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ON HOMOGENEOUS REAL HYPERSURFACES IN A COMPLEX PROJECTIVE SPACE

Dedicated to Professor S. Sasaki on his 60th birthday

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The purpose of this paper is to determine those homogeneous real hypersurfaces in a complex projective space $P_n(C)$ of complex dimension $n(\geqq 2)$ which are orbits under analytic subgroups of the projective unitary group $PU(n+1)$, and to give some characterizations of those hypersurfaces. In §1 from each effective Hermitian orthogonal symmetric Lie algebra of rank two we construct an example of homogeneous real hypersurface in $P_n(C)$, which we shall call a model space in $P_n(C)$. In §2 we show that the class of all homogeneous real hypersurfaces in $P_n(C)$ that are orbits under analytic subgroups of $PU(n+1)$ is exhausted by all model spaces. In §§3 and 4 we give some conditions for a real hypersurface in $P_n(C)$ to be an orbit under an analytic subgroup of $PU(n+1)$ and in the course of proof we obtain a rigidity theorem in $P_n(C)$ analogous to one for hypersurfaces in a real space form.

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1. Model spaces

In this section we shall state several model spaces in a complex projective space $P_n(C)$ with the Fubini-Study metric of constant holomorphic sectional curvature. They are obtained essentially as orbits under the linear isotropy groups of various Hermitian symmetric spaces of rank two. Precisely, let $(\mathfrak{u}, \theta)$ be an effective orthogonal symmetric Lie algebra of compact type. $\mathfrak{u}$ is a compact semisimple Lie algebra and θ is an involutive automorphism of $\mathfrak{u}$ ([3]). Let $\mathfrak{u}=\mathfrak{k}+\mathfrak{p}$ be the decomposition of $\mathfrak{u}$ into the eigenspaces of θ for the eigenvalues $+1$ and -1, respectively. Then $\mathfrak{k}$ and $\mathfrak{p}$ satisfy $[\mathfrak{k}, \mathfrak{k}]\subset\mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}]\subset\mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}]\subset\mathfrak{k}$.

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For the Killing form B of $\mathfrak{u}$ we define a positive definite inner product $\langle\,,\rangle$ on $\mathfrak{p}$ by $\langle X, Y\rangle=-B(X, Y)$ for $X, Y\in\mathfrak{p}$. Let K be the analytic subgroup of the group of inner automorphisms of $\mathfrak{u}$ with Lie algebra ad($\mathfrak{k}$). Then K leaves the subspace $\mathfrak{p}$ of $\mathfrak{u}$ invariant and acts on $\mathfrak{p}$ as an orthogonal transformation group with respect to $\langle\,,\rangle$. We define a representation ρ of K on $\mathfrak{p}$ by $\rho(k)=k|\mathfrak{p}$ for $k\in K$. The differentiation ρ_* of ρ is an isomorphism of $\mathfrak{k}$ into the Lie algebra of the orthogonal group of $\mathfrak{p}$ and satisfies $(\rho_*X)Y=[X, Y]$ for all $X\in\mathfrak{k}$ and all $Y\in\mathfrak{p}$. Let S denote the unit hypersurface in $\mathfrak{p}$ centered at the origin and A be a regular element of $\mathfrak{p}$ in S. Then the orbit $N=\rho(K)A$ of A under $\rho(K)$ is a submanifold of S of codimension $R-1$ ([9]), where R denotes the rank of the orthogonal symmetric Lie algenra $(\mathfrak{u}, \theta)$. Furthermore we assume that $(\mathfrak{u}, \theta)$ is Hermitian and of rank two. Then N is a hypersurface in S. It is known ([3]) that there is an element Z_0 in the center of $\mathfrak{k}$ such that

$$(\rho_*Z_0)^2=-1\,,$$
$$\langle(\rho_*Z_0)X, (\rho_*Z_0)Y\rangle=\langle X, Y\rangle \qquad \text{for } X, Y\in\mathfrak{p}.$$

