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ON THE STABLE JAMES NUMBERS OF COMPLEX PROJECTIVE SPACES

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1. Introduction

For a pointed finite CW-pair $i: A\subset X$ where A is a connected oriented topological manifold, a (stable) map $f: X\to A$ is of type r if the composite $A\xrightarrow{i}X\xrightarrow{f}A$ has degree r. $j(X, A)$ and $j_s(X, A)$ denote the sets of integers r for which there exists a map $f: X\to A$ of type r and a stable map of type r respectively. When $j(X, A)$ forms an ideal $(k(X,A))$ in the ring of integers Z—here $k(X, A)$ denotes the non-negative generator, we call $k(X, A)$ the James number of the pair (X, A). In the stable case $j_s(X, A)$ is always an ideal of Z. So we may define the stable James number $k_s(X, A)$.

James [3] has posed the problem of determining $j(SP^m(S^n), S^n)$, where $SP^m(S^n)$ is the m-fold symmetric product of an n-sphere S^n with a base point x_0 and $i: S^n\to SP^m(S^n)$ is the axial embedding $x\to[x, x_0, \cdots, x_0]$. James showed for example $j(SP^m(S^n), S^n)$ forms an ideal of Z and, for an even dimentional sphere S^{2n}, $k(SP^m(S^{2n}), S^{2n})=0$. On the contrary $k_s(SP^m(S^n), S^n)\neq 0$ for any positive integers m and n. From now on we introduce the notation $k_s^{m,n}$ instead of $k_s(SP^m(S^n), S^n)$.

In this note we give lower bounds and an upper bound of $k_s^{m,2}$. That is, we prove

Theorem. *For positive integers m and n*

(1) $k_s^{m,n}\neq 0$;

(2) $k_s^{m+1,n}$ *is a multiple of* $k_s^{m,n}$;

(3) $k_s^{m,2}$ *is divisible by all the integers* $m, m-1, \cdots, 2$;

(4) $k_s^{2^m-1,2}$ *is divisible by* 2^m *for* $m\geqq 2$;

(5) $k_s^{m,2}$ *is a divisor of* $m!(m-1)!\cdots 2!$, *in particular none of the prime factors of* $k_s^{m,2}$ *is greater than m.*

Corollary. *The above lower estimates* (3) *and* (4) *are best possible for* $m\leqq 4$. *That is*

$$k_s^{1,2}=1,\ k_s^{2,2}=2,\ k_s^{3,2}=k_s^{4,2}=12\,.$$

There is a homeomorphism $SP^m(S^2) \simeq CP^m$, the m-dimensional complex projective space. Under this identification the natural inclusions $S^2 \overset{i}{\subset} SP^m(S^2) \subset SP^{m+1}(S^2)$ become the standard ones $CP^1 \overset{i}{\subset} CP^m \subset CP^{m+1}$. $k_s^{m,2}$ is just the same as Conner-Smith's $d(m)$ [1], Example 4.

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2. Proofs of (1) and (2)

Using the group multiplication of S^1, we know $k_s^{m,1}=1$. So we assume $n \geqq 2$. First we prove (1). Consider the stable Puppe exact sequence

$$\cdots \to \{SP^m(S^n), S^n\} \overset{i^*}{\to} \{S^n, S^n\} \to \{SP^m(S^n)/S^n, S^{n+1}\} \to \cdots,$$

here $\{X, Y\}$ denotes the set of stable homotopy classes of stable maps $X \to Y$. Since $\{S^n, S^n\} \cong Z$, if i^* is non-trivial, then $k_s^{m,n}$=index of image $i^* \neq 0$. So, for our purpose, it suffices to show that $\{SP^m(S^n)/S^n, S^{n+1}\}$ is finite. Notice that $\{SP^m(S^n)/S^n, S^{n+1}\} = \pi_s^{n+1}(SP^m(S^n)/S^n)$ is the reduced framed cobordism group. Let $E_2^{u,v} = \tilde{H}^u(SP^m(S^n)/S^n; G_{-v}) \Rightarrow \pi_s^*(SP^m(S^n)/S^n)$ be the Atiyah-Hirzebruch spectral sequence for $SP^m(S^n)/S^n$, where G_k is the stable k-stem of spheres. Since $\tilde{H}^u(SP^m(S^n)/S^n; Z)=0$ for $u \leqq n+1$, $\sum_{u+v=n+1} E_2^{u,v}$ is finite. Then $\sum_{u+v=n+1} E_\infty^{u,v}$ and hence $\pi_s^{n+1}(SP^m(S^n)/S^n)$ are finite. This implies (1).

