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ON A COMPLEXITY OF A SURFACE IN 3-SPHERE

Dedicated to Professor Ralph H. Fox for his 60th birthday

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0. Introduction

Throughout this paper we shall only be concerned with the combinatorial category, consisting of simplicial complexes and piecewise-linear maps. It is the purpose of the paper to prove intuitively obvious topological theorems which are interesting in the Morse theory of 3-manifolds. The theorems concern "knot types" of embeddings of a closed (=compact, without boundary), connected and orientable surface M_p of genus p into the 3-dimensional sphere S^3.

As widely known, a surface M_p in S^3, denoted by $(M_p\subset S^3)$, is obtained from some 2-spheres by adding handles, Fox [3] and Homma [5]. Using the fact, we shall define a complexity $\langle s, t\rangle$, a pair of natural numbers, for the knot type of the $(M_p\subset S^3)$ in §1. After establishing a canonical representative for the knot type of $(M_p\subset S^3)$ in §2, we first consider some non-existence results in §3. In §4 and §5, we construct some pairs $(M_p\subset S^3)$'s for some complexities $\langle s, t\rangle$'s.

In the paper, homeomorphism is denoted by $\cong$, while $\simeq$ and $\sim$ refer to homotopy and homology, respectively. ∂X, cl (X) and $^\circ X$ denote, respectively, the boundary, the closure and the interior of a manifold X. By D^n and S^{n-1} we shall denote the standard n-cell and the standard (n—1)-sphere ∂D^n, respectively, and particularly, $D^1=[-1, 1]$.

1. Definitions and notation

First let us explain several definitions and notation, and formulate our main theorem.

In general, we shall denote by M a compact orientable surface, and $\#(M)$ and $g(M)$ stand for the number of connected components of M and the total genus of M, respectively.

We shall say that a submaifold X of a manifold Y is *properly embedded* (or simply *proper*) if $X\cap\partial Y=\partial X$.

By $(M\subset M^3)$ we denote a pair of mainfolds such that a 3-manifold M^3 and a properly embedded surface M. Throughout §§1, 2, 3 and 4, we do not give any orientation on M and M^3. Two pairs $(M\subset M^3)$ and $(M'\subset M^3)$ are said to be *congruent*, or *of the same knot type*, if there is a homeomorphism ψ: $M^3\to M^3$ such that $\psi(M)=M'$. We denote the congruence class of a pair $(M\subset M^3)$ by $\langle(M\subset M^3)\rangle$, so the $(M\subset M^3)$ is a representative of $\langle(M\subset M^3)\rangle$.

For a pair $(M\subset M^3)$, a simple loop $J\ (\cong S^1)$ on M is said to be a *co-unknotted* loop, if J bounds a 2-cell $D(J)$ in M^3 with $D(J)\cap M=\partial D(J)=J$, and $D(J)$ will be called an *associated disk*. Especially, a co-unknotted loop J is said to be *essential* if J is not contractible in M, and otherwise, J is *inessential*. Note that J is contractible in M if and only if J bounds a 2-cell on M, see Epstein [2].

We may say a 3-manifold F^3 has *Fox's property*, if for any pair $(M\subset F^3)$ with $g(M)>0$ there exists an essential co-unknotted loop on M, and throughout the paper by F^3 we will denote a 3-mainfold which has Fox's property.

1.1. Proposition. (Kinoshita [8], Fox [3], Homma [5]) *Any orientable 3-manifold whose fundamental group is either finite or a finitely generated free group has Fox's property.* (Refer to Haken [4]).

For a pair $(M\subset M^3)$, let h: $D^1\times D^2\to {}^\circ M^3$ and d: $D^2\times D^1\to {}^\circ M^3$ be embeddings of 3-cells such that

(i) $h(D^1\times D^2)\cap M=h(\partial D^1\times D^2)$,
(ii) $d(D^1\times D^2)\cap M=d(\partial D^2\times D^1)$.

Then we have another embedded surfaces

(i) $M(h)=M-h(\partial D^1\times D^2)\cup h\,(D^1\times\partial D^2)$,
(ii) $M(d)=M-d(\partial D^2\times D^1)\cup d\,(D^2\times\partial D^1)$.

We will say that "$M(h)$ is formed from M by adding a handle h" and similarly, "$M(d)$ is formed from M by adding a dome d". It will be noticed that:

1.2. (i) $^\sharp(M(h))={}^\sharp(M)-1$ and $g(M(h))=g(M)$ if $h(\{0\}\times\partial D^2)\sim 0$ in $M(h)$, and $^\sharp(M(h))={}^\sharp(M)$ and $g(M(h))=g(M)+1$ if $h(\{0\}\times\partial D^2)\not\sim 0$ in $M(h)$.
(ii) $^\sharp(M(d))={}^\sharp(M)+1$ and $g(M(d))=g(M)$ if $d(\partial D^2\times\{0\})\sim 0$ in M, and $^\sharp(M(d))={}^\sharp(M)$ and $g(M(d))=g(M)-1$ if $d(\partial D^2\times\{0\})\not\sim 0$ in M.

Then, as an immediate consequence of 1.1 and 1.2, we have:

1.3. Proposition. *For any $\langle(M_p\subset F^3)\rangle$, there exists a representative $(M_p\subset F^3)$ such that M_p is formed from $\mathcal{P}_0=\Sigma_1\cup\cdots\cup\Sigma_s$, a union of non-intersecting 2-spheres in F^3, by adding one by one $s+p-1$ handles h_1, $\cdots$, h_{s+p-1}.*

Of course, this representative $(M_p\subset F^3)$ in 1.3 is not uniquely determined. If r handles $h_{i1},\cdots,h_{ir}$, $1\leqq i1<\cdots<ir\leqq s+p-1$, are mutually independent, that

is, $h_{i1}(D^1\times D^2)$, $\cdots$, $h_{ir}(D^1\times D^2)$ are mutually disjoint in F^3, we can add these r handles at a time. Therefore, the $(M_p\subset F^3)$ is formed from $\mathcal{P}_0$ by t times, $1\leqq t\leqq s+p-1$, as a process

$$\text{(1.4)}\quad \begin{aligned}\mathcal{P}_0=\textstyle\sum_1\cup\cdots\cup\sum_s&\to\mathcal{P}_1=\mathcal{P}_0(h_{11},\cdots,h_{1r(1)})\\&\to\mathcal{P}_2=\mathcal{P}_1(h_{21},\cdots,h_{2r(2)})\\&\to\cdots\\&\to\mathcal{P}_t=\mathcal{P}_{t-1}(h_{t1},\cdots,h_{tr(t)})=M_p,\end{aligned}$$

where $r(1)+r(2)+\cdots+r(t)=s+p-1$. A handle h_{ij}, $1\leqq i\leqq t$, $1\leqq i\leqq r(i)$, is said to belong to $\mathcal{P}_i$, and we denote $h_{ij}\in\mathcal{P}_i$.

