)$ is a continuously differentiable function. Hence setting $g_{\lambda,1}(x,y)=g_{\lambda,1}^{(1)}(x,y)+\sum_{k=2}^{\infty} g_\lambda^{(k)}(x,y)$, we prove the theorem.

6. Proof of Theorem 1. Now we can prove the Theorem 1. Let x and y be any point in R^1 such that $x \neq y$. Choose an open interval B containing y such that $B \cap \{x\} = \phi$. Then for any Borel set $A \subset B$, we have

$$\int_A g_\lambda(x, z)dz = \int_A E_x[e^{-\lambda\sigma_B} g_\lambda(X_{\sigma_B}, z)]dz .$$

Since g_λ is continuous, we have

$$g_\lambda(x, z) = E_x[e^{-\lambda \sigma_B} g_\lambda(X_{\sigma_B}, z)] , \qquad z \in B . \tag{6.1}$$

By quasi-left continuity of X, letting $B \downarrow \{y\}$, we obtain

$$g_\lambda(x, y) = E_x[e^{-\lambda\sigma_y}] g_\lambda(y, y) , \qquad x \neq y . \tag{6.2}$$

Set $g_{\lambda,0}(x, y)=\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i(x-y)\xi}}{\lambda-\psi(x, \xi)} d\xi$. Then as is seen in the proof of Lemma 5.5, we have $g_{\lambda,0}(x,y)>0$ for any x, y. Next we show $g_\lambda(x, x)>0$. First note that $g_\lambda=g_{\lambda,0}+g_{\lambda,1}\geqq 0$, $g_{\lambda,1}\in C^1$ and $g_{\lambda,0}(x, \ )\leqq g_{\lambda,0}(x, x)$. If $g_\lambda(x, x)=0$ then we have $g_{\lambda,0}(x, y)-g_{\lambda,0}(x, x)\geqq -g_{\lambda,1}(x, y)+g_{\lambda,1}(x, x)$ Therefore

$$\begin{aligned}\varlimsup_{y\uparrow x}\frac{g_{\lambda,0}(x,y)-g_{\lambda,0}(x,x)}{y-x} &\leqq \varlimsup_{y\uparrow x}\frac{-g_{\lambda,1}(x,y)+g_{\lambda,1}(x,x)}{y-x} \\ &= \lim_{y\downarrow x}\frac{-g_{\lambda,1}(x,y)+g_{\lambda,1}(x,x)}{y-x} \leqq \lim_{y\downarrow x}\frac{g_{\lambda,0}(x,y)-g_{\lambda,0}(x,x)}{y-x} .\end{aligned}$$

Since $g_{\lambda,0}(x, \cdot)$ is not differentiable at x and $g_{\lambda,0}(x, \ )\leqq g_{\lambda,0}(x, x)$, we have

$$\lim_{y\downarrow x}\frac{g_{\lambda,0}(x,y)-g_{\lambda,0}(x,x)}{y-x} < \varlimsup_{y\uparrow x}\frac{g_{\lambda,0}(x,y)-g_{\lambda,0}(x,x)}{y-x} .$$

This is a contradiction. Thus we have $g_\lambda(x, x)>0$. Therefore we can choose finite number of open sets $U_1, \cdots, U_l$ such that $g_\lambda(x', y')>\varepsilon>0$ if $x', y' \in U_i$, $i=1, \cdots, l$ and $U_i \cap U_{i+1} \neq \phi$, $i=1, \cdots, l-1$, $\bigcup_{i=1}^{l} U_i \supset [x, y]$. Let $x=x_1<x_2\cdots <x_l<x_{l+1}=y$, $x_{i+1}\in U_i \cap U_{i+1}$, $i=1, \cdots, l-1$. By (6.2), we have

$$g_\lambda(x_i, x_{i+1}) = g_\lambda(x_{i+1}, x_{i+1}) E_{x_i}[e^{-\lambda\sigma_{x_{i+1}}}] \qquad i = 1, \cdots, l .$$

Hence for some $\delta>0$,

$$E_{x_i}[e^{-\lambda\sigma_{x_{i+1}}}] = \frac{g_\lambda(x_i, x_{i+1})}{g_\lambda(x_{i+1}, x_{i+1})} > \delta, \qquad i = 1, \cdots, l .$$

This implies $P_{x_i}(\sigma_{x_{i+1}}<\infty)>\delta, i=1, \cdots, l$. By the strong Markov property, we have $P_x(\sigma_y<+\infty)>0$ for any $x\neq y$, which completes the proof of Theorem 1.

7. Martin boundary theory. In this section, we shall prepare some lemmas from the theory of Martin boundary for the proof of the part (3) of Theorem 2. For the theory of Martin boundary, we refer to Kunita-T.Watanabe [9]. Although almost parallel arguments to [9] hold, we present this section for completeness, since we do not assume the existence of dual process. Let $X=(X_t, P_x, x\in R^1)$ be the process defined in §2. Define $X^0=(X_t^0, P_x^0, x\in R^1\setminus\{0\})$ by $X_t^0=X_t$ if $t<\sigma_0$, $=0$ if $t\geqq\sigma_0$ and $P_x^0(\ )=P_x(\ ,\ \sigma_0<\infty)$. Let G_λ^0 be the resolvent of X_t^0. By the strong Markov property, the density $g_\lambda^0(x,y)$ of $G_\lambda^0(x, dy)$ with respect to dy is given by

$$g_\lambda^0(x,y) = g_\lambda(x,y)-E_x[e^{-\lambda\sigma_0}]g_\lambda(0,y) \tag{7.1}$$
$$= g_\lambda(x, y)-\frac{g_\lambda(x, 0)}{g_\lambda(0, 0)}g_\lambda(0, y)\,, \qquad x\neq 0\,.$$

For any Borel set A of $R^1\setminus\{0\}$, define $\tau_A=\inf\{t\geqq 0; X_t^0\in A\}$. For any bounded measurable function f and for any Borel set A of $R^1\setminus\{0\}$, define

$$H_A^\lambda f(x) = E_x^0[e^{-\lambda\tau_A}f(X_{\tau_A}^0);\ \tau_A<\sigma_0]\,.$$

Set $u_1(x)=E_x[e^{-\lambda\sigma_0};\ \Omega_1^+]$, $u_2(x)=E_x[e^{-\lambda\sigma_0};\Omega_1^-]$ and $u_3(x)=E_x[e^{-\lambda\sigma_0};\Omega_1^\pm]$ for $x\neq 0$. We call f a λ-harmonic function (relative to X^0) in $R^1\setminus\{0\}$ if for any open set A with $\bar{A}$ compact in $R^1\setminus\{0\}$,

$$f(x) = E_x^0[e^{-\lambda\tau_{A^c}}f(X_{\tau_{A^c}}^0); \tau_{A^c}<\sigma_0]\,. \tag{7.2}$$

REMARK 7.1. It follows from the existence of the continuous density of $G_\lambda(x, dy)$ that every λ-excessive function is lower semi-continuous. Indeed by Fatou's lemma, $G_\lambda f(x)$ is lower semi-continuous (f is nonnegative bounded measurable). Since every λ-excessive function f is an increasing limit of $G_\lambda f_n$ (f_n is nonnegative bounded measurable), f is lower semi-continuous. In view of [1, p. 197], we see that there exists a reference measure.

