Suzuki, T.
Osaka J. Math.
11 (1974), 37-48

RELATIVE EFFICIENCY OF THE SEQUENCES OF STATISTICAL TESTS

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(Received April 7, 1973)

1. Introduction. In this paper giving an extension of the theorem on Pitman efficiency in Noether [2], we try to compare two sequences of tests under more general conditions than Noether [2]. Roughly speaking, the idea of the Pitman efficiency is as follows.

DEFINITION. Given two sequences of tests of the same size of the same statistical hypothesis, the Pitman efficiency of the second sequence of tests with respect to the first sequence is given by the ratio n_1/n_2, where n_2 is the sample size of the second test required to achieve the same power for a given alternative $\theta=\pi_{n_2}(\omega_2)$ as is achieved by the first test with respect to the same alternative $\theta=\pi_{n_1}(\omega_1)$ when the sample size n_1. Here $\pi_n(\omega)$ is a parametric function.

In the paper of Noether [2], it was considered only when (a) the sequence $\{T_n\}$ of statisics is asymptotically normally distributed, (b) the test ϕ_n is such one that $\phi_n=1$ or 0 according as $T_n>c_n$ or $T_n<c_n$ with some constant c_n, and (c) the alternatives $\pi_n(\omega)$ are the following one; $\pi_n(\omega)=\theta_0+n^{-\delta}(\omega-\theta_0)$. In this paper, however, it is shown that the Pitman efficiency is also calculable under more general conditions than those.

In Section 2 we investigate on the rate of convergence of alternatives $\{\pi_n(\omega)\}$. Section 3 is devoted to the calculation of the Pitman efficiency.

2. The rates of convergence of alternatives. Throughout this paper we shall use the following notations. Let Θ be a nonempty subset of R^1 and θ_0 a fixed inner point of Θ. Let K $(\neq\{0\})$ be a fixed cone in R^1, and we denote $\Omega=\{\theta+\theta_0;\ \theta\in K\}(=K+\theta_0)$ and $\Theta_1=\Theta\cap\Omega$. For each $n\in N=\{1, 2, \cdots\}$, let $(\boldsymbol{X}_n, \boldsymbol{A}_n)$ be the cartesian product of n copies of a certain measurable space $(\boldsymbol{X}, \boldsymbol{A})$. For each $\theta\in\Theta$ let P_θ be a probability measure on $(\boldsymbol{X}, \boldsymbol{A})$. Let $P_{\theta,n}$ be the product measure of n copies of P_θ. Let a measure space $(\boldsymbol{Y}, \boldsymbol{B}, \mu)$ be given, where $\boldsymbol{Y}$ is a Borel subset of R^r, $\boldsymbol{B}$ is the Borel σ-field in $\boldsymbol{Y}$ and μ is the Lebesgue measure on $(\boldsymbol{Y}, \boldsymbol{B})$.

DEFINITIN 1. Let $\{Q_{\omega,n};\ \omega\in\Omega\}_{n\in N}$ be a squence of families of probability measures and $\{Q_\omega;\ \omega\in\Omega\}$ a family of probability measures on $(\boldsymbol{Y}, \boldsymbol{B})$. Let Ω_0

be a nonempty subset of Ω. We call that $Q_{\omega,n}$ *converges in law to* Q_ω *uniformly in* Ω_0 if and only if

(2.1) $$\sup_{\omega\in\Omega_0}|Q_{\omega,n}(C)-Q_\omega(C)|\to 0 \quad \text{as} \quad n\to\infty,$$

for each measuable convex set C in $\boldsymbol{Y}$.

We use the notation $\mathrm{supp}(P)$ for the support of a probability masure P on $(\boldsymbol{Y}, \boldsymbol{B})$, i.e., the minimum closed set C in $\boldsymbol{Y}$ such that $P(C)=1$.

Condition (M). (a) A family of distributions Q_ω on $(\boldsymbol{Y}, \boldsymbol{B})$ is dominated by μ and their density is denoted by $dQ_\omega/d\mu=g(y:\omega)$.

(b) For any $c\in[0,\infty)$ and any $\omega_1, \omega_2\in\Omega$, the set $\{y; g(y:\omega_1)\geqq c\cdot g(y:\omega_2)\}$ is contained in $\mathfrak{C}$. Here $\mathfrak{C}$ is the family of sets C such that C or C^C is represented as a finite union of mutually disjoint measurable convex sets in $\boldsymbol{Y}$.

(c) There exists an $\omega\in\Omega\setminus\{\theta_0\}$ such that $\mu(\{\mathrm{supp}(Q_\omega)\cap\mathrm{supp}(Q_{\theta_0})\})>0$, where $A\setminus B$ stands for the set $\{\omega; \omega\in A$ and $\omega\notin B\}$.

(d) The family $\{g(y:\omega); \omega\in\Omega\}$ of densities has monotone likelihood ratio with respect to $|\omega-\theta_0|$ in the following sense: There exists a real valued measurable function $T(y)$ on $\boldsymbol{Y}$ such that, for any $\omega, \omega'\in\Omega$ satisfying $|\omega-\theta_0|<|\omega'-\theta_0|$, the distributions Q_ω and $Q_{\omega'}$ are distinct and the ratio $g(y:\omega)/g(y:\omega')$ is a nondecreasing function of $T(y)$.

(e) Q_{ω_n} converges in law to Q_ω whenever $\omega_n\to\omega$.

By a statistic T_n we mean an $(\boldsymbol{A}_n, \boldsymbol{B})$-measurable map from $\boldsymbol{X}_n$ to $\boldsymbol{Y}$. For a finite measure ν on $(\boldsymbol{X}_n, \boldsymbol{A}_n)$ and a statistic T_n we denote by νT_n^{-1} the induced measure by T_n.