Now, to the pair $(M_p\subset F^3)$ we associate a pair $\langle s, t\rangle\in N\times N$ of natural numbers, and we define a total order $<$(or$>$) in $\{\langle s, t\rangle\}\subset N\times N$ as follows:

(1.5) $\langle s, t\rangle<\langle s', t'\rangle$ if $s<s'$ or if $s=s'$ and $t<t'$.

Then, for every congruence class $\langle(M_p\subset F^3)\rangle$ we can define an invariant $\langle s, t\rangle$ as follows:

1.6. Definition. $\langle(M_p\subset F^3)\rangle$ is with *compelxity* $\langle s, t\rangle$ if there exists a representative $(M_p\subset F^3)$ with $\langle s, t\rangle$ and for any representative $(M'_p\subset F^3)$ with $\langle s', t'\rangle$ of $\langle(M_p\subset F^3)\rangle$, $\langle s, t\rangle\leqq\langle s', t'\rangle$.

It is clear that the complexity $\langle s, t\rangle$ is an invariant of a congruence class $\langle(M_p\subset F^3)\rangle$. Now we can state our version of a special case of Proposition 1.1.

1.7. Proposition. *Every* $\langle(M_1\subset F^3)\rangle$ *is with complexity* $\langle 1,1\rangle$.

In the notion of (1.4) and 1.3, if there is a handle h_{ij} such that $h_{ij}(\{0\}\times\partial D^2)$ is inessential on $\mathcal{P}_{i-1}(h_{ij})$, then the handle h_{ij} and one of the 2-spheres $\Sigma_1,\cdots,\Sigma_s$ can be omitted from the definition of complexity. Consequently, we have:

1.8. Proposition. *For every* $p\geqq 1$ *and for every* $\langle(M_p\subset F^3)\rangle$ *with complexity* $\langle s, t\rangle$, $\langle 1, 1\rangle\leqq\langle s, t\rangle\leqq\langle s, s+p-1\rangle\leqq\langle p, 2p\rangle$.

More sharp statements will be given later.

We call a disk-sum of p copies of $D^2\times S^1$ a *solid-torus of genus* p, and denote it by $p(D^2\times S^1)$. Since $p(D^2\times S^1)$ is embeddable in any 3-manifold, the following is obvious.

1.9. Proposition. *For any* $p\geqq 1$, *there exists a* $\langle(M_p\subset F^3)\rangle$ *with complexity* $\langle 1, 1\rangle$.

In §4, we will prove the following:

1.10. Theorem. *For any* $p\geqq 2$, *there exists a* $\langle(M_p\subset F^3)\rangle$ *with complexity* $\langle s, t\rangle$ *such that* $\langle s, t\rangle>\langle 1, 1\rangle$. (Refer to Kneser [10]).

2. Handle-isotopy and canonical representative

Let $(M \subset F^3)$ be a pair and let h: $D^1 \times D^2 \to F^3$ be a handle for M. Let $D^2_{-1} = h(\{-1\} \times D^2)$ and $D^2_1 = h(\{1\} \times D^2)$, and let D'^2 be a 2-cell on M with $D'^2 \cap (D^2_{-1} \cup D^2_1) = \phi$, and let γ be a simple arc on M with $\gamma \cap (D'^2 \cup D^2_{-1} \cup D^2_1) = \gamma \cap (\partial D'^2 \cup \partial D^2_1) = \partial\gamma$.
Then, sliding the end D^2_1 of the handle h along γ in a regular neighborhood $N(D^2_1 \cup \gamma \cup D'^2; F^3)$, we have a new handle h': $D^1 \times D^2 \to F^3$ for M such that $h'(\{1\} \times D^2) = D'^2$ and $h'(\{-1\} \times D^2) = D^2_{-1}$. It is easily seen that this deformation of a handle can be extended to an ambient isotopy of F^3; and so $\langle\langle (M(h) \subset F^3) \rangle\rangle = \langle\langle (M(h') \subset F^3) \rangle\rangle$. We will call this deformation a *handle-isotopy* (*along* γ). Of course, the above remains valid if D^2_{-1} is substituted for D^2_1.

An immediate consequence is:

2.1. Lemma. *In the notation of* (1.4), *a handle h_{ij} can be deformable by a handle-isotopy along a simple arc γ on M_p if the associated loop h_{ij} $(\{0\} \times \partial D^2)$ is co-unknotted and $\gamma \subset M_p - h_{ij}(D^1 \times \partial D^2)$. Especially, every handle h_{tj} can be deformable by a handle-isotopy along γ, provided that $\gamma \subset M_p - h_{tj}(D^1 \times \partial D^2)$.*

By successive application of 2.1, we deduce:

2.2. Theorem. *In the notation of* (1.4), *every handle $h_{ij} \in \mathcal{P}_i$ can be deformable by a handle-isotopy along γ by deforming handles belonging to $\mathcal{P}_{i+1} \cup \cdots \cup \mathcal{P}_t$ suitably, provided that $\gamma \subset \mathcal{P}_i - h_{ij}(D^1 \times \partial D^2)$, for $i = 1, \cdots, t$.*

REMARK. In 2.1 and 2.3, the handles belonging to $\mathcal{P}_1 \cup \cdots \cup \mathcal{P}_i$ with h_{kj} $(D^1 \times \partial D^2) \cap \gamma \neq \phi$ may be considered to be changed by the handle-isotopy, but there may be no confusion if we denote them by the same symbols.

