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INVOLUTIONS AND CIRCLE ACTIONS WHICH BORD

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In [7; §7] Fuichi Uchida demonstrates that the Thom-Gysin sequence enables one to determine that any smooth principal circle action on $S^{2m+1}\times S^{2n+1}$ bords as a free S^1 action. In this paper similar results are proved for free involutions, Uchida's results on circle actions extended in several directions, and some results are elicited on the bordism of arbitrary involutions on products of two spheres.

Denote by $\hat{N}_*(G)$ and $\hat{\Omega}_*(G)$ respectively the bordism of smooth principal G actions on closed smooth mainfolds and the bordism of smooth principal orientation preserving G actions on closed smooth oriented mainfolds. In section 1 it is shown that

Theorem 1.3. *If T is a smooth fixed point free involution on the product of two spheres $S^m\times S^n$, then $[S^m\times S^n, T]$ bords in $\hat{N}_*(Z_2)$.*

In section 2 it is shown that

Theorem 2.1. *Any smooth principal circle action on $S^m\times S^n$ bords in $\hat{N}_*(S^1)$.*

Note that Uchida's result [7; th. 7.3.] is a corollary to this theorem. Also in section 2 one finds

Theorem 2.4. *Any principal circle action on $(S^{2j+1})^{2k}$ bords in $\hat{N}_*(S^1)$.*

Theorem 2.5. *If $RP(n_1)\times\cdots\times RP(n_r)$ is a product of real projective spaces so that at least two of the n_j are odd, then any free circle action on $RP(n_1)\times\cdots\times RP(n_r)$ bords in $\hat{N}_*(S^1)$.*

Further, there are corollaries to each of these results giving modified oriented analogues.

In section 3 the bordism of arbitrary smooth involutions on products of spheres is examined. Let $N_*(Z_2)$ be the bordism of unrestricted smooth involutions. Since there is an injection from $N_*(Z_2)$ into $\oplus N_*(BO_k)$ given by classifying the normal bundle to the fixed set [3; §28], it suffices to consider the bordism of the fixed set and its normal bundle. Hence an involution on a single sphere bords since the Smith results [1; chapt. 3] yield that the fixed set is itself a Z_2 cohomology sphere and hence the total Stiefel Whitney class of the normal bundle is trivial. Using the results of Bredon on the cohomology of the fixed set of involutions on products of two spheres [1; chapt. 7] the following theorem is exhibited.

Theorem 3.1. *If T is a smooth involution on $S^m \times S^n$, $[S^m \times S^n, T]$ bords in $N_*(Z_2)$ unless $(m, n)=(3, 4), (5, 8), (6, 8), (6, 9), (7, 8), (7, 9)$ or $(7, 10)$.*

The author would like to thank R.E. Stong whose questions led to several of the results of this paper.

1. Free involutions on the product of two spheres. If M admits a free involution T, then there is the Gysin exact sequence in cohomology for the principal Z_2 bundle $M \to M/T$. Following Uchida, this exact sequence is used to determine the cohomology algebra of $(S^m \times S^n)/T$ and hence the bordism of $[S^m \times S^n, T]$.

Lemma 1.1. *If c in $H^*(M; Z_2)$ is a 1-dimensional cohomology element, and x is any element of $H^*(M, Z_2)$ such that $xc^{q+1}=0$, then $Sq^i(x)\cdot c^{q+k}=0$ for $k \geqslant 1$.*

Proof. $Sq^1(xc^{q+k})=0$ but by the Cartan formula $Sq^1(xc^{q+k})=Sq^1(x)\cdot c^{q+k}+x\cdot Sq^1(c^{q+k})$ where the second summand is zero since $Sq^i(c^j)=\binom{j}{i}c^{j+i}$. One inducts on i with the Cartan formula giving the induction step. ****

Let $m \leqslant n$, $M=S^m \times S^n$, and T be a free involution on M.
Then

Lemma 1.2. *$H^*(M/T; Z_2)=Z_2[c]/c^{k+1} \oplus \Lambda[x]$ as modules where $k=m$ or n, $\dim x=m+n-k$, and c is the first Stiefel Whitney class of the principal Z_2 bundle $M \to M/T$.*