DEFINITION 2. Let $\{\pi_n\}_{n\in N}$ be a sequence of mappings from Ω to Θ_1. A sequence $\{T_n\}_{n\in N}$ of statistics is said to be *of type* (*L*) *relative to* $\{\pi_n\}$ (or $\{\pi_n\}$ is called an *accessible sequence* of $\{T_n\}$ (a) if $P_{\pi_n(\omega),n}\,T_n^{-1}$ converges in law to a certain probability measure Q_ω on $(\boldsymbol{Y}, \boldsymbol{B})$ as n tends to infinity uniformly in a neighborhood of each $\omega\in\Omega$, (b) if the family $\{Q_\omega; \omega\in\Omega\}$ of limit distributions satisfies Condition (M).

DEFINITION 3. Let $\{\pi_n\}_{n\in N}$ be a sequence of mappings from Ω to Θ_1 such that $\pi_n(\omega)\to\theta_0$ as $n\to\infty$ for each fixed $\omega\in\Omega$. The *rate of convergence* of $\{\pi_n\}$ is defined as the class of sequences $\{k_n\}_{n\in N}$ of positive numbers such that for every $\omega\in\Omega\setminus\{\theta_0\}$

(2.2) $$0<\liminf_{n\to\infty} k_n|\pi_n(\omega)-\theta_0|\leqq\limsup_{n\to\infty} k_n|\pi_n(\omega)-\theta_0|<\infty.$$

Denote by ϕ_ω^α the most powerful level α test for testing a simple hypothesis Q_{θ_0} against an alternative Q_ω. For a function f and a probability measure P, $E[f; P]$ stands for the expectation of f under P.

Lemma 1. *Suppose that the family $\{Q_\omega\}$ satisfies Condition (M). Let α be any number satisfying $0<\alpha<1$, and ω be any point in Ω. If a sequence $\{Q_{\omega',n}\}$ of probability measures on $(\boldsymbol{Y}, \boldsymbol{B})$ converges in law to $Q_{\omega'}$, then we have*

$$\lim_{n\to\infty} E[\phi_\omega^\alpha; Q_{\omega',n}] = E[\phi_\omega^\alpha; Q_{\omega'}]. \tag{2.3}$$

Proof. First we observe that for any $c\in[0, \infty)$ and $\omega_1, \omega_2\in\Omega$, the set $\{y; g(y:\omega_1)\leqq c\cdot g(y:\omega_2)\}$ is contained in $\mathfrak{C}$. This follows directly from the fact:

$$\begin{aligned}\{y; g(y:\omega_1)\leqq cg(y:\omega_2)\} &= \{y; g(y:\omega_2)\geqq \frac{1}{c}g(y:\omega_1)\} \qquad \text{if} \quad c>0\\ &= \{y; g(y:\omega_1)\geqq 2g(y:\omega_1)\} \qquad \text{if} \quad c=0.\end{aligned}$$

Since the class $\mathfrak{C}$ is closed under the formations of complement and finite intersection, we have

$$\{y; g(y:\omega_1) = cg(y:\omega_2)\}\in\mathfrak{C}, \quad \{y; g(y:\omega_1)>cg(y:\omega_2)\}\in\mathfrak{C}. \tag{2.4}$$

On the other hand, according to the Neyman-Pearson lemma, ϕ_ω^α is given by

$$\begin{aligned}\phi_\omega^\alpha(y) &= 1 \quad \text{if} \quad g(y:\omega)>cg(y:\theta_0)\\ &= d \quad \text{if} \quad g(y:\omega)=cg(y:\theta_0)\\ &= 0 \quad \text{if} \quad g(y:\omega)<cg(y:\theta_0)\end{aligned} \tag{2.5}$$

where c and $d(0\leqq d\leqq 1)$ are some constants. From (2.4) we have $\{y; g(y:\omega)> cg(y:\theta_0)\}\in\mathfrak{C}$ and $\{y; g(y:\omega)=cg(y:\theta_0)\}\in\mathfrak{C}$, and hence we have

$$\begin{aligned}\lim_{n\to\infty} E[\phi_\omega^\alpha; Q_{\omega',n}] &= \lim_{n\to\infty} Q_{\omega',n}(\{y; g(y:\omega)>cg(y:\theta_0)\})\\ &\quad +d\cdot[\lim_{n\to\infty} Q_{\omega',n}(\{y; g(y:\omega) = c\cdot g(y:\theta_0)\})]\\ &= Q_{\omega'}(\{y; g(y:\omega)>c\cdot g(y:\theta_0)\})\\ &\quad +d\cdot Q_{\omega'}(\{y; g(y:\omega) = c\cdot g(y:\theta_0)\})\\ &= E[\phi_\omega^\alpha; Q_{\omega'}].\end{aligned} \tag{2.6}$$

The proof of the lemma is completed.

Denote by $\beta(\omega:\alpha)$ the power of the most powerful level α test for testing Q_{θ_0} against Q_ω; $\beta(\omega:\alpha)=E[\phi_\omega^\alpha; Q_\omega]$.

Lemma 2. (cf. Lehmann [1]) *Let $\alpha\in(0, 1)$. If $\{Q_\omega; \omega\in\Omega\}$ satisfies Condition (M), then $\beta(\omega:\alpha)<\beta(\omega':\alpha)$ whenever ω and $\omega'\in\Omega$ satisfy $|\omega-\theta_0| < |\omega'-\theta_0|$ and $\beta(\omega':\alpha)<1$.*

Denote by Ψ the family of functions ψ from $[0, \infty]$ to $[0, \infty]$ satisfying the following conditions (a) to (d).