2.3. Theorem. *For any $\langle\langle (M_p \subset F^3) \rangle\rangle$ with complexity $\langle s, t \rangle$, we can take a canonical representative $(M^*_p \subset F^3)$ as follows:*

(0) *M^*_p consists of s 2-spheres $\Sigma_1 \cup \cdots \cup \Sigma_s$ and $s+p-1$ handles.*

(i) *In the $s+p-1$ handles, there are just p handles, say $h_{\sigma_1}, \cdots, h_{\sigma_p}$, such that h_{σ_i} $(\partial D^1 \times D^2)$ is contained in one of $\Sigma_1, \cdots, \Sigma_s$ $i = 1, \cdots, p$. (So, $h_{\sigma_i}(\{0\} \times \partial D^2) \not\sim 0$ on M^*_p.)*

(ii) *In the $s+p-1$ handles, there are just $s-1$ handles, say $h_{\tau_1}, \cdots, h_{\tau_{s-1}}$, such that $h_{\tau_i}(\{-1\} \times D^2)$ and h_{τ_i} $(\{1\} \times D^2)$ are contained in different 2-spheres of $\Sigma_1, \cdots, \Sigma_s$, $i = 1, \cdots, s-1$. (So, h_{τ_i} $(\{0\} \times \partial D^2) \sim 0$ on M^*_p.)*

Proof. Let $(M_p \subset F^3)$ be a representative of $\langle\langle (M_p \subset F^3) \rangle\rangle$ which consists of s 2-spheres $\Sigma_1 \cup \cdots \cup \Sigma_s$ and $s+p-1$ handles $h_1, \cdots, h_{s+p-1}$. For brevity, we will call a handle h_i *c-handle* if h_i connects two of $\Sigma_1, \cdots, \Sigma_s$, that is, h_i $(\{-1\} \times D^2)$ and $h_i(\{1\} \times D^2)$ are contained in different 2-spheres of $\Sigma_1, \cdots, \Sigma_s$.

If in the $s+p-1$ handles, there are exactly $s-1$ c-handles, we are finished, and so we assume that there are more than $s-1$ c-handles. Suppose that there are at least two c-handles for Σ_1; and let h_{λ_1} and h_{μ_1} be such c-handles which connect Σ_{i_1} with Σ_1 and Σ_{j_1} with Σ_1, respectively. We may assume that $h_{\lambda_1}\in \mathcal{P}_{u_1}$ and $h_{\mu_1}\in\mathcal{P}_{v_1}$, and $1\leqq u_1\leqq v_1\leqq t$. Then, we can take a simple arc γ on $P_{v1}-h_{\mu_1}(D^1\times\partial D^2)$ such that γ runs from Σ_1 to Σ_{i_1} through $h_{\lambda_1}(D^1\times\partial D^2)$. By 2,2, there is a handle-isotopy along γ so that h_{μ_1} connects Σ_{i_1} with Σ_{j_1}. Note that if $i_1\neq j_1$ then h_{μ_1} must be now a c-handle, and if $i_1=j_1$ then h_{μ_1} is not a c-handle now.

Repeating the procedure, we may assume that there is only one c-handle, say h_{τ_1}, for Σ_1, and that h_{τ_1} connects Σ_2 with Σ_1.

Next we observe Σ_2. Since M_p is connected, there are some c-handles for Σ_2 other than h_{τ_1} if $s>2$. Let h_{λ_2} and h_{μ_2} be c-handles such that h_{λ_2} connects Σ_{i_2} with Σ_2 and h_{μ_2} connects Σ_{j_2} with Σ_2, and we may assume that $h_{\lambda_2}\in\mathcal{P}_{u_2}$, $h_{\mu_2}\in\mathcal{P}_{v_2}$ and $1\leqq u_2\leqq v_2\leqq t$. Then, we have a handle-isotopy so that h_{μ_2} connects Σ_{i_2} with Σ_{j_2}. Repeating the procedure, we may assume that there is exactly one c-handle, say h_{τ_2}, for Σ_2 other than h_{τ_1}, and that h_{τ_2} connects Σ_3 with Σ_2.

By the repetition of the procedure, we can assume that there is only one c-handle for each of Σ_1 and Σ_s, and there are exactly two c-handles for Σ_i, for $i=2, \cdots, s-1$. Thus, we have a required representative $(M_p^*\subset F^3)$ which satisfies (0) and (ii), and so (i).

On a surface M_p, we can choose a system of $2p$ simple loops $\{a_1, \cdots, a_p\}\cup\{b_1, \cdots, b_p\}$ such that $a_i\cap b_i$ consists of one crossing point and $a_i\cap a_j=\phi$, $b_i\cap b_j=\phi$, $a_i\cap b_j=\phi$ for $i\neq j$. We will call such a system *canonical.*

2.4. Corollary. (Homma [5]) *For any* $(M_p\subset F^3)$, *there exists a canonical system* $\{a_1, \cdots, a_p\}\cup\{b_1, \cdots, b_p\}$ *on* M_p *such that* $a_1\cup\cdots\cup a_p$ *are the boundaries of mutually disjoint 2-cells* $D_1^2\cup\cdots\cup D_p^2$.

Of course, this canonical system is not uniquely determined, and a_i is not always co-unknotted.

3. Non-existence results

In this section, we will give some non-existence theorems by contrast to Theorem 1.10.

3.1. Theorem. *For any* $p\geqq 2$ *and* $s\geqq 2$, *there is no* $\langle(M_p\subset F^3)\rangle$ *with complexity* $\langle s, 1\rangle$.

Proof. Suppose that there exists a $\langle(M_p\subset F^3)\rangle$ with complexity $\langle s, 1\rangle$ for $p\geqq 2$ and $s\geqq 2$. Then, by Theorem 2.3 there exists a canonical representative $(M_p^*\subset F^3)$ of the $\langle(M_p\subset F^3)\rangle$, and let $h_{\tau_1}, \cdots, h_{\tau_{s-1}}$ be handles of type (ii) in 2.3. From the definition of complexity, all handles of M_p^* belong to $\mathcal{P}_1$, *i.e.* all handles are mutually independent. So, $\mathcal{P}_0(h_{\tau_1}, \cdots, h_{\tau_{s-1}})$ must be a 2-sphere, hence the $\langle(M_p\subset F^3)\rangle$ is with complexity $\langle 1,1\rangle$, which contradicts our hypothesis.

In general, we claim:

3.2. Proposition. *For any* $p\geqq 2$ *and* $s\geqq 2$, *if* $\langle(M_p\subset F^3)\rangle$ *is with complexity* $\langle s, t\rangle$, *then for a canonical representative* $(M_p^*\subset F^3)$ *of the* $\langle(M_p\subset F^3)\rangle$ *in* Theorem 2.3, *every handle* $h_{1j}\in\mathcal{P}_1$ *is of type* (i) *in* 2.3.