Thus we may regard $\mathfrak{p}$ as a complex vector $(n+1)$-space $\boldsymbol{C}^{n+1}$ with complex structure $I=\rho_*Z_0$ and Hermitian inner product $\langle\,,\rangle$, where $2(n+1)=\dim\mathfrak{p}$. Let π be the canonical projection of $\mathfrak{p}-\{0\}=\boldsymbol{C}^{n+1}-\{0\}$ onto $P_n(\boldsymbol{C})$ and V be a vector field on $\mathfrak{p}$ defined by $V_X=I(X)$, $X\in\mathfrak{p}$. Since the 1-parameter subgroup $\rho(\exp\boldsymbol{R}Z_0)$ of $\rho(K)$ induces V and leaves N invariant, it is easy to prove htat the image $M=\pi(N)$ of N by π becomes a real hypersurface in $P_n(\boldsymbol{C})$. We assert that $\rho(K)$ is an analytic subgroup of the unitary group $U(n+1)$ of $\mathfrak{p}$ with respect to I and ρ mapps the group C_0 of K generated by Z_0 onto the center of $U(n+1)$ isomorphically. In fact, for any $k\in K$ we have

$$I\circ\rho(k)=(\mathrm{ad}\,Z_0)|\mathfrak{p}\circ k|\mathfrak{p}=k|\mathfrak{p}\circ(\mathrm{ad}\,Z_0)|\mathfrak{p}=\rho(k)\circ I\,.$$

The second assertion is evident. It follows that the group $G=\rho(K)/\rho(C_0)$ is a compact analytic subgroup of $PU(n+1)=U(n+1)/\rho(C_0)$ which acts on M transitively as a transformation group of isometries of M. We shall call this M a model space in $P_n(\boldsymbol{C})$. We can say that a real hypersurface $\hat{M}$ in $P_n(\boldsymbol{C})$ obtained from another regular element of $\mathfrak{p}$ in S is of the same type as M in the sense that both M and $\hat{M}$ are orbits in $P_n(\boldsymbol{C})$ under the same subgroup G of $PU(n+1)$. Thus it turned out that each effective Hermitian orthogonal symmetric Lie algebra of compact type and of rank two produces real hypersurfaces of the same type in $P_n(\boldsymbol{C})$. By virture of a complete classification theorem of effective Hermitian orthogonal symmectric Lie algebras we obtain the following list of model spaces of different type in $P_n(\boldsymbol{C})$. The first case in the Table is the only case where $(\mathfrak{u}, \theta)$ is reducible, which was found by N. Tanaka ([8]).

Table

$\mathfrak{u}$	$\mathfrak{k}$	dim M
$\mathfrak{su}(p+1)+\mathfrak{su}(q+1)$ $p\geqq q\geqq 1,\ p>1$	$\mathfrak{s}(\mathfrak{u}(p)+\mathfrak{u}(1))+\mathfrak{s}(\mathfrak{u}(q)+\mathfrak{u}(1))$	$2(p+q)-3$
$\mathfrak{su}(m+2)$ $m\geqq 3$	$\mathfrak{s}(\mathfrak{u}(m)+\mathfrak{u}(2))$	$4m-3$
$\mathfrak{o}(m+2)$ $m\geqq 3$	$\mathfrak{o}(m)+\boldsymbol{R}$	$2m-3$
$\mathfrak{o}(10)$	$\mathfrak{u}(5)$	17
E_6	$\mathfrak{o}(10)+\boldsymbol{R}$	29

2. Orbits under analytic subgroups of $PU(n+1)$

In §1 we saw that each model space is an orbit in $P_n(\boldsymbol{C})$ under an anlytic subgroup of the identity component $PU(n+1)$ of the group of all isometries of $P_n(\boldsymbol{C})$. Conversely we have

Theorem 2.1. *If M is a real hypersurface in $P_n(\boldsymbol{C})$ being an orbit an analytic subgroup G of $PU(n+1)$, then M is congruent to one of model spaces with respect to the group of all isometries of $P_n(\boldsymbol{C})$*

In order to prove Theorem 2.1 we need some preparations.

Lemma 2.2. *Let $(\mathfrak{u}, \theta)$ be an effective orthogonal symmetric Lie algebra of compact type and the other notations as in* §1. *If H is an analytic subgroup of K such that $\rho(H)$ acts on an orbit $N=\rho(K)A$ transitively, then so is kHk^{-1} for any $k\in K$.*

Proof. Choosing an element h of H such that $\rho(k^{-1})A=\rho(h)A$, we have $\rho(kHk^{-1})A=\rho(kH)\rho(k^{-1})A=\rho(kH)\rho(h)A=\rho(k)\rho(H)A=\rho(k)N=N$. Q.E.D.