Lemma 7.1. *u_i, $i=1, 2, 3$ are λ-harmonic (relative to X^0) in $R^1\setminus\{0\}$.*

Proof. First consider the case $i=1$. Note that $\tau_{A^c}<\sigma_0$ on Ω_1. Indeed by Remark 7.1, we can apply Theorem 4.2 of [16]. Therefore we have

$$E_x[e^{-\lambda\tau_{A^c}}; X_{\tau_{A^c}-}\in \bar{A}, X_{\tau_{A^c}}= 0] = 0\,.$$

Thus we get $P_x(\tau_{A^c}=\sigma_0,\Omega_1)=0$. By the strong Markov property, we have

$$u_1(x) = E_x^0[e^{-\lambda\tau_{A^c}}u_1(X_{\tau_{A^c}}^0);\ \tau_{A^c}<\sigma_0]\,.$$

By the same way, we obtain (7.2) for $u := 2, 3$, which proves the lemma.

Lemma 7.2. (1) *Let A be a compact set of $R^1\setminus\{0\}$ and $B(\subset R^1\setminus\{0\})$ be a neighborhood of A. Then we have*

$$\inf_{x\in A}\int_B g_\lambda^0(y, x)dy > 0\,. \tag{7.3}$$

(2) (i) *For any open set A in $R^1\setminus\{0\}$ and for any $y\in A$, we have*

$$H_A^\lambda g_\lambda^0(x, y) = g_\lambda^0(x, y)\,.$$

(ii) *For any $y\in R^1\setminus\{0\}$, $g_\lambda^0(\ , y)$ is λ-harmonic (relative to X^0) in $R^1\setminus\{0, y\}$.*
(3) *Let $x_0\in R^1\setminus\{0\}$ be fixed. Let V be an open interval of $R^1\setminus\{0\}$ containing x_0 with $\bar V\subset R^1\setminus\{0\}$. Then there exists an open interval $U(x_0)(\subset V)$ containing x_0 such that*

$$g_\lambda^0(x, y) > E_x^0[e^{-\lambda\tau_{V^c}}g_\lambda^0(X_{\tau_{V^c}}^0, y);\ \tau_{V^c}<\sigma_0] \qquad \textit{for any } x, y\in U(x_0).$$

Proof. (1) By (7.1), $g_\lambda^0(x, y)$ is continuous in $(R^1\times R^1)\setminus\{0\}$ and

$$g_\lambda^0(x, x) = g_\lambda(x, x) - \frac{g_\lambda(x, 0)}{g_\lambda(0, 0)}g_\lambda(0, x) > g_\lambda(x, x) - g_\lambda(0, x) > 0\,.$$

Therefore $\inf_{x\in A}\int_B g_\lambda^0(y, x)dy>0$. Thus (1) is proved.
(2) (i). Let A be any open set in $R^1\setminus\{0\}$ and f be any continuous function of compact support such that $f=0$ on A^c. Then by the strong Markov property we have

$$\int_{R^1\setminus\{0\}} E_x^0[e^{-\lambda\tau_A}g_\lambda^0(X_{\tau_A}^0, y);\ \tau_A<\sigma_0]f(y)dy = E_x^0\left[\int_0^\infty e^{-\lambda t}f(X_t^0)dt;\ \tau_A<\sigma_0\right]$$
$$= \int_{R^1\setminus\{0\}} g_\lambda^0(x, y)f(y)dy\,.$$

Since $g_\lambda^0(x, y)$ is continuous in $(R^1\times R^1)\setminus\{0\}$, we get

$$E_x^0[e^{-\lambda\tau_A}g_\lambda^0(X_{\tau_A}^0, y);\ \tau_A<\sigma_0] = g_\lambda^0(x, y) \qquad \text{for } y\in A \text{ and } x\neq 0\,.$$

For the proof (ii), let A be an open set in $R^1\setminus\{0, y\}$ with $\bar A$ compact in $R^1\setminus\{0, y\}$. Then by (i), for $y'\in(\bar A)^c$ we have

$$H_{(\bar A)^c}^\lambda g_\lambda^0(x, y') = g_\lambda^0(x, y')\,.$$

Thus (2) is proved.
(3) For any $x, y\neq 0$ $(x\neq y)$, we have

$$g^0_\lambda(x,y) = g^0_\lambda(y,y)E^0_x[e^{-\lambda\tau_y}; \tau_y < \sigma_0] .$$

Since by the right continuity of paths, $P_x\{\tau_y>0\}=1$, we have

$$g^0_\lambda(x,y) < g^0_\lambda(y,y) \tag{7.4}$$

Applying the Riemann-Lebesgue theorem to $g^{(1)}_\lambda(x,y)$, we have $\lim_{|x|\to\infty} g^{(1)}_\lambda(x,y)=0$, uniformly on any compact set of y. Therefore by virtue of (7.1), (5.26) and Lemma 5.7, we have

$$\lim_{|z|\to\infty} g^0_\lambda(z,y) = 0 \qquad \text{uniformly in } y:\ |x_0-y| < 1 . \tag{7.5}$$

Therefore it follows from (7.4), (7.5) and the continuity of g^0_λ that there exist constants δ_0 and $\varepsilon_0 > 0$ such that

$$\inf_{z\notin V} [g^0_\lambda(y,y) - g^0_\lambda(z,y)] \geqq \varepsilon_0 \qquad \text{for } y:\ |x_0-y| < \delta_0 .$$

By the continuity of g^0_λ, for any ε_1: $0<\varepsilon_1<\varepsilon_0$, there exists a constant $\delta>0$ $(\delta<\delta_0)$ such that for $|x-x_0|<\delta$, $|y-x_0|<\delta$,

$$g^0_\lambda(y,y) - g^0_\lambda(x,y) < \varepsilon_1 .$$

Therefore we have

$$E^0_y[e^{-\lambda\tau_{V^c}} g^0_\lambda(X^0_{\tau_{V^c}}, y); \tau_{V^c} < \sigma_0] \leqq g^0_\lambda(y,y) - \varepsilon_0 < g^0_\lambda(x,y) .$$

So it is sufficient to put $U(x_0)=\{x;\ |x-x_0|<\delta\} \cap V$, which proves the Lemma 7.2.