Remainder of the paper, we consider only pairs $(M\subset S^3)$'s. For a pair $(M\subset S^3)$, the residual space S^3-M consists of $^{\#}(M)+1=q$ non-intersecting 3-manifolds. We denote the closures of these manifolds in S^3 by $W_1(M\subset S^3)\cup\cdots\cup W_q(M_q\subset S^3)$ or simply by $W_1\cup\cdots\cup W_q$, and call the disjoint union of them the *closed complement of* $(M\subset S^3)$. We record the following well-known theorem due to J.W. Alexander.

3.3. Proposition. (Alexander [1]) *For any pair* $(M_0\subset S^3)$, $W_1(M_0\subset S^3)\cong W_2(M_0\subset S^3)\cong D^3$. (Remember that $M_0\cong S^2$.)

3.4. Theorem. *For any* $p\geqq 2$, *there is no* $\langle(M_p\subset S^3)\rangle$ *with complexity* $\langle p, 2p-1\rangle$.

Proof. Assume the contrary, then there is a $\langle(M_p\subset S^3)\rangle$ with complexity $\langle p, 2p-1\rangle$ for $p\geqq 2$, and let $(M_p^*\subset S^3)$ be a canonical representative of it in Theorem 2.3.

Let $\mathcal{P}_0=\Sigma_1\cup\cdots\cup\Sigma_p$ be the 0-th step of M_p^*, and let $W_1(\mathcal{P}_0\subset S^3)\cup\cdots\cup W_{p+1}(\mathcal{P}_0\subset S^3)$ be the closed complement. It will be noticed that for every Σ_k there exists only one handle of type (i) in 2.3. From the definition of complexity, to each step $\mathcal{P}_i$, $i=1,\cdots,2p-1$, only one handle, say h_i, belongs.

For clarity, the proof will be divided into five steps.

Step 1: From Proposition 3.2, h_1 is of type (i) in 2.3. Without loss of generality, we may assume that $h_1(\partial D^1\times D^2)$ is contained in Σ_1 and $h_1(D^1\times D^2)$ is contained in $W_1(\mathcal{P}_0\subset S^3)$. Moreover, we may assume that $W_1(\mathcal{P}_0\subset S^3)\cap W_2(\mathcal{P}_0\subset S^3)=\Sigma_1$ by 3.3. Then, the complement of $(\mathcal{P}_1\subset S^3)=(\mathcal{P}_0(h_1)\subset S^3)$ consists of

$$\begin{aligned}W_1(\mathcal{P}_1\subset S^3) &= \mathrm{cl}(W_1(\mathcal{P}_0\subset S^3)-h_1(D^1\times D^2)),\\ W_2(\mathcal{P}_1\subset S^3) &= W_2(\mathcal{P}_0\subset S^3)\cup h_1(D^1\times D^2),\\ W_k(\mathcal{P}_1\subset S^3) &= W_k(\mathcal{P}_0\subset S^3)\text{ for } k=3,\cdots,p+1.\end{aligned}$$

Step 2: If h_2 is of type (ii) in 2.3, then $h_2(\{0\}\times\partial D^2)$ is contractible on $\mathcal{P}_1(h_2)$ because $\mathcal{P}_1$ consists of only one closed orientable surface of genus 1 and $p-1$ 2-spheres. So, $\langle(M_p\subset S^3)\rangle$ must be with complexity smaller than or equal to $\langle p-1, 2p-2\rangle$, which contradicts our hypothesis. We know that h_2 is of type (i) in 2.3, and $h_2(\partial D^1\times D^2)$ is contained in one of $\Sigma_2\cdots,\Sigma_p$, say Σ_2. Since $h_1(D^1\times {}^\circ D^2)\cap h_2(D^1\times D^2)\neq\phi$, $h_2(D^1\times D^2)\subset W_2(\mathcal{P}_1\subset S^3)$, and we may assume that $W_2(\mathcal{P}_1\subset S^3)\cap W_3(\mathcal{P}_1\subset S^3)=\Sigma_2$. Then, the closed complement of $(\mathcal{P}_2\subset S^3)=$

$(\mathcal{P}_1(h_2)\subset S^3)$ consists of

$$\begin{aligned} W_1(\mathcal{P}_2\subset S^3) &= W_1(\mathcal{P}_1\subset S^3), \\ W_2(\mathcal{P}_2\subset S^3) &= \mathrm{cl}(W_2(\mathcal{P}_1\subset S^3)-h_2(D^1\times D^2)), \\ W_3(\mathcal{P}_2\subset S^3) &= W_3(\mathcal{P}_1\subset S^3)\cup h_2(D^1\times D^2), \\ W_k(\mathcal{P}_2\subset S^3) &= W_k(\mathcal{P}_1\subset S^3) \text{ for } k=4, \cdots, p+1. \end{aligned}$$

Step 3: If h_3 is of type (ii) 2.3, there are four cases to be considered: h_3 connects i) $\Sigma_1(h_1)$ with $\Sigma_2(h_2)$, ii) $\Sigma_1(h_1)$ with Σ_k, $k=3, \cdots, p$, iii) $\Sigma_2(h_2)$ with Σ_k, $k=3, \cdots, p$, and iv) Σ_k with Σ_j, $k\neq j$, $k, j=3, \cdots, p$. Since $h_2(D^1\times {}^\circ D^2)\cap h_3(D^1\times D^2)\neq\phi$, $h_3(D^1\times D^2)\subset W_3(\mathcal{P}_2\subset S^3)$. For $\Sigma_1(h_1)\cap W_3(\mathcal{P}_2\subset S^3)=\phi$, the case i) cannot occur actually. Moreover, every case of ii), iii) and iv) cannot occur by the same reason as that of *Step* 2. Now, we know that h_3 is of type (i) in 2.3, and $h_3(\partial D^1\times D^2)$ is contianed in one of $\Sigma_3, \cdots, \Sigma_p$, say Σ_3. Note that $\Sigma_1\cup\Sigma_2\cup\Sigma_3$ are considered to be concentric. We may assume that $W_3(\mathcal{P}_3\subset S^3)\cap W_4(\mathcal{P}_2\subset S^3)=\Sigma_3$. Then, the closed complement of $(\mathcal{P}_3\subset S^3)=(\mathcal{P}_2(h_3)\subset S_3)$ consists of

$$\begin{aligned} W_1(\mathcal{P}_3\subset S^3) &= W_1(\mathcal{P}_2\subset S^3),\ W_2(\mathcal{P}_3\subset S^3) = W_2(\mathcal{P}_2\subset S^3), \\ W_3(\mathcal{P}_3\subset S^3) &= \mathrm{cl}(W_3(\mathcal{P}_3\subset S^3)-h_3(D^1\times D^2)), \\ W_4(\mathcal{P}_3\subset S^3) &= W_4(\mathcal{P}_2\subset S^3)\cup h_3(D^1\times D^2), \\ W_k(\mathcal{P}_3\subset S^3) &= W_k(\mathcal{P}_2\subset S^3) \text{ for } k=5, \cdots, p+1. \end{aligned}$$