Lemma 2.3. *Let $(\mathfrak{u}, \theta)$ be an irreducible effective orthogonal symmetric Lie algebra of compact type and of rank two and H be an analytic subgroup of K such that $\rho(H)$ acts on N transitively. Suppose that there is a $\rho(H)$-invariant complex structure I on $\mathfrak{p}$ such that $I=\rho_* Z_0$ for some $Z_0\in\mathfrak{k}$. If H is not semisimple, then $(\mathfrak{u}, \theta)$ is Hermitian.*

Proof. Assume that $(\mathfrak{u}, \theta)$ is not Hermitian. Then $\mathfrak{k}$ is semisimple. We assert that $\mathfrak{k}$ and $\mathfrak{u}$ have the same rank. In fact, if the rank of $\mathfrak{k}$ is smaller than that of $\mathfrak{u}$, then there is a Cartan subalgebra $c(\mathfrak{k})+c(\mathfrak{p})$ of $\mathfrak{u}$ such that $c(\mathfrak{k})$ is a Cartan subalgebra of $\mathfrak{k}$ containing Z_0, and $\{0\}\neq c(\mathfrak{p})\subset\mathfrak{p}$. Then $\rho_* Z_0$ vanishes on $c(\mathfrak{p})$, which contradicts $\rho_* Z_0=I$. By a complete classification theorem of effective orthogonal symmetric Lie algebras we know that the possile set of paris $(\mathfrak{u}, \mathfrak{k})$ satisfying these conditions is $\{(G_2, \mathfrak{o}(4)), (\mathfrak{sp}(2+n), \mathfrak{sp}(2)+\mathfrak{sp}(n))\}$.

The case where $\mathfrak{u}=G_2$ and $\mathfrak{k}=\mathfrak{o}(4)$. Since $\rho(H)$ acts on N transitively, dim $H\geq$ dim $N=$ dim $\mathfrak{p}-2=6$. Hence $\mathfrak{h}=\mathfrak{k}$ since dim $\mathfrak{o}(4)=6$, where $\mathfrak{h}$ denotes the Lie algebra of H. This contradicts the fact that $\mathfrak{o}(4)$ is semisimple.

The case where $\mathfrak{u}=\mathfrak{sp}(2+n)$ and $\mathfrak{k}=\mathfrak{sp}(2)+\mathfrak{sp}(n)$. In this case we shall derive a contradiction by determining a concrete expression of $\mathfrak{h}$. We denote by $\boldsymbol{H}$ the real algebra of quaternions and by $1, i, j, k$ the units of $\boldsymbol{H}$. We identify $\boldsymbol{C}$ with the subalgebra $\boldsymbol{R}\cdot 1+\boldsymbol{R}\cdot i$ of $\boldsymbol{H}$. The set of all martices of degree n with coefficients in $\boldsymbol{H}$ will be denoted by $M_n(\boldsymbol{H})$. Then we have

$$\mathfrak{u} = \mathfrak{sp}(2+n) = \{X\in M_{2+n}(\boldsymbol{H});\ {}^tX=-\bar{X}\}\ ,$$

$$\mathfrak{k} = \mathfrak{sp}(2)+\mathfrak{sp}(n) = \left\{\begin{pmatrix} X & O \\ O & Y\end{pmatrix};\ X\in\mathfrak{sp}(2),\ Y\in\mathfrak{sp}(n)\right\}.$$

We choose as a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ the following one

$$\mathfrak{t} = \left\{U(x_1, \cdots, x_{n+2}) = \begin{pmatrix} ix_1 & & 0 \\ & \ddots & \\ 0 & & ix_{n+2}\end{pmatrix};\ x_1, \cdots, x_{n+2}\in\boldsymbol{R}\right\}.$$

Then $U_r=U(0, \cdots, 1, \cdots, 0)$ (0 except for r-th), $1\leqq r\leqq n+2$, forms a base of $\mathfrak{t}$. A base ω_r, $1\leqq r\leqq n+2$, of the dual space $\mathfrak{t}^*$ of $\mathfrak{t}$ is defined by $\omega_r(U_s)=\delta_{rs}$, $1\leqq s\leqq n+2$. For an element $\alpha\in\mathfrak{t}^*$ we put

$$\mathfrak{u}_\alpha = \{X\in\mathfrak{u}^C;\ [U, X] = 2\pi i\alpha(U)X \quad \text{for all } U\in\mathfrak{t}\}\ ,$$

where $\mathfrak{u}^C$ denotes the complexification of $\mathfrak{u}$. If $\mathfrak{u}_\alpha \neq \{0\}$ then α is called *a root of* $\mathfrak{u}$ *with respect to* $\mathfrak{t}$. The set of nonzero roots of $\mathfrak{u}$ with respect to $\mathfrak{t}$ is denoted by Δ. We put

$$\Delta_{\mathfrak{k}} = \{\alpha\in\Delta;\ \mathfrak{u}_\alpha\subset\mathfrak{k}^C\},\ \Delta_{\mathfrak{p}} = \{\alpha\in\Delta;\ \mathfrak{u}_\alpha\subset\mathfrak{p}^C\}\ .$$