(a) ψ is monotone decreasing in (c, ∞) for sufficiently large $c>0$.

(b) For any $\rho>0$, $\lim_{x\to\infty} \psi(\rho x)/\psi(x)=a_\psi(\rho)$ exists.

(c) $\lim_{\rho\to 0} a_\psi(\rho) = \infty$, $\lim_{\rho\to\infty} a_\psi(\rho)=0$. (For the sake of convenience we define $a_\psi(0)=\infty$ and $a_\psi(\infty)=0$.)

(d) $a_\psi(\rho)$ is a continuous and monotone strictly decreasing function of ρ.

REMARK. From the properties (a), (b), (c) and (d) mentioned above, it follows that for any ψ and $\psi'\in\Psi$

$$a_\psi(1) = 1,\ a_\psi(\rho)>0 \text{ for any } 0<\rho<\infty,\ \lim_{x\to\infty} \psi(x) = 0 \tag{2.7}$$
$$\text{and } \lim_{\substack{n\in N\\ n\to\infty}} \psi(n)/\psi'(n) = \lim_{x\to\infty} \psi(x)/\psi'(x)\,.$$

Denote by $\hat{\Psi}$ the class of the families $\{\pi_\nu\}_{\nu>0}$ having positive continuous parameter ν of mappings from Ω to Θ_1 such that

$$\begin{aligned} \pi_\nu(\omega) &= \theta_0+\psi(\nu)(\omega-\theta_0) && \text{if } \theta_0+\psi(\nu)(\omega-\theta_0)\in\Theta_1 \\ &= \theta_0 && \text{otherwise,} \end{aligned} \tag{2.8}$$

with some $\psi\in\Psi$.

Theorem 1. *Suppose that $\{\pi_\nu\}_{\nu>0}$ and $\{\pi'_\nu\}_{\nu>0}$ are two elements of $\hat{\Psi}$ and that a sequence $\{T_n\}_{n\in N}$ of statistics is of type (L) relative to $\{\pi_n\}_{n\in N}$ and also to $\{\pi'_n\}_{n\in N}$. Then $\lim_{n\to\infty} [\pi_n(\omega)-\theta_0]/[\pi'_n(\omega)-\theta_0]$ exists and positive finite for any $\omega\in\Omega\setminus\{\theta_0\}$, and hence the rates of convergence of $\{\pi_n\}_{n\in N}$ and of $\{\pi'_n\}_{n\in N}$ coincide with each other.*

Proof. Let $\pi_n(\omega)=\theta_0+\psi(n)(\omega-\theta_0)$ and $\pi'_n(\omega)=\theta_0+\psi'(n)(\omega-\theta_0)$ for sufficiently large $n\in N$, where ψ, $\psi'\in\Psi$. Define $\rho_n=\psi'(n)/\psi(n)$. In order to prove the theorem it is sufficient to show that $\lim_{n\to\infty} \rho_n$ exists and $0<\lim_{n\to\infty} \rho_n<\infty$. First we show that $\liminf_{n\to\infty} \rho_n>0$. Suppose that $\liminf_{n\to\infty} \rho_n=0$ then take a subsequence $\{\rho_{n_i}\}$ of $\{\rho_n\}$ such that $\rho_{n_i}\to 0$. For any point ω in Ω, let ϕ be the most powerful level α test for testing Q_{θ_0} against Q_ω and ϕ' that for testing Q'_{θ_0} against Q'_ω. Here Q'_ω is the limiting distribution of $P_{\pi'_n(\omega),n}T_n^{-1}$. Then from Lemma 1 and the property of uniform convergence of $P_{\pi_n(\cdot),n}T_n^{-1}$ we have

$$\begin{aligned} E[\phi'; Q'_\omega] &= \lim_{i\to\infty} E[\phi'; P_{\theta'_i(\omega),n_i} T_{n_i}^{-1}] \\ &= \lim_{i\to\infty} E[\phi'; P_{\theta_i(\omega_i),n_i} T_{n_i}^{-1}] \\ &= E[\phi'; Q_{\theta_0}] \\ &= \alpha\,, \end{aligned} \tag{2.9}$$

where $\omega_i=\theta_0+\rho_{n_i}(\omega-\theta_0)$, $\theta'_i(\omega)=\pi'_{n_i}(\omega)$ and $\theta_i(\omega_i)=\pi_{n_i}(\omega_i)$. Thus $E[\phi'; Q'_\omega]=\alpha$

for every $\omega\in\Omega$. But this does not hold unless $\Omega=\{\theta_0\}$ from Lemma 2. Hence $\liminf_{n\to\infty}\rho_n>0$. Similarly we have $\limsup_{n\to\infty}\rho_n<\infty$.