Lemma 7.3. *Let* $\{\mu_n\}$ *be a sequence of measures on* $R^1\setminus\{0\}$. *Define* $G^0_\lambda\mu_n$ *by*

$$G^0_\lambda\mu_n(x) = \int_{R^1\setminus\{0\}} g^0_\lambda(x,y)\mu_n(dy) .$$

If there exists a locally integrable function v *such that* $G^0_\lambda\mu_n \leqq v$, *then we have*

(i) *There exists a subsequence* $\{\mu_{n_k}\}$ *of* $\{\mu_n\}$ *such that* $\{\mu_{n_k}\}$ *converges to some* μ *weakly.*

(ii) *If for the above* $\{\mu_{n_k}\}$, $G^0_\lambda\mu_{n_k}$ *converges to some* u *almost everywhere with respect to Lebesgue measure, then*

$$\lim_{\alpha\uparrow\infty} \alpha G^0_\alpha u \geqq G^0_\lambda\mu .$$

Especially if u *is* λ*-excessive, then* $u \geqq G^0_\lambda\mu$.

(iii) *If for any nonnegative bounded measurable function* f *with compact support and for any* $\varepsilon>0$, *there exists a compact set* A *in* $R^1\setminus\{0\}$ *such that*

$$\int_{A^c} \hat{G}^0_\lambda f(y)\mu_n(dy) < \varepsilon, \qquad n=1,2,\ \cdots, \tag{7.6}$$

then $\lim_{\alpha\uparrow\infty} \alpha G_\alpha^0 u = G_\lambda^0 \mu$, *where* $\hat{G}_\lambda^0 f(y) = \int_{R^1\setminus\{0\}} g_\lambda^0(x,y) f(x)dx$. *Note that if there exists a compact set* A_1 *such that* $\operatorname{supp}(\mu_n) \subset A_1$, $n=1, 2, \cdots$, *then the condition in* (iii) *is fulfilled.*

Proof. For the assertion (i) it is enough to show that, for each compact set, $\mu_n(A)$ is bounded. Let B be a compact neighborhood of A. By Lemma 7.2, $c = \inf_{x\in A} \int_B g_\lambda^0(y, x)dy > 0$. Hence

$$\infty > \int_B v(x)dx \geqq \int_B G_\lambda^0 \mu_n(x)dx \geqq \int_A \left\{ \int_B g_\lambda^0(x, y)dx \right\} \mu_n(dy) \geqq c\mu_n(A) .$$

For (ii), let μ_{n_k} converge weakly to μ and let f be a nonnegative bounded measurable function of compact support. Using the facts that $v \geqq G_\lambda^0 \mu_{n_k}$ and $\hat{G}_\lambda^0 f$ is nonnegative continuous, we have

$$\begin{aligned}\int_{R^1\setminus\{0\}} f(x)u(x)dx &= \lim_{k\to\infty} \int_{R^1\setminus\{0\}} f(x) G_\lambda^0 \mu_{n_k}(x)dx = \lim_{k\to\infty} \int_{R^1\setminus\{0\}} \hat{G}_\lambda^0 f(y) \mu_{n_k}(dy) \\ &\geqq \int_{R^1\setminus\{0\}} \hat{G}_\lambda^0 f(y)\mu(dy) = \int_{R^1\setminus\{0\}} f(x) G_\lambda^0 \mu(x)\, dx .\end{aligned}$$

Therefore $u \geqq G_\lambda^0 \mu$ a.e. rf*. On the other hand, for any $\alpha > 0$,

$$u \equiv \lim_{k\to\infty} G_\lambda^0 \mu_{n_k} \geqq \varliminf_{k\to\infty} \alpha G_\alpha^0 (G_\lambda^0 \mu_{n_k}) \geqq \alpha G_\alpha^0 (\lim_{k\to\infty} G_\lambda^0 \mu_{n_k}) = \alpha G_\alpha^0 u \qquad \text{a.e. } dx .$$

So we have

$$\begin{aligned}&\int_{R^1\setminus\{0\}} f(x) \lim_{\alpha\uparrow\infty} \alpha G_\alpha^0 u(x)dx \\ &= \lim_{\alpha\uparrow\infty} \int_{R^1\setminus\{0\}} \alpha \hat{G}_\alpha^0 f(y)u(y)dy \geqq \lim_{\alpha\uparrow\infty} \int_{R^1\setminus\{0\}} \alpha \hat{G}_\alpha^0 f(y) G_\lambda^0 \mu(y)dy \\ &\geqq \int_{R^1\setminus\{0\}} f(y) \varliminf_{\alpha\uparrow\infty} \alpha G_\alpha^0 (G_\lambda^0 \mu)(y)dy = \int_{R^1\setminus\{0\}} f(y) G_\lambda^0 \mu(y)dy .\end{aligned}$$

Hence we have

$$\lim_{\alpha\uparrow\infty} \alpha G_\alpha^0 u(x) \geqq G_\lambda^0 \mu(x) \qquad \text{a.e. } dx . \tag{7.7}$$

Since the both sides of (7.7) are λ-excessive, we have $\lim_{\alpha\uparrow\infty} \alpha G_\alpha^0 u \geqq G_\lambda^0 \mu$. Finally we shall prove (iii). Let f be a nonnegative bounded measurable function with compact support. Then by the assumption, for any £>0, there exists a compact set A in $R^1\setminus\{0\}$ such that (7.6) holds. So we have

$$\lim_{k\to\infty} \int_{R^1\setminus\{0\}} \hat{G}_\lambda^0 f(y)\mu_{n_k}(dy) \leqq \lim_{k\to\infty} \int_A \hat{G}_\lambda^0 f(y) \mu_{n_k}(dy) + \lim_{k\to\infty} \int_{A^c} \hat{G}_\lambda^0 f(y) \mu_{n_k}(dy)$$

$$\leqq \int_{R^1\setminus\{0\}} \hat{G}^0_\lambda f(y)\mu(dy) + \lim_{k\to\infty} \int_{A^c} \hat{G}^0_\lambda f(y)\mu_{n_k}(dy) \leqq \int_{R^1\setminus\{0\}} \hat{G}^0_\lambda f(y)\mu(dy) + \varepsilon.$$

Therefore we have

$$\int_{R^1\setminus\{0\}} f(x)u(x)dx \leqq \int_{R^1\setminus\{0\}} \hat{G}^0_\lambda f(y)\mu(dy) + \varepsilon\,.$$

Hence $u \leqq G^0_\lambda \mu$ a.e. dx. This implies that $\lim_{\alpha\uparrow\infty} \alpha G^0_\alpha u \leqq G^0_\lambda \mu$. Thus the proof is completed.