Then we easily find (cf. [7])

$$\Delta_{\mathfrak{k}} = \{\pm\omega_1\pm\omega_2,\ \pm 2\omega_r (1\leqq r\leqq n+2),\ \pm\omega_r\pm\omega_s (3\leqq r<s\leqq n+2)\}\ ,$$
$$\Delta_{\mathfrak{p}} = \{\pm\omega_1\pm\omega_r,\ \pm\omega_2\pm\omega_r,\ (3\leqq r\leqq n+2)\}\ .$$

Since for any Cartan subalgebra $\mathfrak{t}'$ of $\mathfrak{h}$ there is an element k_0 of K such that Ad(k_0) mapps $\mathfrak{t}'$ into $\mathfrak{t}$(cf. [5]), we may assume by Lemma 2.2 that $Z_0\in\mathfrak{t}$. For any $\alpha\in\Delta_{\mathfrak{p}}$ and any $X_\alpha\in\mathfrak{u}_\alpha$ we have

$$I(X_\alpha) = [Z_0, X_\alpha] = 2\pi i\alpha(Z_0)X_\alpha\ ,$$

which implies that $2\pi i\alpha(Z_0)$ is an eigenvalue of I. Hence $\alpha(z_0)=\pm 1$ for any $\alpha\in\Delta_{\mathfrak{p}}$, where we put $z_0=2\pi Z_0$. It follows that $z_0=\pm U_1\pm U_2$ or $\pm U_3\pm\cdots\pm U_{n+2}$. Since the Weyl group $W_{\mathfrak{k}}$ of $\mathfrak{k}$ is generated by the reflections of $\Delta_{\mathfrak{k}}$, there is an element w of $W_{\mathfrak{k}}$ such that $w(z_0)=U_1+U_2$ or $U_3+\cdots+U_{n+2}$. Hence we may assume again by Lemma 2.2 that $z_0=U_1+U_2$ or $U_3+\cdots+U_{n+2}$. First let $z_0=U_1+U_2$. A subalgebra $\mathfrak{h}'=\{X\in\mathfrak{k};\ [X, Z_0]=0\}$ of $\mathfrak{k}$ contains $\mathfrak{h}$. By a simple calculation we find

$$\mathfrak{h}'=\left\{\left(\begin{array}{cc|c} ia & z & \\ -\bar{z} & ib & 0 \\ \hline \multicolumn{2}{c|}{0} & * \end{array}\right);\ a, b\in \boldsymbol{R},\ z\in \boldsymbol{C}\right\}.$$

If we put

$$A=\left(\begin{array}{c|c|c} 0 & \begin{matrix} i & 0 \\ 0 & 2i \end{matrix} & 0 \\ \hline \begin{matrix} i & 0 \\ 0 & 2i \end{matrix} & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

then A is a regular element of $\mathfrak{p}$. This can be easily checked from the fact that $A\in\mathfrak{u}_{\omega_1-\omega_3}+\mathfrak{u}_{\omega_2-\omega_4}\subset\mathfrak{p}$. It is be easily calculated that the centralizer $\mathfrak{k}(A)$ of A in $\mathfrak{k}$ is given by

$$\mathfrak{k}(A)=\left\{\left(\begin{array}{c|c|c} \begin{matrix} a & 0 \\ 0 & b \end{matrix} & 0 & 0 \\ \hline 0 & \begin{matrix} -iai & 0 \\ 0 & -ibi \end{matrix} & 0 \\ \hline 0 & 0 & * \end{array}\right);\ a, b\in \boldsymbol{R}i+\boldsymbol{R}j+\boldsymbol{R}k\right\}$$

Therefore the following subspace of $\mathfrak{k}$ is not contained in $\mathfrak{h}'+\mathfrak{k}(A)$

$$\left\{\left(\begin{array}{c|c|c} \begin{matrix} 0 & q \\ -\bar{q} & 0 \end{matrix} & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right);\ q\in \boldsymbol{R}i+\boldsymbol{R}k\right\}.$$

On the other hand, since the tangent space of N at A coincides with the subspace $[\mathfrak{k}, A]=[\mathfrak{h}', A]$ of $\mathfrak{p}$, we see that $\mathfrak{k}=\mathfrak{h}'+\mathfrak{k}(A)$. This is a contradiction. Similarly we have a contradiction also in the case where $z_0=U_3+\cdots+U_{n+2}$. Q.E.D.