Proof. The Thom-Gysin sequence of $M \to M/T$,

$$H^*(M/T) \xrightarrow{\pi^*} H(M) \xrightarrow{\delta} H^*(M/T), \qquad H^*(M/T) \xleftarrow{\cdot c} H^*(M/T)$$

yields the desired result modulo understanding the image of $\delta: H^n(M) \to H^n(M/T)$ in the case where the dimension of x is m. Denote by β the generator of $H^n(M)$. It is sufficient to show that $\delta(\beta)=c^n$.

Suppose this is not case. Suppose $\delta(\beta)=y$ where $y=xc^{n-m}$ or $y=xc^{n-m}+c^n$. Note that $y\cdot c=0$ and $H^{n+m}(M/T)=Z_2[c^{m+n}]$. Now since $\pi^*(x)=\alpha$ in $H^m(M)$ and δ is an $H^*(M/T)$ module homomorphism, $\delta(\alpha\cdot\beta)=x\cdot y=c^{m+n}$. Now by applying lemma 1.1 one sees that $Sq^m(y)=x\cdot y$. It follows that V_m, the m^{th} Wu class, is either x or $x+c^m$.

Hence the m^{th} Stiefel Whitney class, $W_m(M/T)=Sq(1+v_1+\cdots+v_m)=v_m+$ powers of $c=x+$powers of c. Thus $\pi^*(W_m)\neq 0$. Now $\pi^*(\tau(M/T))=\tau(M)$ where $\tau(M)$ is the tangent bundle to M. Since $W(M)=1$ one has a contradiction.$***$

Theorem 1.3. *If T is a smooth involution on $M=S^m\times S^n$, $[M, T]=0$ in $\hat{N}_*(Z_2)$.*

Proof. Since $\hat{N}_*(Z_2)$ is isomorphic to $N_*(BO_1)$ it suffices to compute the bordism of the principal Z_2 bundle $M\to M/T$. One first claims that the characteristic ring of $H^*(M/T)$ is contained in the subalgebra generated by c. This follows from lemma 1.2 and the fact that the i^{th} Wu class V_i is the unique element of $H^i(M/T)$ such that $Sq^i(x)=V_i\cdot x$ for x in $H^{n+m-i}(M/T)$. Hence each $V_i=c^i$ or 0 which implies the same is true for W_i. Since $c^{n+m}=0$, it follows that $[M, T]=0$ in $\hat{N}_*(Z_2)$.$***$

Corollary 1.4. *If T is a free orientation preserving involution on $M=S^m\times S^n$, then $[M, T]=0$ in $\hat{\Omega}_*(Z_2)$.*

Proof. There is a generalized Rohlin exact sequence $\hat{\Omega}_*(Z_2)\xrightarrow{2}\hat{\Omega}_*(Z_2)\xrightarrow{\rho}\hat{N}_*(Z_2)$ (see [2; (16.2)]). Further, via the augmentation split exact sequence $\hat{\Omega}_*(Z_2)\cong\Omega_*\oplus\tilde{\Omega}_*(BZ_2)$ where $\tilde{\Omega}_*(BZ_2)$ is torsion of order 2 [6; pp. 4-5]. Hence $[M,T]=[M/T\times Z_2$, interchange$]+[N, S]$ where $[N]=0$ in Ω_*. Thus $2[M/T]=0$ implying that $[M, T]$ is of order 2. One need only note that ρ is monic on torsion of order two.

2. Free circle actions on products of spheres and products of projective spaces. For a mainfold M let $\phi:S^1\times M\to M$ denote a smooth principal circle action on M. One has easily

Theorem 2.1. $[S^m\times S^n, \phi]=0$ *in* $\hat{N}_*(S^1)$.