Let $\liminf_{n\to\infty}\rho_n=a$ and $\limsup_{n\to\infty}\rho_n=b$. Then $0<a\leqq b<\infty$, and there exist subsequences $\{\rho_{n_i}\}$ and $\{\rho_{n'_j}\}$ of $\{\rho_n\}$ such that $\rho_{n_i}\to a$ and $\rho_{n'_j}\to b$. Let $\omega_i=\theta_0+\rho_{n_i}(\omega-\theta_0)$, $\omega'_j=\theta_0+\rho_{n'_j}(\omega-\theta)$, $\tilde{\omega}=\theta_0+a(\omega-\theta_0)$ and $\hat{\omega}=\theta_0+b(\omega-\theta)$. Then, again from Lemma 1 and the property of uniform convergence of $\{P_{\pi_n(\cdot),n}T_n^{-1}\}$, we have for each $\omega\in\Omega$

$$
\begin{aligned}
(2.10)\qquad E[\phi: Q_{\tilde{\omega}}] &= \lim_{i\to\infty} E[\phi; P_{\theta_i(\omega_i),n_i}T_{n_i}^{-1}] \\
&= \lim_{i\to\infty} E[\phi; P_{\theta'_i(\omega),n_i}T_{n_i}^{-1}] \\
&= E[\phi; Q'_{\omega}] \\
&= \lim_{j\to\infty} E[\phi; P_{\bar{\theta}_j(\omega),n'_j}T_{n'_j}^{-1}] \\
&= \lim_{j\to\infty} E[\phi; P_{\theta^*_j(\omega'_j),n'_j}T_{n'_j}^{-1}] \\
&= E[\phi; Q_{\hat{\omega}}]
\end{aligned}
$$

where $\theta_i(\omega_i)=\pi_{n_i}(\omega_i)$, $\theta'_i(\omega)=\pi'_{n_i}(\omega)$, $\bar{\theta}_j(\omega)=\pi'_{n'_j}(\omega)$ and $\theta^*_j(\omega'_j)=\pi_{n'_j}(\omega'_j)$. Thus $E[\phi; Q_{\tilde{\omega}}]=E[\phi; Q_{\hat{\omega}}]$, and hence form Lemma 2 it follows that $|\tilde{\omega}-\theta^0|=|\hat{\omega}-\theta_0|$. Therefore, from the definition of $\tilde{\omega}$ and $\hat{\omega}$ we have $a=b$. This completes the proof.

3. The relative efficiency of tests. For a number s $(0\leqq s\leqq\infty)$ and $\psi\in\Psi$ we denote by $\rho_\psi(s)$ the number satisfying the equation

$$(3.1)\qquad a_\psi(\rho_\psi(s)) = s\,.$$

Notice that, by the property of a_ψ, the equation (3.1) has a unique solution for each s satisfying $0\leqq s\leqq\infty$.

Lemma 3. *Let ψ and ψ^* be two elements of Ψ, and c be any positive number.*

(a) *If* $\lim_{\substack{(x,y)\in D\\ x,y\to\infty}} \psi(y)/\psi(x)=p(0\leqq p\leqq\infty)$ *then* $\lim_{\substack{x,y\in D\\ x,y\to\infty}} x/y=\rho_\psi(p)^{-1}$,

where $D\subset(0,\infty)\times(0,\infty)$ is a set snch that for any $M>0$ there exists (x,y) in D satisfying $x>M$ and $y>M$. (Such a set D will be called a set of D-type in the following).

Define

$$(3.2)\qquad D(c) = \{(x,y);\ \psi^*(y)/\psi(x) = c\}\,,$$

which is not empty and a set of D-type by the properties of ψ and ψ^*.

(b) *If* $\lim_{x\to\infty}\psi^*(x)/\psi(x)=\infty$ *then* $\lim_{\substack{(x,y)\in D(c)\\ x,y\to\infty}} x/y=0.$

(c) *If* $\lim_{x\to\infty}\psi^*(x)/\psi(x)=0$ *then* $\lim_{\substack{(x,y)\in D(c)\\ x,y\to\infty}} x/y=\infty$.

(d) *If* $\lim_{x\to\infty}\psi^*(x)/\psi(x)=\lambda$ $(0<\lambda<\infty)$ *then* $\lim_{\substack{(x,y)\in D(c)\\ x,y\to\infty}} x/y=\rho_\psi(c/\lambda)^{-1}$.

Proof. First, we prove the part (a) of the lemma. Let $\lim_{\substack{(x,y)\in D\\ x,y\to\infty}}\psi(y)/\psi(x)=p$, and let $\{(x_i, y_i)\}_{i\in N}\subset D$ be any sequence such that $x_i\to\infty$ and $y_i\to\infty$ as $i\to\infty$. Suppose that $\limsup_{i\to\infty} y_i/x_i>\rho_\psi(p)$, then there exists a number ρ_1 such that $\rho_1>\rho_\psi(p)$ and $y_i/x_i\geqq\rho_1$ for infinitely many i's. Therefore we have

$$
\begin{aligned}
(3.3)\qquad p = \lim_{i\to\infty}\psi((y_i/x_i)x_i)/\psi(x_i) &\leqq \lim_{i\to\infty}\psi(\rho_1 x_i)/\psi(x_i)\\
&= a_\psi(\rho_1)\\
&< a_\psi(\rho_\psi(p)) = p\,,
\end{aligned}
$$

which is a contradiction. Thus $\limsup_{i\to\infty} y_i/x_i\leqq\rho_\psi(p)$. Similary, we have $\liminf_{x\to\infty} y_i/x_i\geqq\rho_\psi(p)$. Hence we have $\lim_{i\to\infty} y_i/x_i=\rho_\psi(p)$. This completes the proof of the part (a).