Now we are in a position to prove Theorem 2.1.

Proof of Theorem 2.1. Let $\boldsymbol{C}^{n+1}$ be a complx vector $(n+1)$-space with complex structure I' and Hermitian inner product $\langle\ ,\ \rangle'$, and $\pi'\colon \boldsymbol{C}^{n+1}-\{0\}\to$

$P_n(\boldsymbol{C})$ be the canonical projection. Let S' denote the unit hypersphere in $\boldsymbol{C}^{n+1}$ centered at the origin. Then it is evident that the subset $N=\pi'^{-1}(M)\cap S'$ of S' becomes a hypersurface in S' in a natural manner. Moreover N is an orbit under an anlytic subgroup of $U(n+1)$. In fact, if we denote by $\mathfrak{g}$ the Lie algebra of G and by $\mathfrak{z}$ the center of $\mathfrak{u}(n+1)$, then the direct sum $\mathfrak{g}+\mathfrak{z}$ is a subalgebra of $\mathfrak{su}(n+1)+\mathfrak{z}=\mathfrak{u}(n+1)$ and hence of $\mathfrak{o}(2n+2)$. Let $\hat{H}$ be the analytic subgroup of $O(2n+2)$ with Lie algebra $\mathfrak{g}+\mathfrak{z}$. Then N coincides with an orbit under $\hat{H}$, which proves our assertion. On the other hand, W.Y. Hsiang and H.B. Lawson Jr. [4] classified those compact analytic subgroups of $O(m+1)$ up to conjugation which have orbits of codimension one in an m-sphere and are not subgroups of another compact analytic subgroups of $O(m+1)$ with the same orbits. As a result of their classification we know that those groups except for reducible ones coincide exactly with the linear isotropy groups of various irreducible symmetric spaces of rank two. Since $\hat{H}$ includes the center of $U(n+1)$, $\hat{H}$ is reducible as a subgroup of $O(2n+2)$ if and only if $\hat{H}$ is reducible as a subgroup of $U(n+1)$. If $\hat{H}$ is reducible, then it can be easily shown that N is a product of two spheres. Hence $\hat{H}$ is conjugate to a subgroup of a subgroup of the following form of $U(n+1)$ in $O(2n+2)$

$$\begin{pmatrix} U(r) & O \\ O & U(n+1-r) \end{pmatrix}, \quad 1\leqq r\leqq n .$$

In other words, there is an orthogonal symmetric Lie algebra $(\mathfrak{u}, \theta)$ of the first type in the Table and a $\boldsymbol{R}$-linear isomrphism of $\boldsymbol{C}^{n+1}$ onto $\mathfrak{p}$ with sends I', $\langle\ ,\ \rangle'$ and N to I, $\langle\ ,\ \rangle$ and an orbit $N_0=\rho(K)A$ in S, resectively. Thus M is a model space in $P_n(\boldsymbol{C})$ of the first type. If $\hat{H}$ is irreducible, then $\hat{H}$ is compact by a theorem of M. Goto ([2]). Then above theorem of Hsiang and Lawson implies that there is an irreducible effective orthogonal symmetic Lie algebra $(\mathfrak{u}, \theta)$ of compact type and of rank two such that we can identify $\boldsymbol{C}^{n+1}$ with $\mathfrak{p}$ as $\boldsymbol{R}$-linear spaces, $\langle\ ,\ \rangle'$ with $\langle\ ,\ \rangle$ and N with an orbit $N_0=\rho(K)A$ in S under the linear isotropy representation of $(\mathfrak{u}, \theta)$, and such that $\rho(K)$ coincides with the idenity component of the group of all orthogonal transformations of $\mathfrak{p}$ leaving N_0 invariant, in particular, $\rho_*(\mathfrak{k})$ contains I' which can be regarded a complex structure of $\mathfrak{p}$. We put $H=\rho^{-1}(\hat{H})$, which is a compact analytic subgroup of K. Then H and $(\mathfrak{u}, \theta)$ satisfy the condition of Lemma 2.3 and so $(\mathfrak{u}, \theta)$ is Hermtian. Since an irreducible group $\hat{H}$ of $O(2n+2)$ commutes elementwise with both I and I', we have $I=\pm I'$ by Schur's lemma. If $I=-I'$, by taking $-Z_0$ instead of Z_0 we have $I=I'$. Hence we may set $I=I'$. Thus the above identification: $\boldsymbol{C}^{n+1}\equiv\mathfrak{p}$ induces the identification of two complex projective spaces $P_n(\boldsymbol{C})$ under which $M=\pi(N_0)$. Q.E.D.