Lemma 7.4. *Let A be an open set in $R^1\setminus\{0\}$ with $\bar{A}$ compact in $R^1\setminus\{0\}$. Then there exists a measure μ_i ($i=1, 2, 3$) concentrated in A such that*

$$H^\lambda_A u_i(x) = \int_{\bar{A}} g^0_\lambda(x, y)\mu_i(dy), \qquad i = 1, 2, 3\,.$$

Proof. First we show for the case $i=1$. Since u_1 is λ-excessive, setting $f_n(x)=n[u_1(x)-nG^0_{\lambda+n}u_1(x)]$ we have $u_1(x)=\lim_{n\to\infty} G^0_\lambda f_n(x)$. We shall first prove Lemma 7.4 for $v_n(x)=G^0_\lambda f_n(x)$. Let B be a compact set in $R^1\setminus\{0\}$, then by the strong Markov property

$$H^\lambda_{B^c} v_n(x) = E^0_x\left[\int_{\tau_{B^c}}^{\sigma_0} e^{-\lambda t} f_n(X^0_t)dt; \tau_{B^c} < \sigma_0\right] \to 0 \quad \text{as} \quad B \uparrow R^1\setminus\{0\}\,.$$

So we have $H^\lambda_{B^c} H^\lambda_A v_n(x) \leqq H^\lambda_{B^c} v_n(x) \to 0$ as $B \uparrow R^1\setminus\{0\}$. Since $H^\lambda_A v_n$ is λ-excessive, by setting $f_{n,m}(x)=m[H^\lambda_A v_n(x)-mG^0_{\lambda+m}H^\lambda_A v_n(x)]$ we have $H^\lambda_A v_n(x)= \lim_{m\to\infty} G^0_\lambda f_{n,m}(x)$. Set $v_{n,m}=G^0_\lambda f_{n,m}$ and $\mu_{n,m}(dy)=f_{n,m}(y)dy$. Then using Lemma 7.2, we have

$$H^\lambda_{B^c} v_{n,m}(x) = H^\lambda_{B^c} G^0_\lambda \mu_{n,m}(x) = \int_B H^\lambda_{B^c} g^0_\lambda(x, y)\mu_{n,m}(dy) + \int_{B^{cc}} H^\lambda_{B^c} g^0_\lambda(x, y)\mu_{n,m}(dy)$$
$$\geqq \int_{B^c} H^\lambda_{B^c} g^0_\lambda(x, y)\mu_{n,m}(dy) = \int_{B^c} \quad (x, y)\mu_{n,m}(dy).$$

For any nonnegative bounded measurable function / with compact support,

$$\int_B \hat{G}^0_\lambda f(y)\mu_{n,m}(dy) = \int_B \left\{\int_{R\setminus\{0\}} g^0_\lambda(x, y)f(x)dx\right\}\mu_{n,m}(dy)$$
$$\leqq \int_{R^1\setminus\{0\}} H^\lambda_{B^c} v_{n,m}(x)f(x)dx \leqq \int_{R^1\setminus\{0\}} H^\lambda_{B^c} H^\lambda_A v_n(x)f(x)dx$$
$$\leqq \int_{R^1\setminus\{0\}} H^\lambda_{B^c} v_n(x)f(x)dx \to 0 \quad \text{as} \quad B \uparrow R^1\setminus\{0\}\,.$$

Therefore by Lemma 7.3, there exists a measure μ_n on $R^1\setminus\{0\}$ such that

$H_A^\lambda v_n(x)=G_\lambda^0\mu_n(x)$. Next we shall show $\mu_n((\bar{A})^c)=0$. Let us suppose that there exists a compact subset D of $(\bar{A})^c$ such that $\mu_n(D)>0$. Then there exists a point $x_0\in D$ such that for any neighborhood Q containing x_0, $\mu_n(Q)>0$. Let $V(\subset(\bar{A})^c)$ be an open interval containing x_0 with $\bar{V}\subset R^1\setminus\{0\}$. Then by Lemma 7.2, there exists an open interval $U(x_0)(\subset V)$containing x_0 such that for any $x, y\in U(x_0)$,

$$E_x^0[e^{-\lambda\tau_{V^c}}g_\lambda^0(X_{\tau_{V^c}}^0,y);\ \tau_{V^c}<\sigma_0]<g_\lambda^0(x, y)$$

Therefore

$$\int_{U(x_0)\cap D}g_\lambda^0(x_0, y)\mu_n(dy)>E_{x_0}^0\left[\int_{U(x_0)\cap D}e^{-\lambda\tau_{V^c}}g_\lambda^0(X_{\tau_{V^c}}^0, y)\mu_n(dy)\right].$$

On the other hand $g_\lambda^0(\ \ , y)$ is λ-excessive (relative to X^0), we have

$$\int_{(R^1\setminus\{0\})\setminus(U(x_0)\cap D)}g_\lambda^0(x_0, y)\mu_n(dy)\geqq\int_{(R^1\setminus\{0\})\setminus(U(x_0)\cap D)}E_x^0[e^{-\lambda\tau_{V^c}}g_\lambda^0(X_{\tau_{V^c}}^0, y)]\mu_n(dy).$$

Therefore we have

$$v_n(x_0)>E_{x_0}^0[e^{-\lambda\tau_{V^c}}v_n(X_{\tau_{V^c}}^0);\ \tau_{V^c}<\sigma_0].$$

This contradicts Lemma 7.1. So $\mu_n((\bar{A})^c)=0$. Thus Lemma 7.4 is proved for $v_n(x)=G_\lambda^0 f_n(x)$. Again applying Lemma 7.3 to $G_\lambda^0\mu_n=H_A^\lambda v_n$, we have

$$H_A^\lambda u_1(x)=G_\lambda^0\mu(x)\qquad\text{for some } \mu \text{ with supp } (\mu)\subset\bar{A}.$$

The proof for $u_i(x), i=2,3$ is similar to that of u_1, which completes the proof.