Secondly, we prove the part (b). Let $\lim_{x\to\infty}\psi^*(x)/\psi(x)=\infty$. Then, from the equality $c=\psi^*(y)/\psi(x)=[\psi^*(y)/\psi(y)][\psi(y)/\psi(x)]$ it follows that

$$
(3.4)\qquad \lim_{\substack{(x,y)\in D(c)\\ x,y\to\infty}}\psi(y)/\psi(x)=0\,.
$$

Let $\{(x_i, y_i)\}_{i\in N}\subset D(c)$ be any sequence such that $x_i\to\infty$ and $y_i\to\infty$ as $i\to\infty$. Suppose that $\liminf_{i\to\infty} y_i/x_i=\rho_0<\infty$, then $a_\psi(\rho_0+1)>0$ from (2.7). But, taking account of (3.4) we have

$$
\begin{aligned}
(3.5)\qquad a_\psi(\rho_0+1) &= \lim_{x\to\infty}\psi((\rho_0+1)x)/\psi(x)\\
&\leqq \lim_{i\to\infty}\psi((y_i/x_i)x_i)/\psi(x_i)\\
&= \lim_{\substack{(x,y)\in D(c)\\ x,x\to\infty}}\psi(y)/\psi(x)\\
&= 0\,.
\end{aligned}
$$

This is a contradiction. Thus have $\liminf_{i\to\infty} y_i/x_i=\infty$, and hence the part (b) was proved.

Obviously, the part (c) follows from the part (b).

Finally we prove the part (d). Let $\lim_{x\to\infty}\psi^*(x)/\psi(x)=\lambda$, $0<\lambda<\infty$. Define $\psi^*(x)/\psi(x)=\lambda_x$ then $\lambda_x\to\lambda$ as $x\to\infty$. By the definition of $D(c)$ we have $\psi(y)/\psi(x)=c/\lambda_y$ for any $(x, y)\in D(c)$. Since c/λ_y converges to c/λ, from the part (a) of this lemma we have

$$\lim_{\substack{(x,y)\in D(c)\\ x,y\to\infty}} x/y = \rho_\psi(c/\lambda)^{-1}. \tag{3.6}$$

This completes the proof of the part (d).

The proof of the lemma is completed.

The following lemma is easily seen, and the proof will be omitted.

Lemma 4. *Suppose that a family $\{Q_\omega;\omega\in\Omega\}$ of probability measures on $(\boldsymbol{Y},\boldsymbol{B})$ satisfies Condition (M). Let α be any number such that $0<\alpha<1$, and let $\beta(\omega:\alpha)$ be as in Lemma 2. Then the function: $\omega\to\beta(\omega:\alpha)$ is continuous on Ω.*

In the followings we shall consider two sequences $\{T_n\}$ and $\{T_n^*\}$ of statistics of type (L) relative to $\{\pi_n\}_{n\in N}$ and to $\{\pi_n^*\}_{n\in N}$, respectively, where $\{\pi_\nu\}_{\nu>0}$ and $\{\pi_\nu^*\}_{\nu>0}$ are elements of $\hat{\Psi}$. Assume that $P_{\pi_n(\omega),n}T_n^{-1}\to Q_\omega$ and $P_{\pi_n^*(\omega),n}T_n^{*-1}\to Q_\omega^*$ in law as $n\to\infty$ uniformly in a neighborhood of each $\omega\in\Omega$. Denote by ϕ and ϕ^* the most powerful level α tests for testing Q_{θ_0} against Q_ω and $Q_{\theta_0}^*$ against Q_ω^*, respectively. The power of the tests ϕ and ϕ^* are denoted by $\beta(\omega:\alpha)$ and $\beta^*(\omega:\alpha)$ respectively. Suppose that we are now concerned with testing the null hypothesis $\theta=\theta_0$ versus the alternative $\theta\in\Theta_1\backslash\{\theta_0\}$. Let $\alpha\in(0, 1)$ be fixed. Define

$$\begin{aligned} &D(\{T_n\},\{T_n^*\},\{\pi_n\},\{\pi_n^*\}) \\ &\quad=\bigcup_{(\omega_1,\omega_2)\in\bar{K}} D(\{T_n\},\{T_n^*\},\{\pi_n\},\{\pi_n^*\};\omega_1,\omega_2), \end{aligned} \tag{3.7}$$

where $\bar{K}\,(=K(\{Q_\omega\},\{Q_\omega^*\}))=\{(\omega_1,\omega_2)\in\Omega\times\Omega;\ \alpha<\beta(\omega_1:\alpha)=\beta^*(\omega_2:\alpha)<1\}$, and $D(\{T_n\},\{T_n^*\},\{\pi_n\},\{\pi_n^*\};\omega_1,\omega_2)=\{(n_1,n_2);\ n_1>0,\ n_2>0,\ \pi_{n_1}(\omega_1)=\pi_{n_2}^*(\omega_2)\}\,(=\bar{D})$.

In the following the notation n_1, n_2 means some positive numbers (not necessarily integers).

REMARK. (1) By Lemma 4, $\bar{K}$ is not empty for any pair $\{Q_\omega;\omega\in\Omega\}$ and $\{Q_\omega^*;\omega\in\Omega\}$ of families of probability measures satisfying Condition (M).

(2) $\bar{D}$ is a set of D-type.

Theorem 2. $\lim_{\substack{(n_1,n_2)\in\bar{D}\\ n_1,n_2\to\infty}} n_1/n_2$, *whenever it exists, does not depend on the choice of the elements $\{\pi_\nu\}_{\nu>0}\in\hat{\Psi}$ and $\{\pi_\nu^*\}_{\nu>0}\in\hat{\Psi}$ such that $\{T_n\}$ and $\{T_n^*\}$ are of type (L) relative to $\{\pi_n\}_{n\in N}$ and $\{\pi_n^*\}_{n\in N}$ respectively.*