3. A rigidity theorem

In this section we shall prove a rigidity theorem on real hypersurfaces in a complex projective space $P_n(C)$ to give a characterization of model spaces. Hereafter let M be a connected Riemannian manifold of dimension $2n-1(\geqq 3)$. We denote by $F(M)$ the bundle of orthonormal frames of M. Then $F(M)$ is a principal fibre bundle over M with structure group $O(2n-1)$. An element u of $F(M)$ can be expressed by $u=(p: e_1, \cdots, e_{2n-1})$, where p is a point of M and $e_1, \dots, e_{2n-1}$ is an ordered orthonomal base of the tangent space of M at p. The projection of $F(M)$ onto M is denoted by π. *The canonical forms* $\theta_1, \cdots, \theta_{2n-1}$ of $F(M)$ are the linear diffrential forms on $F(M)$ defined by

$$\pi_* X = \sum_i \theta^i(X) e_i{}^{2)} ,$$

where X is a tangent vector of $F(M)$ at $u=(p: e_1, \cdots, e_{2n-1})$ and π_* is a differential mapping of π. *The connection forms* θ^i_j of $F(M)$ are the linear differential forms on $F(M)$ uniquely determined by the following conditions:

$$\theta^i_j + \theta^j_i = 0 \quad \text{and} \quad d\theta^i + \sum_j \theta^i_j \wedge \theta^j = 0 . \tag{3.1}$$

The curvature forms Θ^i_j of the connection are defined by

$$\Theta^i_j = d\theta^i_j + \sum_k \theta^i_k \wedge \theta^k_j . \tag{3.2}$$

Hereafter let $P_n(C)$ have constant holomorphic sectional curvature $4c$. The bundle of orthonormal frames of $P_n(C)$ is denoted by $F(P)$. If we denote by $\tilde{\theta}^A$, $\tilde{\theta}^A_B$ and Θ^A_B the canonical forms, the connection forms and the curvature forms of $F(P)$ respectively, then Θ^A_B are given by

$$\Theta^A_B = c\tilde{\theta}^A \wedge \tilde{\theta}^B + c \sum_{C,D} (\tilde{J}^A_C \tilde{J}^B_D + \tilde{J}^A_B \tilde{J}^C_D)\, \tilde{\theta}^C \wedge \tilde{\theta}^D, \tag{3.3}$$

where the tensor field $\tilde{J}=(\tilde{J}^A_B)$ on $F(P)$ denotes the complex structure of $P_n(C)$, that is, $\tilde{J}(\tilde{e}_A)=\sum_B \tilde{J}^B_A \tilde{e}_B$ at $(\tilde{p}: \tilde{e}_1, \cdots, \tilde{e}_{2n}) \in F(P)$. Moreover $\tilde{J}$ satisfies

$$\tilde{J}^A_B + \tilde{J}^B_A = 0 , \tag{3.4}$$

$$\sum_C \tilde{J}^A_C \tilde{J}^C_B = -\delta^A_B , \tag{3.5}$$

$$d\tilde{J}^A_B = \sum_C \tilde{J}^A_C \tilde{\theta}^C_B - \sum_C \tilde{J}^C_B \tilde{\theta}^A_C . \tag{3.6}$$

The equation (3.6) means that $\tilde{J}$ is parallel.

An isometry φ of $P_n(C)$ induces a diffeomorphism of $F(P)$ leaving the forms $\tilde{\theta}^A$, $\tilde{\theta}^A_B$ and Θ^A_B invariant in an obvious manner, which is also denoted by the same letter φ.

2) In the following the indices i, j, k, l run from 1 to $2n$–1 and the indices A, B, C, D run from 1 to $2n$.

Let ι be an isometric immersion of M into $P_n(C)$. For an orthonormal frame $u=(p: e_1, \cdots, e_{2n-1})$ of M there exists a unique tangent vector $\tilde{e}_{2n}$ to $P_n(C)$ at $\iota(p)$ such that $\tilde{u}=(\iota(p): \iota_* e_1, \cdots, \iota_* e_{2n-1}, \tilde{e}_{2n})$ is an orthonormal frame of $P_n(C)$ compatible with the orientation of $P_n(C)$ determined by $\tilde{J}$. This mapping $u \to \tilde{u}$ of $F(M)$ into $F(P)$ is also denoted by the same letter ι. Then denoting by ι^* the dual mapping of ι_* we have $\theta^i=\iota^*\tilde{\theta}^i$ and $\iota^*\tilde{\theta}^{2n}=0$, from which we know $\theta^i_j=\iota^*\tilde{\theta}^i_j$ and $0=\iota^* d\tilde{\theta}^{2n}=-\sum_i \iota^*\tilde{\theta}^{2n}_i \wedge \theta^i$. By Cartan's lemma we may write as

$$\phi_i \equiv \iota^*\tilde{\theta}^{2n}_i = \sum_j H_{ij}\theta^j, \qquad H_{ij} = H_{ij}. \tag{3.7}$$