Using the estimates in Lemma 5.6, we can construct the Martin boundary Δ of the process X_t^0. Set

$$\kappa(x, y)=\frac{g_\lambda^0(x, y)}{g_\lambda^0(c, y)}.$$

Let A be the set of infinite sequences $\{y_n\}$ such that $\{y_n\}$ does not converge to any point in $R^1\setminus\{0\}$ and for which $\kappa(x, y_n)$ converges. We call $\{y_n\}$, $\{z_n\}$ $(\in\tilde{\Delta})$ equivalent if $\lim_{n\to\infty}\kappa(x, y_n)=\lim_{n\to\infty}\kappa(x, z_n)$. Define Δ as the set of all equivalence class of A. Since $\lim_{y\to 0}\kappa(x, y)=\frac{g_\lambda(x, 0)}{g_\lambda(c, 0)}$ by Lemma 5.6, any $\{y_n\}$ $(\in\tilde{\Delta})$ which converges to 0 are equivalent. So the origin belongs to Δ and there exists no points of Δ except the origin in $[-N, N]$ $(N>0)$.

Lemma 7.5. *Let* $\{\mu_n\}$ *be a sequence of (Radon) measures on* $(R^1\setminus\{0\})\cup\Delta$ *such that* $\text{supp}(\mu_n)$ *are contained in* $[-M, M]$ $(M>0)$ *and* $\{\mu_n[-M, M]\}$ *is* *bounded. Let μ be a weak limit of $\{\mu_{n_k}\}$. If*

$$v_n = \int_{(R^1\backslash\{0\})\cup\Delta} \kappa(\cdot, \eta)\mu_n(d\eta)$$

is dominated by a locally integrable function and if v_{n_k} converges a.e. dx *to a function u, then*

$$\lim_{\alpha\uparrow\infty} \alpha G_\alpha^0 u = \int_{(R^1\backslash\{0\})\cup\Delta} \kappa(\ , \eta)\mu(d\eta)\,.$$

Proof. Let / be a continuous function with compact support, then

$$\int_{R^1\backslash\{0\}} f(x)u(x)dx = 52 \int_{R^1\backslash\{0\}} f(x)\left\{\int_{(R^1\backslash\{0\})\cup\Delta} \kappa(x, \eta)\mu_{n_k}(d\eta)\right\}dx$$

$$= \int_{(R^1\backslash\{0\})\cup\Delta} \left\{\int_{R^1\backslash\{0\}} f(x)\kappa(x, \quad , \quad \int_{R^1\backslash\{0\}} \quad \int_{(R^1\backslash\{0\})\cup\Delta} \quad , \eta)\mu(d\eta)\right\}dx\,.$$

Thus $u(x) = \int_{(R^1\backslash\{0\})\cup\Delta} \kappa(x, \eta)\mu(d\eta)$ a.e. dx. The rest of the proof is the same as that of Lemma 7.3 (iii). This completes the proof.

After above preparation we can prove the following lemma.

Lemma 7.6. *There exist nonnegative constants c_1, c_2, c_3 such that*

$$E_x[e^{-\lambda\sigma_0}\ \Omega_1^+] = c_1\frac{g}{g_\lambda(0, 0)} \qquad \textit{for any } x\neq 0\,,$$

$$E_x[e^{-\lambda\sigma_0}; \Omega_1^-] = c_2\frac{g_\lambda(x, 0)}{g_\lambda(0, 0)} \qquad \textit{for any } x\neq 0\,,$$

$$E_x[e^{-\lambda\sigma_0}; \Omega_1^\pm] = c_3\frac{g_\lambda(x, 0)}{g_\lambda(0, 0)} \qquad \textit{for any } x\neq 0\,.$$

Proof. First we show this lemma for $u_1(x) = E_x[e^{-\lambda\sigma_0}\ \Omega_1^+]$. For any open set A in $(R^1\backslash\{0\})\cup\Delta$ with $\bar{A}$ compact, we can choose a sequence $\{A_n\}$ of open sets in $R^1\backslash\{0\}$ with A_n compact in $R^1\backslash\{0\}$ such that $A_n\uparrow A\cap(R^1\backslash\{0\})$. We denote $[A]$ for $A\cap(R^1\backslash\{0\})$ in the rest of the proof. By Lemma 7.4, there exists a measure v_n such that

$$H_{A_n}^\lambda u_1(x) = \int_{\bar{A}_n} g_\lambda^0(x, y)v_n(dy)\,.$$

Define μ_n by $\mu_n(dy) = g_\lambda^0(c\,y)v_n(dy)$, then $\operatorname{supp}(\mu_n)\subset\bar{A}_n$ and $\{\mu_n((R^1\backslash\{0\})\cup\Delta)\}$ is bounded. Therefore there exists a subsequence $\{\mu_{n_k}\}$ of $\{\mu_n\}$ wihch converges to some μ_A. On the other hand since

$$H_{A_n}^\lambda u_1(x) = \int_{\bar{A}_n} \kappa(x, y)\mu_n(dy) \to H_{[A]}^\lambda u_1(x) \quad \text{as} \quad n\to\infty\,,$$

and $H_{A_n}^{\lambda} u_1(x) \leqq 1$ by Lemma 7.5 we have

$$\lim_{\alpha \uparrow \infty} \alpha G_\alpha^0 H_{[A]}^{\lambda} u_1 = \int_{\overline{[A]}} \kappa(\cdot, \eta) \mu_A(d\eta) .$$

Because $H_{[A]}^{\lambda} u_1(x)$ is λ-excessive (relative to X^0), we get

$$H_{[A]}^{\lambda} u_1(x) = \int_{\overline{[A]}} \kappa(x, \eta) \mu_A(d\eta) .$$

Next let A be $\{0\} \in \Delta$. Choose a sequence A_n of open sets in $(R^1 \setminus \{0\}) \cup \Delta$ such that $A_n \downarrow A$. Then since $H_{[A_n]}^{\lambda} u_1(x)$ is a monotone decreasing sequence and $\{\mu_{A_n}((R^1 \setminus \{0\}) \cup \Delta)\}$ is bounded, it follows from Lemma 7.5 that there exists a measure μ_0 such that

$$\lim_{\alpha \uparrow \infty} \alpha G_\alpha^0 (\lim_{n \to \infty} H_{[A_n]}^{\lambda} u_1)(x) = \int_A \kappa(x, \eta) \mu_0(d\eta) .$$

On the other hand, in view of the proof of Lemma 7.1, we have $\lim_{n \to \infty} H_{[A_n]}^{\lambda} u_1(x) = u_1(x)$. So we get

$$\lim_{\alpha \uparrow \infty} \alpha G_\alpha^0 u_1(x) = \int_A \kappa(x, \eta) \mu_0(d\eta) .$$

Since u_1 is λ-excessive, we have

$$u_1(x) = \int_A \kappa(x, \eta) \mu_0(d\eta) = \mu_0(\{0\}) \kappa(x, 0) .$$

Thus we proved the lemma for u_1. By the same way, we can prove the lemma for u_i, $i=2, 3$. The proof of lemma is completed.