Proof. Let the sequence $\{T_n\}$ be of type (L) also relative to $\{\pi'_\nu\}_{\nu>0}\in\hat{\Psi}$. Assume that $P_{\pi'_n(\omega),n}T_n^{-1}\to Q'_\omega$ in law as $n\to\infty$ uniformly in a neighborhood of each $\omega\in\Omega$. Denote by $\beta'(\omega:\alpha)$ the power of the most powerful level α test for testing Q'_{θ_0} against Q'_ω. In order to prove our theorem, it is sufficient to show that

$$(3.8)\qquad \lim_{\substack{(n_1,n_2)\in D' \\ n_1,n_2\to\infty}} n_1/n_2 = \lim_{\substack{(n_1,n_2)\in \bar{D} \\ n_1,n_2\to\infty}} n_1/n_2\,,$$

where $D'=D(\{T_n\}, \{T_n^*\}, \{\pi_n'\}, \{\pi_n^*\})$.
Let $\pi_\nu(\omega)=\theta_0+\psi(\nu)(\omega-\theta_0)$ and $\pi_\nu'(\omega)=\theta_0+\psi'(\nu)(\omega-\theta_0)$ for sufficiently large $\nu>0$. From Theorem 1 we have

$$(3.9)\qquad \lim_{n\to\infty} \psi'(n)/\psi(n) = a\,, \qquad 0<a<\infty.$$

Thus from (2.7) we have

$$(3.10)\qquad \lim_{x\to\infty} \psi'(x)/\psi(x) = a\,.$$

For each $\omega\in\Omega$, let $\bar{\pi}(\omega)=\theta_0+a(\omega-\theta_0)$. Define $(\omega)_\nu=\theta_0+(\psi'(\nu)/\psi(\nu))(\omega-\theta_0)$ for each $\nu>0$ and $\omega\in\Omega$. Then, for each $\omega\in\Omega$

$$(3.11)\qquad \pi_\nu'(\omega)=\pi_\nu((\omega)_\nu)\,,\ \text{and}\ (\omega)_\nu\to\bar{\pi}(\omega)\ \text{as}\ \nu\to\infty.$$

Hence from the assumption, we have

$$(3.12)\qquad P_{\pi_n'(\omega),n}T_n^{-1}\to Q_\omega'\ \text{and}\ P_{\pi_n'((\omega)_n),n}T_n^{-1}\to Q_{\bar{\pi}(\omega)}\ \text{as}\ n\to\infty$$

in law for each $\omega\in\Omega$.
Therefore $Q_\omega'=Q_{\bar{\pi}(\omega)}$, and hence

$$(3.13)\qquad \beta'(\omega:\alpha)=\beta(\bar{\pi}(\omega):\alpha)\qquad \text{fro ech}\ \omega\in\Omega\,.$$

Now, suppose that the following two equations (3.14) and (3.15) hold at the same time:

$$(3.14)\qquad \beta(\omega_1:\alpha)=\beta'(\omega_1':\alpha)=\beta^*(\omega_2:\alpha)$$

and

$$(3.15)\qquad \pi_{n_1}(\omega_1)=\pi_{n_1'}'(\omega_1')=\pi_{n_2}^*(\omega_2)\,.$$

Then, taking account of (3.13), from (3.14) we have

$$(3.16)\qquad \beta(\omega_1:\alpha)=\beta(\bar{\pi}(\omega_1'):\alpha)\,.$$

Hence, from Lemma 2 we have

$$(3.17)\qquad |\omega_1-\theta_0|=|\bar{\pi}(\omega_1')-\theta_0|\,.$$

On the other hand, (3,15) implies

$$(3.18)\qquad \psi(n_1)(\omega_1-\theta_0)=\psi'(n_1')(\omega_1'-\theta_0)\,.$$

From (3.11) we have $\pi_{n_1'}'(\omega_1')=\pi_{n_1'}((\omega_1')_{n_1'})$, and $(\omega_1')_{n_1'}\to\bar{\pi}(\omega_1')$ as $n_1'\to\infty$. Thus, from (3.17) and (3.18) we have

(3.19) $$\psi(n_1)|\bar{\pi}(\omega_1')-\theta_0| = \psi(n_1')|(\omega_1')_{n_1'}-\theta_0| .$$

Therefore

(3.20) $$\lim_{\substack{(n_1, n_2)\in \bar{D} \\ (n_1', n_2)\in D'}} \psi(n_1')/\psi(n_1) = 1 .$$

By Lemma 3. (a) we then have

(3.21) $$\lim_{\substack{(n_1, n_2)\in \bar{D} \\ (n_1', n_2)\in D'}} n_1/n_1' = 1 .$$

Hence we have

(3.22) $$\lim_{\substack{(n_1', n_2)\in D' \\ n_1', n_2\to\infty}} n_1'/n_2 = \lim_{\substack{(n_1, n_2)\in \bar{D} \\ n_1, n_2\to\infty}} n_1/n_2 .$$

This completes the proof of the theorem.

Let $\alpha\in(0, 1)$ be a fixed number, and let $\{\phi_n\}_{n\ni N}$ and $\{\phi_n^*\}_{n\in N}$ be two sequences of tests such that

(3.23) $$\lim_{n\to\infty} E[\phi_n; P_{\theta_0,n}] = \lim_{n\to\infty} E[\phi_n^*; P_{\theta_0,n}] = \alpha .$$

Let Γ be a class of families $\{\gamma_\nu\}_{\nu>0}$ of mappings from Ω to Θ_1.