The quadratic form $\sum_i \phi_i \theta^i$ is called *the second fundamental form of* (M, ι). Put $J^i_j=\tilde{J}^i_j \circ \iota$ and $f_i=\tilde{J}^{2n}_i \circ \iota$. The pair (J, f) is called *the almost Grayan structure of* (M, ι). From (3.2), (3.3) and (3.7) we have the equation of Gauss

$$\Theta^i_j = \phi_i \wedge \phi_j + c\theta^i \wedge \theta^j + c\sum_{k,l}(J^i_k J^j_l + J^i_j J^k_l)\theta^k \wedge \theta^l. \tag{3.8}$$

From (3.3) and (3,7) we have the equation of Codazzi

$$d\phi_i + \sum_j \phi_j \wedge \theta^j_i = c\sum_{j,k}(f_j J^i_k + f_i J^j_k)\theta^j \wedge \theta^k. \tag{3.9}$$

Moreover (J, f) satisfies

$$J^i_j + J^j_i = 0, \tag{3.10}$$

$$\sum_k J^i_k J^k_j - f_i f_j = -\delta^i_j, \quad \sum_j J^i_j f_j = 0, \quad \sum_i f_i^2 = 1, \tag{3.11}$$

$$dJ^i_j = \sum_k J^i_k \theta^k_j - \sum_k J^k_j \theta^i_k - f_i \phi_j + f_j \phi_i, \tag{3.12}$$

$$df_i = \sum_j f_j \theta^j_i - \sum_j J^j_i \phi_j.$$

Thus an isometric immersion ι of M into $P_n(C)$ induces three tensor fields $H=(H_i)$ of type (0,2), $J=(J^i_j)$ of type (1,1) and $f=(f_i)$ to type (0,1) on $F(M)$. For another isometric immersion $\hat{\iota}$ of M into $P_n(C)$ we shall denote the differential forms and the tensor fields on $F(M)$ induced by $\hat{\iota}$ by the same symbol but with a roof $\wedge$ overhead.

Lemma 3.1. *Let $\iota, \hat{\iota}$ be two isometric immersions of M into $P_n(C)$. If $H=\hat{H}$, then $J=\hat{J}$ and $f=\hat{f}$, or $J=-\hat{J}$ and $f=-\hat{f}$.*

Proof. Since $\phi_i=\hat{\phi}_i$ and $\Theta^i_j=\hat{\Theta}^i_j$, we have from (3.8) and (3.9)

$$\sum_{k,l}(J^i_k J^j_l + J^i_j J^k_l)\theta^k \wedge \theta^l = \sum_{k,l}(\hat{J}^i_k \hat{J}^j_l + \hat{J}^i_j \hat{J}^k_l)\theta^k \wedge \theta^l, \tag{3.13}$$

$$\sum_{j,k}(f_j J^i_k + f_i J^j_k)\theta^j \wedge \theta^k = \sum_{j,k}(\hat{f}_j \hat{J}^i_k + \hat{f}_i \hat{J}^j_k)\theta^j \wedge \theta^k. \tag{3.14}$$

Compare the coefficients of $\theta^i \wedge \theta^j$ in (3.13) to get

$$(J^i_j)^2 = (\hat{J}^i_j)^2.$$

Here we define a subbundle F' of $F(P)$ by

$$F' = \{(\tilde{p}: \tilde{e}_1, \cdots, \tilde{e}_{2n-1}, \tilde{e}_{2n}) \in F(P); \tilde{J}\tilde{e}_{2n-1} = \tilde{e}_{2n}\}$$

and restrict the forms the tensor fields under consideration to the subbundle $\hat{\iota}^{-1} F'$ of $F(M)$. Then $\hat{J}^i_{2n-1}=0$ for all i and $\hat{f}_{2n-1}=1$, so $\hat{f}_i=0$ for $1 \leqq i \leqq 2n-2$. Hence $J^i_{2n-1}=0$ for all i and so $f_{2n-1}=\pm 1$ by (3.11). Thus $f_i=0$ for $1 \leqq i \leqq 2n-2$. Put $i=2n-1$ in (3.14) to get

$$J^j_k = f_{2n-1}\hat{J}^j_k \quad \text{for} \quad 1 \leqq i,\ k \leqq 2n-2\,.$$

Since $f_{2n-1}=\pm 1$, we showed that Lemma 3.1 holds on F' and hence on $F(M)$. Q.E.D.