Step 4: Repeating the same arguments in *Step* 3, we may assume that

(i) $h_1, \cdots, h_p$ are of type (i) in 2.3, and $\mathcal{P}_p$ consists of closed orientable surfaces $\Sigma_1(h_1), \cdots, \Sigma_p(h_p)$ of genus 1,

(ii) the closed complement of $(\mathcal{P}_p\subset S^3)$ consists of

$$\begin{aligned} W_1(\mathcal{P}_p\subset S^3) &= \mathrm{cl}(W_1(\mathcal{P}_0\subset S^3)-h_1(D^1\times D^2)), \\ W_k(\mathcal{P}_p\subset S^3) &= \mathrm{cl}(W_k(\mathcal{P}_0\subset S^3)\cup h_{k-1}(D^1\times D^2)-h_k(D^1\times D^2)) \end{aligned}$$

for $k=2, \cdots, p$,

$$W_{p+1}(\mathcal{P}_p\subset S^3) = W_{p+1}(\mathcal{P}_0\subset S^3)\cup h_p(D^1\times D^2).$$

In particular, it will be noticed that $\Sigma_1\cup\cdots\cup\Sigma_p$ are concentric.

Step 5: After *Step* 4, we know that all handles $h_{p+1},\cdots, h_{2p-1}$ belonging to $\mathcal{P}_{p+1}\cup\cdots\cup\mathcal{P}_{2p-1}$ are of type (ii) in 2.3. Since $h_p(D^1\times {}^\circ D^2)\cap h_{p+1}(D^1\times D^2)\neq\phi$, $h_{p+1}(D^1\times D^2)\subset W_{p+1}(\mathcal{P}_p\subset S^3)$. But, since $\partial W_{p+1}(\mathcal{P}_p\subset S^3)=\Sigma_p(h_p)$, h_{p+1} cannot be of type (ii) in 2.3, so the $\langle(M_p\subset S^3)\rangle$ must be with complexity smaller than $\langle p, 2p-1\rangle$.

After all, we obtain a desired contradiction, and completes the proof of Theorem 3.4.

We note the following, which is easily derived from the same argument as above.

3.5. Proposition. *For any* $p\geqq 2$ *and* $t\geqq 2$, *if* $\langle(M_p\subset S^3)\rangle$ *is with complexity* $\langle p, t\rangle$, *then for a canonical representative* $(M_p^*\subset S^3)$ *of the* $\langle(M_p\subset S^3)\rangle$ *in* 2.3, *every handle* $h_{tj}\in\mathcal{P}_t$ *is of type* (ii) *in* 2.3.

We summarize our results Proposition 1.8 and Theorems 3.1 and 3.4 as follows:

3.6. Proposition. *For every* $p\geqq 1$ *and for every* $\langle(M_p\subset S^3)\rangle$ *with complexity* $\langle s, t\rangle$, *the positive integers* p, s *and* t *satisfy one of the followings*:

(1) $s=1$, $1\leqq t\leqq p$,
(2) $2\leqq s\leqq p-1$, $2\leqq t\leqq s+p-1$,
(3) $s=p$, $2\leqq t\leqq 2p-2$.

4. Some existence results

In this section, we will give some existence theorems, and Theorem 1.10 is a direct consequence of these results.

4.1. Theorem. *For any* $p\geqq 2$, *there exists a* $\langle(M_p\subset S^3)\rangle$ *with complexity* $\langle s, t\rangle$ *such that* $s\geqq 2$ *and* $t\geqq 2$.

Proof. The following Fig. 1 shows the case $p=2$, which is due to Homma [5]. First, we will show that the $\langle(M_2\subset S^3)\rangle$ in Fig. 1 is with complexity $\langle 2, 2\rangle$. From the construction, W_1 is homeomorphic to V_{F^*} of Suzuki [12, Fig. 2]. So, we conclude that $\pi_1(W_1)$ is indecomposable with respect to free products and

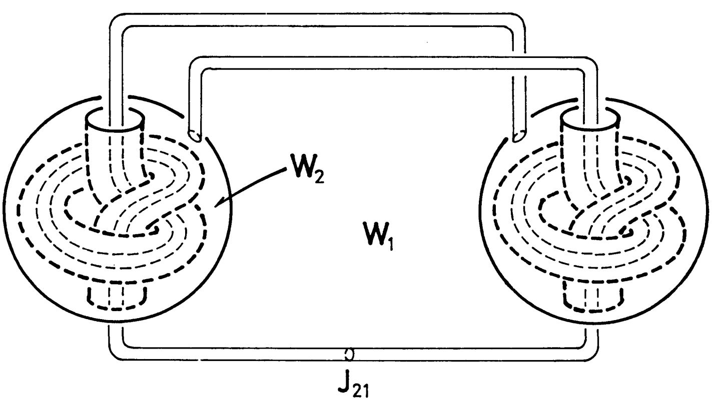

Fig.1 : $(M_2\subset S^3)$

not free, that is, there is no essential co-unknotted loop J on M_2 with $J\simeq 1$ in W_1 by the bounded Kneser's theorem, Jaco [6], see [12, §2]. On the other hand, we know that W_2 is a disk-sum of two copies of a closed complement V_K of the so-called clover-leaf knot. Since $\pi_1(V_K)$ is indecomposable with respect to free products and not free, we know that the essential co-unknotted loop J_{21} on M_2 is unique up to isotopy by [12, Cor. 3.5]. Now, we can easily conclude that the $\langle(M_2\subset S^3)\rangle$ is with complexity $\langle 2, 2\rangle$.