DEFINITION 4. The *Γ-asymptotic relative efficiency of $\{\phi_n^*\}$ with respect to* $\{\phi_n\}$ is defined to be

(3.24) $$e(\{\phi_n^*\}, \{\phi_n\} : \Gamma) = \lim_{i\to\infty} [n_i]/[n_i^*] ,$$

if the right hand side of (3.24) exists and has the same value for any $\{\gamma_\nu\}$ and $\{\gamma_\nu^*\}$ in Γ, and any two points ω and ω^* in $\Omega\backslash\{\theta_0\}$, and any two sequences $\{n_i\}_{i\in N}$ and $\{n_i^*\}_{i\in N}$ of positive numbers such that $n_i \uparrow \infty$ and $n_i^* \uparrow \infty$ and that

(3.25) $$\gamma_{n_i}(\omega) = \gamma_{n_i^*}^*(\omega) \qquad \text{for every } i\in N, \text{ and}$$

(3.26) $$\lim_{i\to\infty} E[\phi_{[n_i]}; P_{\theta_i(\omega),[n_i]}] = \lim_{i\to\infty} E[\phi_{[n_i^*]}^*; P_{\theta_i^*(\omega^*),[n_i^*]}]$$
$$(\theta_i(\omega) = \gamma_{n_i}(\omega), \theta_i^*(\omega^*) = \gamma_{n_i^*}^*(\omega^*))$$

where the limits in both sides of (3.26) exist and equal neither zero nor one. Here for a real number a we denote by $[a]$ the maximum integer less than or equal to a.

Theorem 3. *Suppose that* $(a)\pi_\nu(\omega)=\theta_0+\psi(\nu)(\omega-\theta_0)$ *and* $\pi_\nu^*(\omega)=\theta_0+\psi^*(\nu)(\omega-\theta_0)$ *for sufficiently large* $\nu>0$, $(b)\lim_{\nu\to\infty}\psi^*(\nu)/\psi(\nu)=\lambda(0\leqq\lambda\leqq\infty)$, *and* $(c)Q_\omega^*=Q_{\pi(\omega)}$ $(\omega\in\Omega)$ *where* $\pi(\omega)=\theta_0+c(\omega-\theta_0)$ *with some* $c\in R^1$. *Let* $\hat{\Psi}(a)$ *be the set of families* $\{\pi_\nu\}_{\nu>0}\in\hat{\Psi}$ *such that* $\{\pi_n\}_{n\in N}$ *is an accessible sequence of* $\{T_n\}_{n\in N}$. *Then we have*

$$(3.27)\qquad e(\{\phi^*(T_n^*)\},\ \{\phi(T_n)\}:\hat{\Psi}(a)) = \rho_\psi(|c|/\lambda)^{-1}.$$

Proof. Let $\beta_n(\phi(T_n): \pi_n(\omega)) = E[\phi(T_n); P_{\pi_n(\omega),n}]$ and $\beta_n(\phi^*(T_n^*): \pi_n^*(\omega)) = E[\phi^*(T_n^*); P_{\pi_n^*(\omega),n}]$ for each $\omega \in \Omega$. From Lemma 1 we have

$$(3.28)\qquad \beta_n(\phi(T_n): \pi_n(\omega)) \to \beta(\omega: \alpha) \quad \text{and} \quad \beta_n(\phi^*(T_n^*): \pi_n^*(\omega)) \to \beta^*(\omega: \alpha)$$

as $n \to \infty$, for each $\omega \in \Omega$. From our assumption we have

$$(3.29)\qquad \beta^*(\omega: \alpha) = \beta(\pi(\omega): \alpha) \qquad \text{for each } \omega \in \Omega .$$

Thus, by Lemma 2 it holds that

$$(3.30)\qquad \alpha < \beta(\omega_1: \alpha) = \beta^*(\omega_2: \alpha) < 1 \ \text{ implies } |\omega_1 - \theta_0| = |c|\,|\omega_2 - \theta_0| .$$

On the other hand,

$$(3.31)\qquad \pi_{n_1}(\omega_1) = \pi_{n_2}^*(\omega_2) \ \text{ implies } \psi(n_1)(\omega_1 - \theta_0) = \psi^*(n_2)(\omega_2 - \theta_0) .$$

We note here that in (3.31) n_1 and n_2 are not necessarily integers. Combining (3.30) with (3.31), we have

$$(3.32)\qquad |c|\,\psi(n_1) = \psi^*(n_2)$$

for any $(n_1, n_2) \in \bar{D}$. Hence $\bar{D} \subset D(|c|)$, where $D(|c|) = \{(n_1, n_2);\ n_1 > 0,\ n_2 > 0,\ \psi^*(n_2)/\psi(n_1) = |c|\}$. We then have by Lemma 3,

$$(3.33)\qquad \lim_{\substack{(n_1, n_2) \in \bar{D} \\ n_1, n_2 \to \infty}} n_1/n_2 = \lim_{\substack{(n_1, n_2) \in D(|c|) \\ n_1, n_2 \to \infty}} n_1/n_2 = \rho_\psi(|c|/\lambda)^{-1} .$$

We note here that by Theorem 2 the left hand side of (3.33) does not depend on the choice of $\{\pi_\nu\}_{\nu>0} \in \hat{\Psi}(a)$ and $\{\pi_\nu^*\}_{\nu>0} \in \hat{\Psi}(a)$. Thus, taking account of (3.28) the left hand side of (3.33), by definition, gives $\hat{\Psi}(a)$-asymptotic relative efficiency of $\{\phi^*(T_n^*)\}_{n\in N}$ with respect to $\{\phi(T_n)\}_{n\in N}$. This completes the proof of the theorem.