From the equality $k_s^{m,n}$=index of image i^*, (2) is obvious. Thus (1) and (2) follow.

3. Proof of (3)

We use the complex K-theory. Let η_m be the canonical complex line bundle over CP^m and $g_C = \eta_1 - 1 \in \tilde{K}(S^2)$ be the Bott generator. Then $K(CP^m)$ is the truncated polynomial ring with generator $\eta_m - 1$ and the relation $(\eta_m - 1)^{m+1} = 0$. Choose $f \in \{CP^m, S^2\}$ such that $i^*(f) = k_s^{m,2}\iota$, where ι denotes the identity map of S^2. Let $f^*: K^*(S^2) \to K^*(CP^m)$ and $f^*: H^*(S^2; Q) \to H^*(CP^m; Q)$ be the induced homomorphisms. Put

$$f^*(g_C) = \sum_{j=1}^{m} a_j(\eta_m - 1)^j$$

where $a_j \in Z$. Since $i^*(f) = k_s^{m,2}\iota$, we have $a_1 = k_s^{m,2}$. Let $t \in H^2(CP^m; Z)$ be the first Chern class of η_m. We apply the Chern character, ch, for $\tilde{K}^*(CP^m)$ and $\tilde{K}^*(S^2)$. Then

$$a_1 t = k_s^{m,2} t = (f^* \circ ch)(g_C) = (ch \circ f^*)(g_C) = \sum_{j=1}^{m} a^j(\exp(t) - 1)^j$$

that is

$$t=\sum_{j=1}^{m}(a_j/a_1)(\exp(t)-1)^j$$

in $H^*(CP^m; Q)$, where we used the fact that ch is "stable". On the other hand

$$t=\log(1+(\exp(t)-1))=\sum_{j=1}^{\infty}((-1)^{j+1}/j)(\exp(t)-1)^j.$$

Hence

$$k_s^{m,2}=a_1=(-1)^{j+1}ja_j;\ j=1, 2, \cdots, m.$$

This implies (3).

4. Proof of (4)

In this section we use KO-theory. We introduce the following notations: η_H=the canonical symplectic line bundle over S^4; $g_H=\eta_H-1\in\widetilde{KSp}(S^4)$; $g_R=g_H\wedge g_H\in\widetilde{KO}(S^8)$; $\rho: K^*(\ )\to KO^*(\ )$, the real restriction; $\varepsilon: KO^*(\ )\to K^*(\ )$, the complexification; $\mu_3=\rho(g_C^3\wedge(\eta_m-1))\in\widetilde{KO}^{-6}(CP^m)$; $\mu_0=\rho(\eta_m-1)\in\widetilde{KO}(CP^m)$. We require the following theorem of Fujii [2]:

$\widetilde{KO}^{-6}(CP^m)$ is the free module with basis μ_3, $\mu_3\mu_0$, $\cdots$, $\mu_3\mu_0^{u-1}$, and also, in case m is odd, $\mu_3\mu_0^u$(if $m\equiv 3$ mod 4) or τ(if $m\equiv 1$ mod 4), where $2\tau=\mu_3\mu_0^u$ and $u=[m/2]$ ([ ] is the Gauss notation).