By the same way as that of the proof [12, Th. 5.2], using the pair $(M_2\subset S^3)$ we can construct required pair $(M_p\subset S^3)$ for any $p>2$. The following Fig. 2 illustrates the case $p=3$.

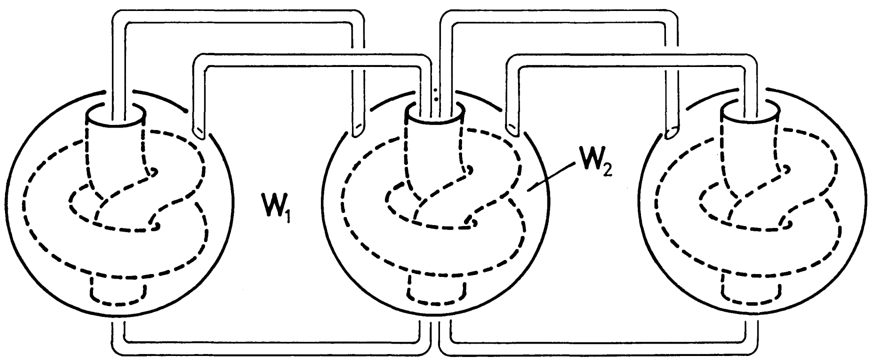

Fig.2 : $(M_3\subset S^3)$

From the construction, the closed complement W_1 in Fig. 2 is homeomorphic to V_{F1} of [12, Fig. 3]. Thus, $\pi_1(W_1)$ is also indecomposable and not free, that is, there is no essential co-unknotted loop J on M_3 with $J\simeq 1$ in W_1. On the other hand, W_2 in Fig. 2 is a disk-sum of three copies of V_K, the closed complement of the clover-leaf knot. So, for any essential co-unknotted loop J on M_3, we conclude that $J\sim 0$ on M_3, Jaco [6], see [12, Prop. 2.15]. Now, we can easily conclude that the $\langle(M_3\subset S^3)\rangle$ satisfies the required condition.

The proof of the case $p>3$, which is omitted here, is the same as that of the case $p=3$.

REMARK. It can be shown by a long geometric proof that the $\langle(M_3\subset S^3)\rangle$ in Fig. 2 is with complexity $\langle 3. 2\rangle$. In fact, the author suspects, but cannot prove, that the every class $\langle(M_p\subset S^3)\rangle$ obtained in the proof of 4.1 is with complexity $\langle p, 2\rangle$.

As shown in the proof of 4.1, for every essential co-unknotted loop J on M_2 in Fig. 1, $J\sim 0$ on M_2. With reference to Proposition 1.1, we record the following, but the proof is omitted.

4.2. Proposition. *For any pair* $(M_p \subset F^3)$, *there exists an essential co-unknotted loop* J *on* M_p *with* $J \sim 0$ *on* M_p, *provided that* $p \geqq 2$.

4.3. Theorem. *For any* $p \geqq 2$, *there exists a* $\langle (M_p \subset S^3) \rangle$ *with complexity* $\langle 1, t \rangle$ *such that* $t \geqq 2$.

Proof. The following $(M_2' \subset S^3)$ in Fig. 3 shows the case $p=2$. From the construction, it is easy to check that W_1' is homeomorphic to the W_1 of the

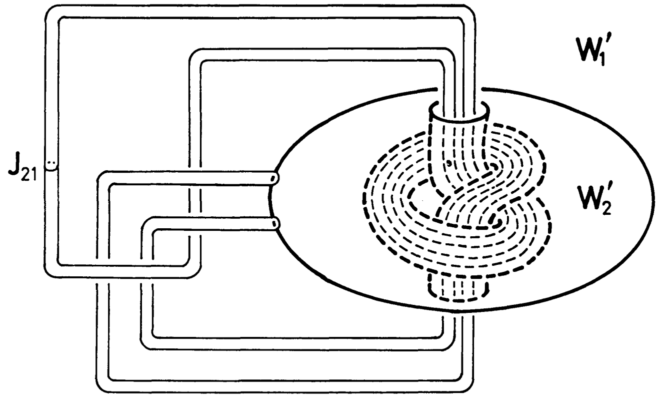

Fig. 3 : $(M_2' \subset S^3)$

$(M_2 \subset S^3)$ in Fig. 1; so there is no essential co-unknotted loop J on M_2' with $J \simeq 1$ in W_1'. On the other hand, W_2' is a disk-sum of $D^2 \times S^1$ and V_K. By [12, Cor. 3.6], the essential co-unknotted loop J_{21} on M_2' with $J_{21} \not\sim 0$ on M_2', is unique up to isotopy, and now we can conclude that the $\langle (M_2' \subset S^3) \rangle$ is with complexity $\langle 1, 2 \rangle$.

The following $(M_3' \subset S^3)$ in Fig. 4 shows the case $p=3$, which is obtained from the $(M_2' \subset S^3)$ by adding a handle h_{31}, where $h_{31}(D^1 \times D^2)$ is shown by an arc in the figure. In the other cases $p>3$, we can construct required pairs inductively using this $(M_3' \subset S^3)$, and so on.

From the construction, the W_1' in Fig. 4 is a disk-sum of the W_1 of the $(M_2 \subset S^3)$ in Fig. 1 and $D^2 \times S^1$. We have the essential co-unknotted loop $J_{31} = h_{31}(\{0\} \times \partial D^2)$ on M_3' with $J_{31} \not\sim 0$ on M_3', and J_{31} is unique up to isotopy by [12, Cor. 3.6]. On the other hand, the W_2' in Fig. 4 is a disk-sum of V_K and the W_1 of the $(M_2 \subset S^3)$ in Fig. 1. So, there is no essential co-unknotted loop J on M_3' with $J \simeq 1$ in W_2' and $J \not\sim 0$ on M_3'. Since the $(M_2' \subset S^3)$ is obtained from the $(M_3' \subset S^3)$ by adding a dome along the 2-cell $h_{31}(\{0\} \times D^2)$, we can conclude that the $\langle (M_3' \subset S^3) \rangle$ is with complexity $\langle 1, 3 \rangle$.