8. Proof of Theorem 2. In this section we shall prove Theorem 2 by using the estimates established in §5. Before entering the proof, we note that σ_0 is an accessible stopping time on Ω_1. Indeed let R_n be the hitting time of $A_n = \left\{x;\, E_x[e^{-\lambda\sigma_0}] > 1 - \frac{1}{n}\right\}$ and let $\Lambda = \{R_n < \sigma_0, \forall n\}$. Since $E_x[e^{-\lambda\sigma_0}]$ is λ-excessive (relative to X) and every λ-excessive function is lower semi-continuous (see Remark 7.1), A_n contains an open set which contains the origin. Therefore in view of the proof of Lemma 7.1, $\Lambda \supset \Omega_1$. On the other hand, by Proposition (4.12) of [1. Chapter IV] $\lim R_n = T$ on Λ a.s. Thus σ_0 is accessible on Ω_1.

Proof of (1). Let X_t^0, G_λ^0 and g_λ^0 be as in the beginning of §7. For $x \neq 0$, put

$$F_+(x) = \lim_{y \downarrow 0} \frac{g_\lambda^0(x, y)}{g_\lambda^0(a, y)} \quad \text{and} \quad F_-(x) = \lim_{y \uparrow 0} \frac{g_\lambda^0(x, y)}{g_\lambda^0(b, y)} .$$

Then by Lemma 5.2 we have

$$F_+(x) = \lim_{y \downarrow 0} \frac{g_\lambda(x, y)g_\lambda(0, 0) - g_\lambda(x, 0)g_\lambda(0, y)}{g_\lambda(a, y)g_\lambda(0, 0) - g_\lambda(a, 0)g_\lambda(0, y)}$$

$$(8.1) \qquad = \frac{g_\lambda(0, 0)\dfrac{\partial g_\lambda}{\partial y}(x, 0) - g_\lambda(x, 0)\dfrac{\partial g_\lambda}{\partial y}(0, 0+)}{g_\lambda(0, 0)\dfrac{\partial g_\lambda}{\partial y}(a, 0) - g_\lambda(a, 0)\dfrac{\partial g_\lambda}{\partial y}(0, 0+)} .$$

$$F_-(x) = \frac{g_\lambda(0, 0)\dfrac{\partial g_\lambda}{dy}(x, 0) - g_\lambda(x, 0)\dfrac{\partial g_\lambda}{\partial y}(0, 0-)}{g_\lambda(0, 0)\dfrac{\partial g_\lambda}{\partial y}(b, 0) - g_\lambda(b, 0)\dfrac{\partial g_\lambda}{\partial y}(0, 0-)} .$$

We can choose constants a and b such that the denominators do not vanish by virtue of Lemma 5.2. In fact, if for all $a \neq 0$

$$g_\lambda(0, 0)\frac{\partial g_\lambda}{dy}(a, 0) - g_\lambda(a, 0)\frac{\partial g_\lambda}{dy}(0, 0+) = 0 ,$$

then we have $g_\lambda(0, 0)\left(\dfrac{\partial g_\lambda}{dy}(0+, 0) - \dfrac{\partial g_\lambda}{dy}(0, 0+)\right) = 0$. Hence by (5.2a) and (5.2b), we have $\sigma(0)^{-2}g_\lambda(0, 0)=0$. This is a contradiction.

By (8.1), (5.2a) and (5.2b), we get

(8.2)

$$F_+(0+) = \frac{2g_\lambda(0, 0)}{\sigma(0)^2\left\{g_\lambda(0, 0)\dfrac{\partial g_\lambda}{\partial y}(a, 0) - g_\lambda(a, 0)\dfrac{\partial g_\lambda}{dy}(0, 0+)\right\}} \neq 0, \quad F_+(0-) = 0 ,$$

$$F_-(0-) = \frac{-2g_\lambda(0, 0)}{\sigma(0)^2\left\{g_\lambda(0, 0)\dfrac{\partial g_\lambda}{\partial y}(b, 0) - g_\lambda(b, 0)\dfrac{\partial g_\lambda}{\partial y}(0, 0-)\right\}} \neq 0, \quad F_-(0+) = 0 .$$

Since $F_+(X_t^0)$ is a λ-excessive function relative to X^0, $e^{-\lambda t}F_+(X_t^0)$ is a supermartingale[1] having left limit with probability one. Therefore

$$(8.3) \qquad P_x^0\{\lim_{t \uparrow \sigma_0} e^{-\lambda t}F_+(X_t^0) \text{exists}/\Omega_1\} = 1 \qquad \text{for } x \neq 0 .$$

Combining (8.2) and (8.3), we have $P_x(\Omega_1^\pm | \Omega_1) = 0$. Hence we have $P_x(\Omega_1^+ \cup \Omega^- | \Omega_1) = 1$ for any $x \neq 0$, which completes the proof of (i).