Choose $f\in\{CP^m, S^2\}$ such that $i^*(f)=k_s^{m,2}\iota$. Let $f^*: KO^*(S^2)\to KO^*(CP^m)$ be the induced homomorphism. By Fujii's theorem we may write

$$f^*(g_R)=\begin{cases}\sum_{j=0}^{(m/2)-1}a_j\mu_3\mu_0^j & \text{if } m\equiv 0 \bmod 2\\ \sum_{j=0}^{[m/2]}a_j\mu_3\mu_0^j & \text{if } m\equiv 3 \bmod 4\\ \sum_{j=0}^{[m/2]-1}a_j\mu_3\mu_0^j+a_{[m/2]}\tau & \text{if } m\equiv 1 \bmod 4\end{cases}$$

where $a_j\in Z$. In case $m=1$, we have $\mu_3=2g_R\in\widetilde{KO}^{-6}(S^2)$. This and $i^*(f)=k_s^{m,2}\iota$ imply $2a_0=k_s^{m,2}$. We write ch for $ch\circ\varepsilon$. Then we have

$$ch(\mu_3)=\exp(t)-\exp(-t)=2\sinh(t)$$

and

$$ch(\mu_0)=\exp(t)+\exp(-t)-2=2(\cosh(t)-1).$$

Since

$$2a_0t=k_s^{m,2}t=(f^*\circ ch)(g_R)=(ch\circ f^*)(g_R),$$

we obtain

$$(\sharp)\ a_0 t = \begin{cases} \sinh(t) \sum_{j=0}^{(m/2)-1} 2^j a_j(\cosh(t)-1)^j & \text{if } m\equiv 0 \bmod 2 \\ \sinh(t) \sum_{j=0}^{[m/2]} 2^j a_j(\cosh(t)-1)^j & \text{if } m\equiv 3 \bmod 4 \\ \sinh(t)\{ \sum_{j=0}^{[m/2]-1} 2^j a_j(\cosh(t)-1)^j & \\ \qquad + 2^{[m/2]-1} a_{[m/2]}(\cosh(t)-1)^{[m/2]}\}, & \text{if } m\equiv 1 \bmod 4 \end{cases}$$

in $H^*(CP^m; Q)$. In case $m\equiv 0 \bmod 2$, if we differentiate the two sides of ($\sharp$) by t, then we have

$$j!a_0 = (-1)^j 2^j \cdot 3\cdot 5 \cdots (2j+1)a_j \quad \text{for } 1\leqq j\leqq m/2-1\,,$$

and elementary calculation shows that we obtain the same information on $k_s^{m,2} = 2a_0$ as (3). This and ($\sharp$) imply that we obtain the same information about $k_s^{4j+1,2}$ and $k_s^{4j+1,2}$ for $j\geqq 1$. Hence,in case $m\equiv 1 \bmod 4$, we obtain nothing more than (3). In case $m\equiv 3 \bmod 4$, that is $m=4j-1$ for some j, we have the same information about $k_s^{4j-1,2}$ and $k_s^{4j,2}$. If j is a power of two, 2^q, from (3) we see that 2^{q+1} divides $k_s^{2^{q+2}-1,2}$, but the aboves imply that 2^{q+2} divides $k_s^{2^{q+2}-1,2}$. Thus (4) follows. Remark, in case j is not a power of two, we obtain nothing more than (3).

5. Proofs of (5) and Corollary

Choose $f_{m-1}\in\{CP^{m-1}, S^2\}$ such that $i^*(f_{m-1})=k_s^{m-1,2}\iota$. Let p_{m-1}: $S^{2m-1} \to CP^{m-1}$ be the canonical fibration and ord (p_{m-1}) be its order as a stable map. The composite $(\mathrm{ord}(p_{m-1}))\iota\circ f_{m-1}\circ p_{m-1}$ is null homotopic. Hence there exists $f\in\{CP^m, S^2\}$ such that $f\circ j=(\mathrm{ord}(p_{m-1}))\iota\circ f_{m-1}\in\{CP^{m-1}, S^2\}$, where j: $CP^{m-1} \subset CP^m$. This implies that $k_s^{m,2}$ is a divisor of $\mathrm{ord}(p_{m-1})\cdot k_s^{m-1,2}$. Inductively we know that $k_s^{m,2}$ is a divisor of $\mathrm{ord}(p_{m-1})\cdot\mathrm{ord}(p_{m-2})\cdots\mathrm{ord}(p_1)k_s^{1,2}$. Obviously $k_s^{1,2} = 1$. By Toda [4], page 1103, $\mathrm{ord}(p_{m-1})$ is a divisor of $m!$. Thus (5) follows. And we complete the proof of Theorem.