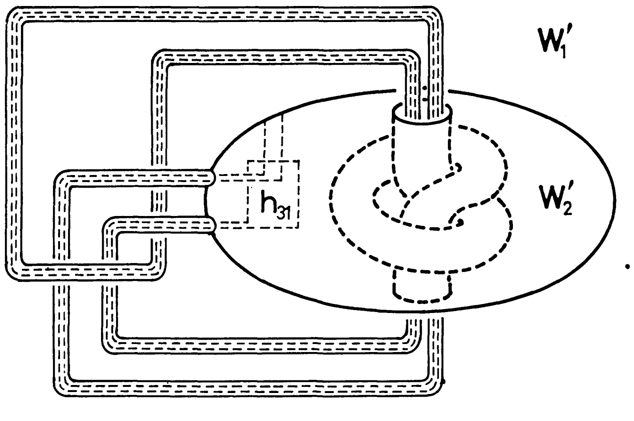

Fig.4 : $(M'_3 \subset S^3)$

In general, if $p=2n$, then W'_1 of $(M'_p \subset S^3)$ is a disk-sum of n copies of the W_1 of the $(M_2 \subset S^3)$ in Fig. 1, and W'_2 of $(M'_p \subset S^3)$ is a disk-sum of $D^2 \times S^1$, V_K and $n-1$ copies of the W_1 of the $(M_2 \subset S^3)$ in Fig. 1. If $p=2n+1$, then W'_1 of $(M'_p \subset S^3)$ is a disk-sum of $D^2 \times S^1$ and n copies of the W_1 of the $(M_2 \subset S^3)$ in Fig. 1, and W'_2 of $(M'_p \subset S^3)$ is a disk-sum of V_K and n copies of the W_1 of the $(M_2 \subset S^3)$ in Fig. 1. So, in every case,as a system of mutually disjoint and homologically independent essential co-unknotted loops on M'_p, we can take a system which consists of exactly one loop, and completing the proof.

REMARK. In the above, an essential co-unknotted loop J on M'_p with $J \not\sim 0$ on M'_p is not unique up to isotopy for $p>3$, but the every $\langle(M'_p \subset S^3)\rangle$ may be with complexity $\langle 1, p\rangle$.

REMARK. In the proof of Theorems 4.1 and 4.3, we based on the Homma's example $(M_2 \subset S^3)$ in Fig. 1. To construct another examples, we refer the reader to Jaco [7], Kinoshita [9] and Suzuki [11], etc..

5. Remarks and questions

In the preceding section, we have constructed some pairs and actually determined its complexity in some of the simplest cases. In more complicated cases we will need much information on 3-manifolds in S^3. While, the author suspects, but cannot prove, that:

5.1. Qusetion. *For positive integers p, s and t satisfying one of the* (1), (2) *and* (3) *in* Proposition 3.6, *does there exist a* $\langle(M_p \subset S^3)\rangle$ *with complexity* $\langle s, t\rangle$?

In fact, Theorems 4.1 and 4.3 and Proposition 1.9 imply that in the case $p=2$ Question 5.1 is affirmative. In the case $p=3$ we can easily give some $\langle(M_3\subset S^3)\rangle$'s with complexity $\langle s, t\rangle\neq\langle 1, 1\rangle, \langle 1, 3\rangle$ and $\langle 3, 2\rangle$. Generally, using $\langle(M_p\subset S^3)\rangle$'s whose complexities have been known, we can construct some kind of $\langle(M_q\subset S^3)\rangle$'s. For example:

5.2. Example. (Fig. 5) *There exists a $\langle(M_3\subset S^3)\rangle$ with complexity $\langle 2, 2\rangle$.*

Proof. Using the pair $(M_2\subset S^3)$ in Fig. 1, we give the following $(M_3''\subset S^3)$ in Fig. 5. From the construction, it is easy to check that W_1'' is a disk-sum of

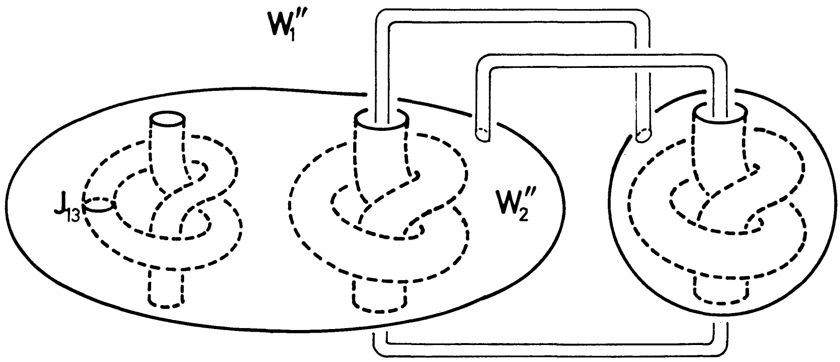

Fig. 5 : $(M_3''\subset S^3)$

$D^2\times S^1$ and the W_1 in Fig. 1, and W_2'' is a disk-sum of three copies of V_K. As homologically non-trivial co-unknotted loops, we have a unique loop J_{13} on M_3''. From the definition of complexity, we can easily conclude that the $\langle(M_3''\subset S^3)\rangle$ is with complexity $\langle 2, 2\rangle$.

5.3. Example. (Fig. 6) *For any $p\geqq 2$, there exists a $\langle(M_p\subset S^3)\rangle$ with complexity $\langle 1,2\rangle$.*

Proof. The case $p=2$ is Theorem 4.3 (Fig. 3). Using the $(M_2'\subset S^3)$ in Fig. 3, we give the following pair $(M_p'{}''\subset S^3)$ in Fig. 6 for $p\geqq 3$. It is easy to check that $W_1'{}''$ is a disk-sum of $(p-2)(D^2\times S^1)$ and the W_1 in Fig. 1, and $W_2'{}''$ is a disk-sum of $D^2\times S^1$ and $p-1$ copies of V_K. So, we can choose at most $p-1$ mutually disjoint and homologically independent essential co-unknotted loops on $M_p'{}''$. Now, we can easily conclude that the $\langle(M_p'{}''\subset S^3)\rangle$ satisfies the required condition.