Next we show (ii). Let $\{\tau_n\}$ be the sequence of stopping times such that $\tau_0 = 0$, $\tau_n = \inf\left\{t \; X_t^0 \in \left[-\frac{1}{n}, \frac{1}{n}\right]\right\}$. Then as noted in the paragraph before the

1) '$\{e^{-\lambda t}F_+(X_t^0)\}_{0 \leq t \leq \sigma_0}$ is a supermartingale' means $\{e^{-\lambda t}F_+(X_t)I_{[0,\sigma_0)}(t), \mathcal{F}_t, P_x\}$ is a supermartingale.

proof of (1), $\tau_n<\sigma_0$, $\forall n$ and $\lim \tau_n=\sigma_0$ on Ω_1 a.s. Set $f_{\tau_n}(\omega)=e^{-\lambda\tau_n}F_+(X^0_{\tau_n}(\omega))$. Then $\{f_{\tau_n}(\omega)\}_{n\geq 0}$ is a nonnegative bounded martingale. So there exists $f_\infty(\omega)=\lim_{n\to\infty} f_{\tau_n}(\omega)$ such that $E^0_x[f_\infty(\omega)/\mathcal{F}^0_{\tau_n}]=f_{\tau_n}(\omega)$[2], (a.s. P^0_x) $n\geq 0$. Hence we have

$$E^0_x[f_\infty(\omega)] = E^0_x[f_{\tau_n}(\omega)] .$$

By (8.2), we have

$$f_\infty(\omega)=\begin{cases} e^{-\lambda\sigma_0}F_+(0+) & \text{on } \Omega_1^+, \\ 0 & \text{on } \Omega_1^-. \end{cases}$$

Hence we obtain

$$E^0_x[e^{-\lambda\sigma_0}F_+(0+);\ \ \Omega_1^+] = E^0_x\left[e^{-\lambda t}\frac{g_\lambda(0,0)\frac{\partial g_\lambda}{\partial y}(X^0_{\tau_n},0)-g_\lambda(X^0_{\tau_n},0)\frac{\partial g_\lambda}{\partial y}(0,0+)}{g_\lambda(0,0)\frac{\partial g_\lambda}{\partial y}(a,0)-g_\lambda(a,0)\frac{\partial g_\lambda}{\partial y}(0,0+)}\right] \quad \text{for } n\geq 0 .$$

Therefore we get

$$E^0_x[e^{-\lambda\sigma_0};\Omega_1^+] = \frac{\sigma(0)^2}{2g_\lambda(0,0)}\left[g_\lambda(0,0)\frac{\partial g_\lambda}{\partial y}(x,0)-g_\lambda(x,0)\frac{\partial g_\lambda}{\partial y}(0,0+)\right] .$$

By the same way, we have

$$E^0_x[e^{-\lambda\sigma_0}\ \Omega_1^-] = \frac{\sigma(0)^2}{2g_\lambda(0,0)}\left[-g_\lambda(0,0)\frac{\partial g_\lambda}{\partial v}(x,0)+g_\lambda(x,0)\frac{\partial g_\lambda}{\partial y}(0,0-)\right] .$$

This completes the proof of (1).

Proof of (2). Let us suppose the condition (A) of Lemma 5.5 satisfied. We can choose a constant $c\in R^1\setminus\{0\}$ such that

$$-g_\lambda(0,0)\frac{\partial g_\lambda}{\partial y}(c,0)+g_\lambda(c,0)\frac{\partial g_\lambda}{\partial y}(0,0-)\neq 0 .$$

Put $K(x)=\lim_{y\uparrow 0}\frac{g^0_\lambda(x,y)}{g^0_\lambda(c,y)}$ $x\neq 0$. Then we have

$$K(x)=\frac{-g_\lambda(0,0)\frac{\partial g_\lambda}{\partial y}(x,0)+g_\lambda(x,0)\frac{\partial g_\lambda}{\partial y}(0,0-)}{-g_\lambda(0,0)\frac{\partial g_\lambda}{\partial y}(c,0)+g_\lambda(c,0)\frac{\partial g_\lambda}{\partial y}(0,0-)},$$

$$\lim_{x\uparrow 0} K(x)=\infty \quad\text{and}\quad \lim_{x\downarrow 0} K(x)=0 .$$

As before, $e^{-\lambda t}K(X^0_t)$ is a nonnegative supermartingale having a left limit.

2) $\mathcal{F}^0_t$ is the σ-field generated by X^0_s, $0\leq s\leq t$.

Therefore we have $P_x^0(\lim_{t\uparrow\sigma_0} e^{-\lambda t}K(X_t^0)\text{ exists}/\Omega_1)=1$ for $x\neq 0$. It follows that $P_x(\exists t_n\uparrow\sigma_0,\ X_{t_n}<0/\Omega_1)=0$ for $x\neq 0$. This implies that $P_x(\Omega_1^+/\Omega_1)=1$ for $x\neq 0$.

Next suppose the condition (B) of Lemma 5.5 is fulfilled. We construct a finite measure μ on $(-\infty, 0)$ such that

$$U(x)=\int_{-\infty}^{0}\frac{g_\lambda^0(x, y)}{g_\lambda^0(c, y)}\mu(dy)$$

has the property

$$U(x)<\infty \quad \text{for} \quad x\in R^1\setminus\{0\} \quad \text{and} \quad \lim_{x\uparrow 0} U(x)=\infty .$$

Fix $c\in R^1\setminus\{0\}$. Note that

$$(8.4)\quad \frac{g_\lambda^0(x, -\varepsilon)}{g_\lambda^0(c, -\varepsilon)}=\frac{g_\lambda(0, 0)[g_\lambda(x, -\varepsilon)-g_\lambda(x, 0)]+g_\lambda(x, 0)[g_\lambda(0, 0)-g_\lambda(0, -\varepsilon)]}{g_\lambda(0, 0)[g_\lambda(c, -\varepsilon)-g_\lambda(c, 0)]+g_\lambda(c, 0)[g_\lambda(0, 0)-g_\lambda(0, -\varepsilon)]}.$$

Putting $y=x+\varepsilon$ and using the condition (B), we have

$$\begin{aligned} g_\lambda(x, -\varepsilon)-g_\lambda(x, 0) &= g_\lambda(x, x)-g_\lambda(x, x+(\varepsilon-y))-[g_\lambda(x,x)-g_\lambda(x, x-y)] \\ &\geqq K\varepsilon^\alpha\Big(\log\frac{1}{\varepsilon}\Big)^r \qquad \text{for } 0\leqq y\leqq a\varepsilon,\ 0\leqq\varepsilon\leqq\varepsilon_0, \end{aligned}$$

that is $g_\lambda(x, -\varepsilon)-g_\lambda(x, 0)\geqq K\varepsilon^\alpha\Big(\log\frac{1}{\varepsilon}\Big)^r$ for $-\varepsilon\leqq x\leqq -(1-a)\varepsilon,\ 0\leqq\varepsilon\leqq\varepsilon_0$.

Hence $g_\lambda^0(x, -\varepsilon)\geqq K_1\varepsilon^\alpha\Big(\log\frac{1}{\varepsilon}\Big)^r$. By the same way, we have $g_\lambda^0(c, -\varepsilon)<K_2\varepsilon^\beta$.