Proof. Let γ be the canonical complex line bundle over BS^1 and let $\gamma^2=\gamma\otimes\gamma$. From the cofibration $S(\gamma^2)\to D(\gamma^2)\to(D(\gamma^2), S(\gamma^2))$ one gets a short exact sequence $0\to\hat{N}_*(S^1)\xrightarrow{\rho}\hat{N}_*(Z_2)\to\hat{N}_*(S^1)\to 0$ where ρ is the restriction map from free circle actions to free Z_2 actions. The result follows from theorem 1.3.$***$

Corollary 2.2. *If ϕ is a principal circle action on $S^{2k+1}\times S^{2r+1}$, then $[S^{2k+1}\times S^{2r+1}, \phi]=0$ in $\hat{\Omega}_*(S^1)$.*

Proof. One again has a generalized Rohlin exact sequence $\hat{\Omega}_*(S^1)\xrightarrow{2}\hat{\Omega}_*(S^1)\xrightarrow{\rho}\hat{N}_*(S^1)$. One computes $\Omega_*(S^1)=\Omega_{*-1}(BS^1)$ to learn that the torsion is of order 2 and that $\hat{\Omega}_{ev}(S^1)$ is torsion. Hence the forgetful map ρ is monic on $\hat{\Omega}_{ev}(S^1)\to\hat{N}_{ev}(S^1)$.$***$

One considers next principal S^1 actions on even powers of odd dimensional spheres.

Lemma 2.3. *Let $N=(S^{2j+1})^{2k}$. For any principal S^1 bundle $N\to N/S^1$, $H^*(N/S^1)=Z_2[u]/u^{j+1}\otimes\Lambda(x_1)\otimes\Lambda(x_2)\otimes\cdots\otimes\Lambda(x_{2k-1})$ as a ring where the degree of x_i is $2j+1$ and where u is the second Stiefel Whitney class of the principal S^1 bundle.*

Proof. From the Z_2 cohomology Thom-Gysin sequence for the appropriate principal S^1 bundle one learns that $H^{2j+1}(N/S^1)\to H^{2j+1}(N)$ can not be epic since $\dim(N/S^1)<\dim N$. Hence $H^{2j+1}(N/S^1)$ must be generated by $2k-1$ elements, $x_1, x_2, \cdots, x_{2k-1}$, which map onto $2k-1$ of the generators of $H^{2j+1}(N)$. If $r\equiv 0$ $(2k+1)$ and $0\leqslant i\leqslant 2j+1$, it follows that multiplication by u from $H^{r+i}(N/S^1)$ into $H^{r+i+2}(N/S^1)$ is the zero homomorphism if i is odd or if $i=2j$ and is an isomorphism otherwise. Hence the fact that the Z_2 rank of $H^{q(2j+1)}(N)$ is $\binom{2k}{q}$ and the fact that $\binom{2k-1}{q-1}+\binom{2k-1}{q}=\binom{2k}{q}$ implies that the rank of $H^{q(2j+1)+i}(N/S^1)$ is $\binom{2k-1}{q}$ if i is even and is 0 if i is odd. Since q dimensional monomials in the $x_1, \cdots, x_{2k-1}$ generate a submodule of $H^{q(2j+1)}(N/S^1)$ of dimension $\binom{2k-1}{q}$ the result follows.$_{***}$

Theorem 2.4. *A principal S^1 action on $(S^{2j+1})^{2k}$ bords in $\hat{N}_*(S^1)$.*

Proof. In the notation of lemma 2.3, it is enough to consider the bordism class of $N\to N/S^1$ in $N_*(BS^1)$. Now u generates the image of $H^*(BS^1)$ in $H^*(N/S^1)$ and $Sq^1(u)=0$. It follows from lemma 2.3 that $Sq^1: H^{ev}(N/S_1)\to H^{od}(N/S^1)$ is zero. Since $W_{2k+1}=Sq^1(W_{2k})$ for an oriented mainfold and dimension of N/S^1 is odd, the result follows.$_{***}$

Corollary 2.5. $[(S^{2j+1})^{2k}, \phi]=0$ *in* $\hat{\Omega}_*(S^1)$ *for all principal S^1 actions* ϕ.$_{***}$

Now consider $P=RP(n_1)\times\cdots\times RP(n_r)$. For a principal S^1 action on P one considers the Thom-Gysin sequence with Z_2 coefficients of $\pi: P\to P/S^1$. One has $0\to H^1(P/S^1)\xrightarrow{\pi^*}H^1(P)\to H^0(P/S^1)\xrightarrow{\cdot u}H^2(P/S^1)\xrightarrow{\pi^*}$ where u is $W_2(P\to P/S^1)$. Since $H^*(P)$ is generated by 1 dimensional elements and since $\dim(P/S^1)<\dim(P)$, $H^1(P/S^1)\to H^1(P)$ is not epic implying $u=0$ and π^* is monic.