Examples 5.2 and 5.3 also suggest an interesting point: The complexity of a $\langle(M_p\subset S^3)\rangle$ is connected with its prime decompositions, [12, Th. 1.6]. In

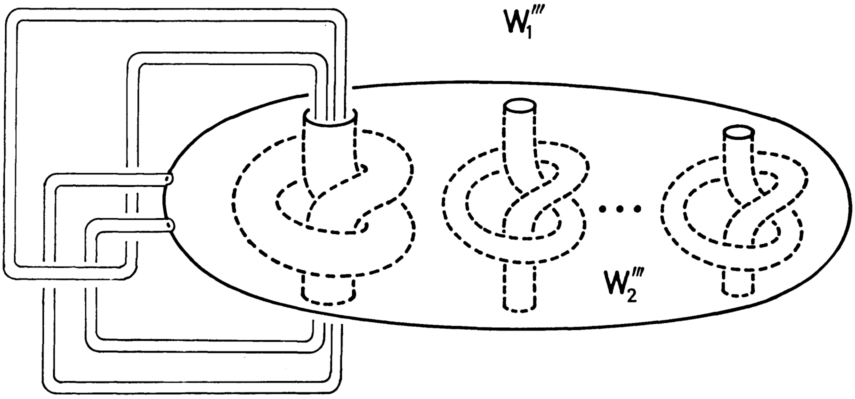

Fig. 6 : $(M_p''' \subset S^3)$

remainder of the paper, we consider only pairs $(M_p \subset S^3)$'s such that M_p is oriented and S^3 has the right-handed orientation. In the sense of [12, Def. 1.11], the complexity $\langle s, t\rangle$ is also an invariant of the congruence class, denote it also by $\langle (M_p \subset S^3)\rangle$, of a pair $(M_p \subset S^3)$. For the other notation, see [12, §1]. From the definitions of complexity and composition of pairs [12], we have at once:

5.4. Proposition. *Let* $\langle s, t\rangle$, $\langle s_1, t_1\rangle$ *and* $\langle s_2, t_2\rangle$ *be the complexities of* $\langle (M_p \subset S^3)\rangle$, $\langle (M_{p1} \subset S^2)\rangle$ *and* $\langle (M_{p2} \subset S^3)\rangle$, *respectively. Suppose that* $(M_p \subset S^3) \cong (M_{p1} \subset S^3) \# (M_{p2} \subset S^3)$.
Then,

$$\langle s, t\rangle \leqq \langle s_1 + s_2 - 1, \text{ max. } \{t_1, t_2\}\rangle.$$

5.5. Question. *Does it hold in the above* 5.4 *the equality*

$$\langle s, t\rangle = \langle s_1 + s_2 - 1, \text{ max. } \{t_1, t_2\}\rangle ?$$

In view of 5.4, we deduce the following:

5.6. Theorem. *Every* $\langle (M_p \subset S^3)\rangle$ *with complexity* $\langle p, t\rangle$ *is prime.*

Proof. The case $p=1$ is obvious from 1.7 and [12, Prop. 1.5], so we assume that $p \geqq 2$. Suppose that there exists a $\langle (M_p \subset S^3)\rangle$ with complexity $\langle p, t\rangle$ that is not prime. Let $(M_p \subset S^3) \cong (M_{p1} \subset S^3) \# (M_{p2} \subset S^3)$ be a non-trivial decomposition, and let $\langle s_1, t_1\rangle$ and $\langle s_2, t_2\rangle$ be the complexities of the $\langle (M_{p1} \subset S^3)\rangle$ and $\langle (M_{p2} \subset S^3)\rangle$, respectively. From Proposition 3.6 and [12, Prop. 1.3], $1 \leqq s_i \leqq p_i < p$, $(i=1, 2)$, and $p_1 + p_2 = p$. Then, $s_1 + s_2 - 1 \leqq p_1 + p_2 - 1 = p - 1 < p$, hence for any t

$$\langle s_1+s_2-1, \text{max. } \{t_1, t_2\}\rangle < \langle p, t\rangle.$$

This contradicts to 5.4, and the proof completes.

5.7. Theorem. *If* Question 5.5 *is affirmative, then every* $\langle(M_p\subset S^3)\rangle$ *with complexity either* $\langle s, s+p-1\rangle$ *or* $\langle p-1, 2p-3\rangle$ *is prime.*

Proof. By virtue of Proposition 3.6, we may assume that $p>s\geqq 1$ and $p\geqq 2$ for $\langle s, s+p-1\rangle$, and $p\geqq 3$ for $\langle p-1, 2p-3\rangle$. As the same way as that of 5.6, we suppose that there exists a $\langle(M_p\subset S^3)\rangle$ with complexity either $\langle s, s+p-1\rangle$ or $\langle p-1, 2p-3\rangle$ that is not prime. Let $(M_p\subset S^3)\cong(M_{p_1}\subset S^3)\,\sharp\,(M_{p_2}\subset S^3)$ be a non-trivial decomposition, and let $\langle s_1, t_1\rangle$ and $\langle s_2, t_2\rangle$ be the complexities of $\langle(M_{p_1}\subset S^3)\rangle$ and $\langle(M_{p_2}\subset S^3)\rangle$, respectively. From Proposition 3.6 and [12, Prop. 1.3], we have $1\leqq s_i\leqq p_i<p$, $(i=1, 2)$, and $p_1+p_2=p$.

Case (i) $\langle s, s+p-1\rangle$: By our assumption, we have $s+1=s_1+s_2$, and so $s_1\leqq s$ and $s_2\leqq s$. Hence, $t_i\leqq s_i+p_i-1<s+p-1$ for $i=1, 2$ by Proposition 3.6 (or 1.8),

These contradict to our assumption.

Case (ii) $\langle p-1, 2p-3\rangle$: By our assumption, $p-1=s_1+s_2-1$. While, by Proposition 3.6 (or 1.8), if $s_i=p_i$ then $t_i\leqq 2p_i-1$, and if $s_i<p_i$ then $t_i\leqq s_i+p_i-1$. So, if $s_i=p_i$ then $t_i\leqq s_i+p_i-1\leqq 2p_i-2=2p-4<2p-3$, and if $s_i<p_i$ then $t_i\leqq s_i+p_i-1\leqq (p-2)+(p-1)-1<2p-3$.

These contradictions complete the proof.

REMARK. Theorems 5.6 and 5.7 are, of course, sufficient conditions for a $\langle(M_p\subset S^3)\rangle$ to be prime, because the examples of prime pairs given in [12, Th. 5.2] are with complexity $\langle 1,1\rangle$. In fact, there may be a prime $\langle(M_p\subset S^3)\rangle$ with every $\langle s, t\rangle$.

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