So we have

$$\frac{g_\lambda^0(x, -\varepsilon)}{g_\lambda^0(c, -\varepsilon)}>K'\varepsilon^{\alpha-\beta}\Big(\log\frac{1}{\varepsilon}\Big)^r \qquad \text{for } -\varepsilon\leqq x\leqq -(1-a)\varepsilon \text{ and } 0\leqq\varepsilon\leqq\varepsilon_0 .$$

We conclude from this that there exists a constant $K''>0$ such that

$$\frac{g_\lambda^0(x, x_n)}{g_\lambda^0(c, x_n)}>K''(1-a)^{n(\alpha-\beta)}\Big(\log\frac{1}{(1-a)^n}\Big)^r \qquad \text{for } x_n\leqq x\leqq x_{n+1},\ n=1, 2, \cdots,$$

where $x_n=-\varepsilon_0(1-a)^n$. Choose a constant b such that $0<b<1$ and $(1-a)^{\alpha-\beta}b>1$. Define a finite measure μ on $(-\infty, 0)$ by $\mu=\sum_{n=1}^{\infty}b^n\delta_{x_n}$. Because

$$\lim_{\varepsilon\to 0}\frac{g_\lambda^0(x, -\varepsilon)}{g_\lambda^0(c, -\varepsilon)}=\frac{g_\lambda(x, 0)}{g_\lambda(c, 0)},$$

we get

$$U(x)=\int_{-\infty}^{0}\frac{g_\lambda^0(x, y)}{g_\lambda^0(c, y)}\mu(dy)<+\infty \qquad \text{for any } x\neq 0.$$

Since $U(x) \geqq \frac{g_\lambda^0(xx_n)}{g_\lambda^0(c, x_n)} b^n = K''(1-a)^{n(\alpha-\beta)} \left(\log \frac{1}{(1-a)^n}\right)^r b^n$ for $x \in [x_n, x_{n+1}]$, we have $\lim_{x \uparrow 0} U(x) = \infty$. Since $e^{-\lambda t} U(X_t^0)$is a nonnegative supermartingale having a left limit. Therefore as before, we have $P_x(\exists t_n \uparrow \sigma_0, X_{t_n} < 0/\Omega_1) = 0$ for $x \neq 0$. Thus the proof of (2) is completed.

Proof of (3). First we show $P_x(\Omega_1^+/\Omega_1) = 0$ or 1. Define $f_t(\omega)$ by $f_t(\omega) = e^{-\lambda t} E^0_{X_t^0(\omega)}[e^{-\lambda \sigma_0}; \Omega_1^+]$. Let τ_n be as in the proof of (1). Since $\{f_{\tau_n}\}_{n \geqq 0}$ is a bounded nonnegative $\mathcal{F}^0_{\tau_n}$ -martingale, it follows that there exists a $f_\infty(\omega) = \lim_{n \to \infty} f_{\tau_n}(\omega)$ such that

$$f_{\tau_n}(\omega) = E_x^0[f_\infty(\omega)/\mathcal{F}^0_{\tau_n}], \qquad n \geqq 0 .$$

On the other hand, by the strong Markov property,

$$E^0_{X^0_{\tau_n}}[e^{-\lambda \sigma_0}; \Omega_1^+] = e^{+\lambda \tau_n} E_x^0[e^{-\lambda \sigma_0} I_{\Omega_1^+}(\omega)/\mathcal{F}_n] \quad \text{on} \quad t < \sigma_0.$$

So we have

$$\lim_{n \to \infty} E^0_{X^0_{\tau_n}}[e^{-\lambda \sigma_0}; \Omega_1^+] = I_{\Omega_1^+}(\omega) .$$

Hence in view of Lemma 7.6, we have

$$f_\infty(\omega) = \begin{cases} e^{-\lambda \sigma_0}, & \text{if } \omega \in \Omega_1^+, \\ 0 \quad , & \text{if } \omega \notin \Omega_1^+, \end{cases}$$

and in view of Lemma 7.6, we have $f_\infty(\omega) - e^{-\lambda \sigma_0} c_1$. This implies that $P_x(\Omega_1^+/\Omega_1) = 1$ or $P_x(\Omega_1^+/\Omega_1) = 0$. By the same way, we have $P_x(\Omega_1^-/\Omega_1) = 1$ or 0 and $P_x(\Omega_1^\pm/\Omega_1) = 1$or 0. Finally we show that $P_x(\Omega_1^+/\Omega_1) = 0$ and $P_x(\Omega_1^-/\Omega_1) = 0$. Let $\sigma_{-\varepsilon} = \inf\{t; X_t = -\varepsilon\}$. For any $x \in R^1 \setminus \{0\}$,

$$P_x(\sigma_{-\varepsilon} < \sigma_0) \geqq E_x[e^{-\lambda \sigma_{-\varepsilon}}; \sigma_{-\varepsilon} < \sigma_0] = \frac{g_\lambda^0(x, -\varepsilon)}{g_\lambda^0(-\varepsilon, -\varepsilon)} .$$

It follows from Lemma 5.6 that $\underline{\lim}_{\varepsilon \downarrow 0} P_x(\sigma_{-\varepsilon} < \sigma_0) > 0$. Hence there exists a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \downarrow 0$ such that

$$P_x(\overline{\lim_{n \to \infty}} \sigma_{-\varepsilon_n} < \sigma_0) > 0 .$$

Set $T_n = \sigma_{-\varepsilon_n} \wedge \sigma_{-\varepsilon_{n+1}} \wedge \cdots, n = 1, 2, \cdots$. Then T_n is increasing in n and $\lim_{n \to \infty} \sigma_0 \wedge T_n = \sigma_0$ a.s. on $\{\sigma_0 < \infty\}$. So we get

$$\overline{\lim_{n \to \infty}} \{\sigma_{-\varepsilon_n} < \sigma_0\} - \{T_n < \sigma_0 \text{ for any } n\} .$$

Therefore

$$P_x(T_n < \sigma_0, T_n \text{ t } \sigma_0) = P_x(\overline{\lim_{n \to \infty}} \{\sigma_{-\varepsilon_n} < \sigma_0\}, T_n \uparrow \sigma_0) > 0 .$$

It follows that $P_x(\Omega_1^+/\Omega_1)\neq 1$. So $P_x(\Omega_1^+/\Omega_1)=0$. Similarly we can show that $P_x(\Omega^-/\Omega_1)=0$. Therefore we have $P_x(\Omega_1^\mp/\Omega_1)=1$. The proof of Theorem 2 is completed.

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