Let $\omega=(i_1, i_2, \cdots, i_k)$ be a k-tuple of nonnegative integers, $n(\omega)=i_1+i_2+\cdots+i_k$, and $W_\omega(P)=W_{i_1}(P)\cdots W_{i_k}(P)$. If n_j is odd, then $W_\omega(RP(n_j))=0$ if $n(\omega)>n_j-1$. For $P=RP(n_1)\times\cdots\times RP(n_r)$ let q be the number of odd n_j. Then $W_\omega(P)=0$ if $n(\omega)>\dim(P)-q$. Now $\tau(P)=\pi^*(\tau(P/S^1))\oplus 1$ so $W_\omega(P/S^1)=0$ if $n(\omega)>\dim(P)-q$.

Theorem 2.6. *If $\phi: S^1\times\prod_{j=1}^{r}RP(n_j)\to\prod_{j=1}^{r}RP(n_j)=P$ is principal and at least two of the n_j are odd, then $[P, \phi]=0$ in $\hat{N}_*(S^1)$.*

Proof. Consider $P \to P/S^1$ in $N_*(BS^1)$. From the arguments above, all numbers are $0._{***}$

Note. $S^1 = S^1/Z_2$ acts freely on $S^5/Z_2 = RP(5)$ with orbit space being $CP(2)$. Since $[CP(2)] \neq 0$ in N_*, all the hypotheses of theorem 2.6 are needed.

Corollary 2.7. *If* $\phi: S^1 \times \prod_{j=1}^{2k} RP(2n_j+1) \to \prod_{j=1}^{2k} RP(2n_j+1) = P$ *is orientation preserving and principal, then* $[P, \phi] = 0$ *in* $\hat{\Omega}_*(S^1)._{***}$

3. Involutions on a product of two spheres. Let $X = S^m \times S^n$, $m \leqslant n$, and let F be the fixed set of an involution T on X. If ν is the normal bundle to F in X, then the monomorphism $N_*(Z_2) \to \oplus N_*(BO_k)$ sends $[X, T]$ into $[F, \nu]$. Now $W(\tau(F)) \cdot W(\nu) = W(\tau(X)/F) = 1$. Hence if f classifies $\nu \to F$, $f^*(\oplus H^*(BO_k))$ is contained in the characteristic ring of $H^*(F)$. Thus $[X, T]$ bords in $N_*(Z_2)$ if $[F]$ bords in $\oplus N_*$.

From Bredon's work [1; p. 410] one knows up to Z_2 cohomology ring the possible fixed sets of Z_2 acting on X. They are:

(1) $F \sim_2 S^q$
(2) $F \sim_2 S^q + S^p \quad 0 \leqslant p \leqslant n,\ 0 \leqslant q \leqslant n$
(3) $F \sim_2 P^3(q) \quad m > q$
(4) $F \sim_2 P^2(q) \# P^2(q) \quad m \geqslant q = 1,2,4,8$
(5) $F \sim_2 pt + P^3(q) \quad m > q$
(6) $F \sim_2 S^p \times S^q \quad 0 \leqslant p \leqslant m,\ 0 \leqslant q \leqslant n$

Consideration of these facts leads one to

Theorem 3.1. *If* T *is an involution on* $X = S^m \times S^n$, $[X, T] = 0$ *in* $N_*(Z_2)$ *unless* $(m, n) = (3, 4), (5, 8), (6, 8), (6, 9), (7, 8), (7, 9)$ or $(7, 10)$.