REMARK. Theorem 3 extends the result in Noether [2] as follows. If a sequence $\{T_n\}$ of statistics satisfies the conditions A, B, C and D in [2], then the sequence $\hat{T}_n = [T_n - a_n]/b_n$ of statistics is of type (L) relative to $\pi_n(\omega) = \theta_0 + n^{-\delta}(\omega - \theta_0)$ where $a_n = E[T_n; P_{\theta_0,n}]$, $b_n = [E[(T_n - a_n)^2; P_{\theta_0,n}]]$ and δ is some positive number. Furthermore, the family $\{\pi_\nu\}$, $\pi_\nu(\omega) = \theta_0 + \nu^{-\delta}(\omega - \theta_0)$, belongs to $\hat{\Psi}$ and the family of limit distributions of $P_{\pi_n(\omega),n}\hat{T}_n^{-1}$ is a normal family on R^1 with mean $c(\omega - \theta_0)^m/m!$ and variance 1, where c is a positive number and m a positive integer. Therefore, if two sequences $\{T_n\}$ and $\{T_n^*\}$ of statistics satisfy the conditions of the theorem given in [2], then the asymptotic relative efficiency in Pitman's sense of $\{\phi_n^*\}_{n\in N}$ with respect to $\{\phi_n\}_{n\in N}$ can be calculated according to Theorem 3. Here ϕ_n and ϕ_n^* are the tests considered in [2].

Finally we shall give two examples which are not standard case.

EXAMPLE 1. Let $\Theta=\Theta_1=(-1, 1)$, $\Omega=R^1$ and $\theta_0=0$. For each $\theta\in\Theta$ let P_θ be a distribution on R^1 such that $\int_{-\infty}^{\infty} x dP_\theta=a(\theta)=|\theta|[\log|\theta|^{-1}]^{1/2}$ and $\int_{-\infty}^{\infty}(x-a(\theta))^2 dP_\theta=b(\theta)$. Assume that $b(\theta)$ is positive and continuous with respect to θ in a neighborhood of 0 in Θ_1. Suppose that the random variables $X_1, X_2, \cdots, X_n$ are independently and identically distributed according to P_θ. Let $T_n=(X_1+X_2+\cdots+X_n)/n^{1/2}$ and $\pi_n(\omega)=\omega/(\mathrm{n}\cdot\log\mathrm{n})^{1/2}$ then $P_{\pi_n(\omega),n}T_n^{-1}$ converges in law to the normal distribution $N(-\omega/\sqrt{2}, b(0))$ uniformly in a neighborhood of ω. Therefore $\{T_n\}$ is of type (L) relative to $\{\pi_n\}$, and the rate of convergence of $\{\pi_n(\omega)\}$ is $\{\rho_n(\mathrm{n}\cdot\log\mathrm{n})^{1/2}\}$ where $\{\rho_n\}$ is any sequence of positive numbers satisfying $0<\liminf_{n\to\infty}\rho_n\leqq\limsup_{n\to\infty}\rho_n<\infty$.

EXAMPLE 2. Let $\Theta=(0, \infty)$ and $\Omega=\Theta_1=[\theta_0, \infty)$ where θ_0 is a fixed point of Θ. Let P_θ be the uniform dstribution on $[0, \theta]$, and let the random variables $X_1, X_2, \cdots, X_n$ be independently and identically distributed according to P_θ. Let $\{n_m\}_{m\geqq 1}$ be a sequence of positive integers such that $n_1<n_2<\cdots\to\infty$. For each $n\in N$, denote by $m(n)$ the number m satisfying $n_m\leqq n<n_{m+1}$. We assume that, for some positive number c, $m(n)/n\to c$ as $n\to\infty$. Denote by $X_{(n),n}$ the maximum of $X_1, X_2, \cdots, X_n$. We now consider two sequences of statistics $\{T_n\}$ and $\{T_n^*\}$, and two seqences of alternatives $\{\pi_n\}$ and $\{\pi_n^*\}$ such that $T_n=\theta_0+n(X_{(n),n}-\theta_0)$, $T_n^*=T_{m(n)}$ and $\pi_n(\omega)=\pi_n^*(\omega)=\theta_0+(1/n)(\omega-\theta_0)$. Let Q_ω be the distribution with the density $dQ_\omega/d\mu=(1/\theta_0)\exp[(y-\omega)/\theta_0]$ $(y\leqq\omega)$, $=0$ $(y>\omega)$. We then have

$$(3.34)\qquad \begin{aligned}\lim_{n\to\infty} P_{\pi_n(\omega),n}T_n^{-1} &= Q_\omega \text{ (in law)},\\ \lim_{n\to\infty} P_{\pi_n^*(\omega),n}T_n^{*-1} &= Q_{\pi(\omega)} \text{ (in law)}\end{aligned}$$

uniformly in a neighborhood of each $\omega\in\Omega$ where $\pi(\omega)=\theta_0+c(\omega-\theta_0)$. Therefore the sequences $\{T_n\}$ and $\{T_n^*\}$ are of type (L) relative to $\{\pi_n\}$ and to $\{\pi_n^*\}$, respectively. The rate of convergence of $\{\pi_n\}$ is $\{\rho_n\cdot n\}$ where $\{\rho_n\}$ is any sequence of positive numbers satisfying $0<\liminf_{n\to\infty}\rho_n\leqq\limsup_{n\to\infty}\rho_n<\infty$. Since $a_\Psi(\rho)=\rho^{-1}$, we have $e(\{\phi^*(T_n^*)\}, \{\phi(T_n)\}:\hat{\Psi}(a))=c$ where ϕ and ϕ^* are the most powerful level α tests for testing Q_{θ_0} against Q_ω and $Q_{\theta_0}^*$ against Q_ω^*, respectively.

4. Acknowledgment. The author wishes to express his appreciation to Professors Goro Ishii and Hirokichi Kudō for their many helpful discussions.

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