We prove Corollary. For $m\leqq 3$, the estimates (3), (4) and (5) imply that $k_s^{1,2}=1$, $k_s^{2,2}=2$ and $k_s^{3,2}=12$. We show $k_s^{4,2}=12$. Choose $f_3\in\{CP^3, S^2\}$ such that $i^*(f_3)=12\iota$. The composite $f_3\circ p_3$: $S^7\to S^2$ represents an element of G_5, five-stem of spheres. It is well known that $G_5=0$. Hence there exists $f\in\{CP^4, S^2\}$ such that the composite $CP^3\subset CP^4 \xrightarrow{f} S^2$ coincides with f_3. This implies that $k_s^{4,2}$ is a divisor of 12. By (2) $k_s^{4,2}$ is a multiple of $k_s^{3,2}=12$. Therefore $k_s^{4,2}=12$. This completes the proof of Corollary.

6. Addendum

The same technique is applicable to the stable James number $d_H(m)=k_s$

(HP^m, S^4) of the pair of symplectic projective spaces. Using the complex K-theory, a lower bound of $d_H(m)$ can be obtained from

$$a_1 t = \sum_{j=1}^{m} 2^j a_j \Big(\sum_{k=1}^{\infty} t^k/(2k)! \Big)^j, \quad a_1 = d_H(m),$$

where $t \in H^4(HP^m; Z)$ is a generator and $a_j \in Z$. For example, we have $12 \mid d_H(2)$. Since the order of the canonical fibration $S^7 \to S^4$ as a stable map is 24, we have $d_H(2)=24$. So that this estimate is not best possible.

The unstable James numbers of the pairs (RP^m, S^1), (CP^m, S^2), (HP^m, S^4) and the stable James number of (RP^m, S^1) are all zero for $m \geqq 2$, where RP^m denotes the m-dimensional real projective space.

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Added in proof. After completed this manuscript, the author has found a paper of J. Ucci "Symmetric maps of spheres of least positive James number, Indiana Univ. Math. J. (1972), 709—714" which gives an upper bound of unstable James numbers $k^{m,n}=k(SP^m(S^n), S^n)$. Combining his estimate with ours, we obtain

Theorem A.
(i) $\beta_2(m) \leqq \nu_2(k_s^{m,2}) \leqq 2\beta_2(m)$,
(ii) $m \leqq \nu_2(k_s^{2^m-1,2}) \leqq 2m-2$ *for* $m \geqq 2$,
(iii) $\nu_p(k_s^{m,2}) = \beta_p(m)$ *for an odd prime* p,
where $\nu_p(n)$ *denotes the exponent of* p *in the prime factorization of* n *and* $\beta_p(m)$ *is defined by* $p^{\beta_p(m)} \leqq m < p^{\beta_p(m)+1}$.

Proof. Identifying $S(S^n)$ with S^{n+1}, $S(SP^m(S^n))$ can be embedded in $SP^m(S^{n+1})$ so that the inclusion $S^{n+1} \xrightarrow{i} SP^m(S^{n+1})$ factorizes as the composition $S(S^n) \xrightarrow{S(i)} S(SP^m(S^n)) \subset SP^m(S^{n+1})$, where $S(X)$ denotes the reduced suspension of a pointed space X. This implies that $k_s^{m,n}$ is a factor of $k_s^{m,n+1}$. By definition, $k_s^{m,n}$ is a factor of $k^{m,n}$ for odd n. So, in particular, $k_s^{m,2}$ is a factor of $k^{m,3}$. Ucci's estimates of $k^{m,3}$ are $\nu_2(k^{m,3}) \leqq 2\beta_2(m)$ and $\nu_p(k^{m,3}) = \beta_p(m)$ for an odd prime p. Therefore we have $\nu_2(k_s^{m,2}) \leqq 2\beta_2(m)$ and $\nu_p(k_s^{m,2}) \leqq \beta_p(m)$ for an odd prime p. On the other hand the estimates (3) and (4) imply that $\beta_p(m) \leqq \nu_p(k_s^{m,2})$ for a prime p and $n \leqq \nu_2(k_s^{2^n-1,2})$ for $n \geqq 2$. Thus Theorem A follows.