Proof. Consider each of Bredon's cases (1)–(6). For involutions where the cohomology ring of the fixed set of T on X is described by one of the cases (1)–(3), it is clear from elementary considerations that the total Stiefel Whitney class of each component of F is 1. Hence $[X, T] = 0$ in $N_*(Z_2)$ for all (m, n).

In case (4) the positive dimensional cohomology is generated by two q dimensional elements u and v with the property that $u^2 = v^2 \neq 0$ and $uv = 0$. Now $Sq^q(u) = u^2 = Sq^q(v)$ so the q^{th} Wu class $V_q(F) = u + v$ and $V_q(F) = W_q(F)$. Hence $W(F) = 1 + W_q$ with $W_q^2 = 0$. One concludes that $[F] = 0$ in N_* and $[X, T] = 0$ in $N_*(Z_2)$.

From Bredon [1; p. 414] case (5) can only occur if $(m, n) = (3, 4), (5, 8), (6, 8), (6, 9), (7, 8), (7, 9)$, or $(7, 10)$.

Now suppose the fixed set is case (6), that is,

$$H^*(F) = \Lambda(u) \otimes \Lambda(v) \text{ with } \dim u = p \text{ and } \dim v = q. \qquad (*)$$

It is clear that if F does not bound then the Steenrod action is given by

$$Sq^p(v) = uv,\ Sq^{q-p}(u) = v. \qquad (**)$$

Hence this theorem will be proved when the following two lemmas have been demonstrated.$_{***}$

Lemma 3.2. *If F is the fixed set of an involution on $X=S^m\times S^n$ with its cohomology ring and associated Steenrod action given by* $(*)$ *and* $(**)$, *then* $(p, q)=(2, 3)$ *with* $8<(m+n)<10$ *or* $(p, q)=(4, 6)$ *with* $16<m+n<20$.

Lemma 3.3. *If F is as in lemma* 3.2, $(m, n)=(7, 10)$.

Proof of lemma 3.2. The fact that $(p, q)=(2, 3)$ or $(4, 6)$ comes from E. E. Floyd's work [3; (3.1)]. Now let ν be the normal bundle to F in X. Since $W(X)=1$, $W(F)=1+u+v=W(\nu)$ implying that dim $\nu\geqslant q$ or $m+n\geqslant p+2q$. If $m+n=p+2q$, dim $\nu=q$. Consider the inclusion $F\rightarrow T(\nu)$ where $T(\nu)$ is the Thom space of ν. The Thom class in $H^*(T(\nu))$ comes back to hit $W_q(\nu)=v$. But $Sq(v)\neq 0$ which contradicts the fact that $F\rightarrow T(\nu)$ factors through X. Hence $m+n>p+2q$.

Now suppose $m+n>2(p+q)$. Consider the sphere bundles of ν and $\tau(F)\oplus(m+n-p-q)$ together with involution given by multiplication by -1 (where k denotes the trivial k plane bundle). Since $W(\nu)=W(\tau(F)\oplus(m+n-p-q))$, $[S(\nu),-1]=[S(\tau(F)\oplus(m+n-p-q), -1]$ in $\hat{N}_*(Z_2)$. Applying [2; (24.1)] one understands that $[S(\tau(F)\oplus(m+n-p-q)), -1]=0$ in $\hat{N}_*(Z_2)$. The Smith homomorphism (see [5, §6]) acting $m+n-(p+q)-1$ times yields $[S(\tau(F)\oplus 1), -1]=0$ in $\hat{N}_*(Z_2)$. However, [2; (24.2)] when applied to the twist involution on $F\times F$ indicates that $[S(\tau\oplus 1)/-1]_Q=[F\times F]\neq 0$ which is a contradiction. It follows that $m+n\leqslant 2(p+q)$.

However, if $m+n=2(p-q)$, applying [2; (24.2)] as before demonstrates that $0=[X]=[S(\nu\oplus 1)/-1]=[S(\tau\oplus 1/-1)]=[F\times F]\neq 0$. $p+2q<m+n<2(p+q)$.$_{***}$

Proof of Lemma 3.3. According to Bredon, the case under consideration occurs only if $X\xrightarrow{i} X\times_{Z_2} EZ_2$ is totally nonhomologous to zero and the homomorphism on cohomology induced by j: $F\times BZ_2\rightarrow X\times_{Z_2} EZ_2$ is monic and is a $H^*(BZ_2)$ module homomorphism [1; chapt. 7]. Also as in Bredon one need only consider the maps on relative cohomology

$$H^*(X, x_0)\xleftarrow{i^*} H^*(X\times_{Z_2} EZ_2, x_0\times BZ_2)\xrightarrow{j^*} H^*(F\times BZ_2, x_0\times BZ_2)$$

for x_0 a point in F. Let a, b, and ab be a basis for $H^*(X, x_0)$ with dim $a=m$, dim $b=n$. Let α represent a and β represent b in $H^*(X\times_{Z_2} EZ_2, x_0\times BZ_2)$. If t generates $H*(BZ_2)$ then

$$j^*(\alpha) = A_0 t^{m-p} \otimes u + B_0 t^{m-q} \otimes v + C_0 t^{m-(p+q)} \otimes uv$$
$$j^*(\beta) = A_1 t^{n-p} \otimes u + B_1 t^{n-q} \otimes v + C_1 t^{n-(p+q)} \otimes uv.$$

Since $j^*(\alpha\beta) \neq 0$, $A_0 B_1 + A_1 B_0 \neq 0$. (!)

Now the cases which lemma 3.2 leaves to be discussed have the property that $m < p+q$ so $C_0 = 0$. From (!) either B_0 or $B_1 \neq 0$. Since $Sq^p(\alpha)$ and $Sq^p(\beta)$ are dependent on α and β over $H^*(BZ_2)$, computing $Sq^p(j^*(\alpha))$ and $Sq^p(j^*(\beta))$ indicates that $C_1 \neq 0$. From the computation of $Sq^{q-p}(j^*(\alpha))$ one understands that $B_0 \neq 0$ since otherwise $Sq^{q-p}(j^*(\alpha))$ is independent of uv while $Sq^{q-p}(\alpha)$ depends on β, a contradiction. A similar argument on $Sq^p(j^*(\alpha))$ then shows that $A_0 \neq 0$.

Now using the fact that β may be replaced by $\beta + t^{n-m}\alpha$ one sees that the only 6-tuple for $\begin{pmatrix} A_0 & B_0 & C_0 \\ A_1 & B_1 & C_1 \end{pmatrix}$ which needs to be considered is $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. Hence it is clear that $q \leqslant m$ and $n \geqslant p+q$ and therefore one need consider only the cases $(p, q) = (2, 3)$ with $(m, n) = (3, 6)$ and $(4, 5)$ and $(p, q) = (4, 6)$ with $(n, m) = (6, 11)$, $(6, 12)$, $(6, 13)$, $(7, 10)$, $(7, 11)$, $(7, 12)$, $(8, 10)$, $(8, 11)$ and $(9, 10)$.

Now let $m + s \leqslant n$. Then $Sq^s(\alpha)$ is dependent on α for dimensional resaons if $s < n - m$ and because $i^*(\beta) = b$ if $s = n - m$. Looking at $Sq^4(j^*(\alpha))$ in cases $(6, 11)$, $(6, 12)$, $(6, 13)$, $(7, 11)$, $(7, 12)$ and $Sq^2(j^*(\alpha))$ in case $(3, 6)$ shows that these cases can not occur. Further $Sq^m(\beta)$ is independent of $\alpha\beta$ so if $n + s \leqslant m + n$, $Sq^s(j^*(\beta))$ is dependent on α and on β. Thus looking at $Sq^4(j^*(\beta))$ in case $(4, 5)$ and $Sq^8(j^*(\beta))$ in case $(8, 10)$, $(8, 11)$, and $(9, 10)$ shows that these cases can not occur.***

Note. To the author's knowledge, it is unknown whether in fact there are involutions on the exceptional cases in theorem 3.1 with the indicated fixed sets.

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