Theorem 3.2. *Let ι, $\hat{\iota}$ be two isometric immersions of M into $P_n(\boldsymbol{C})$. If $H=\hat{H}$, then ι, $\hat{\iota}$ are rigid, that is, there is an isometry φ of $P_n(\boldsymbol{C})$ such that $\varphi \circ \iota = \hat{\iota}$.*

Proof. By Lemma 3.1 we have $J=\hat{J}$ and $f=\hat{f}$, or $J=-\hat{J}$ and $f=-\hat{f}$. First assume that $J=\hat{J}$ and $f=\hat{f}$. This implies that if u is an element of $F(M)$ such that $\iota(u)$ is a unitary frame of $P_n(\boldsymbol{C})$ then $\hat{\iota}(u)$ is also a unitary frame of $P_n(\boldsymbol{C})$. Then there exists a unique element φ of $PU(n+1)$ such that $\varphi(\iota(u))=\hat{\iota}(u)$. Making use of the same method as one of proving a rigidity theorem of hypersurfaces in a real space form, it can be proved that the mapping $u \rightarrow \varphi$ of $F(M)$ into $PU(n+1)$ is constant (cf. [6], [10]). Next assume that $J=-\hat{J}$ and $f=-f$. This implies that $n+1$ is even since for each $u \in F(M)$ the frames $\iota(u)$ and $\hat{\iota}(u)$ of $P_n(\boldsymbol{C})$ determine the same orientation of $P_n(\boldsymbol{C})$. Hence the isometry τ of $P_n(\boldsymbol{C})$ induced from the conjugation of $\boldsymbol{C}^{n+1}$ preserves the orientation of $P_n(\boldsymbol{C})$. It follows that the almost Grayan structure $(\hat{\hat{J}}, \hat{\hat{f}})$ induced by an isometric immersion $\hat{\hat{\iota}}=\tau \circ \iota$ of M into $P_n(\boldsymbol{C})$ is equal to $(-J, -f)$. Since the second fundamental form of (M, ι) coincides with $\sum_{i,j} \hat{H}_{ij}\theta^i\theta^j$, the previous argument shows that there is an element σ of $PU(n+1)$ such that $\sigma \circ \hat{\hat{\iota}} = \hat{\iota} = \sigma \circ \tau \circ \iota$ Q.E.D.

Theorem 3.3 *Let ι be an isometric immersion of M into $P_n(\boldsymbol{C})$. If a group G of isometries of M leaving H invariant acts on M transitively, then $\iota(M)$ is congruent to a model space, that is, there are an isometry φ of $P_n(\boldsymbol{C})$ and a model space M_0 such that $\iota(M)=\varphi(M_0)$.*

Proof. It follows from Theorem 3.2 that for each $g \in G$ there exists a unique element σ_g of $PU(n+1)$ such that $\sigma_g \circ \iota = \iota \circ g$ or $\sigma_g \circ \iota = \tau \circ \iota \circ g$. Hence M is congruent to an orbit under the identity component of a subgroup $\{\sigma_g \in PU(n+1); g \in G\}$ of $PU(n+1)$. Thus Theoem 3.3 was reduced to Theorem 2.1. Q.E.D.

4. The type number of hypersurfaces

In this section we shall consider the problem of the converse of Lemma 3.1

and fix the notation in §3. If ι, $\hat{\iota}$ are two isometric immersions of M into $P_n(C)$, then we have from (3.8)

$$\phi_i \wedge \phi_j = \hat{\phi}_i \wedge \hat{\phi}_j \ \text{ if } \ J = \pm \hat{J}.$$

Then by a theorem of E. Cartan [1] we know that $\phi_i = \pm \hat{\phi}_i$ at the points where the rank of the second fundamenal form of (M, ι) (which is called the type number of (M, ι)) or of $(M, \hat{\iota})$ is not less than 2. So we shall study the type number t of (M, ι). For a nonemtpy open set U of $F(M)$, let m be the maximal value of t on U. Then t takes the constant m on an open subset U_0 of U, or equivalently the number of linearly independent ones of $\phi_1, \cdots, \phi_{2n-1}$ is equal to m on U_0. In a while restrict the forms and the tensor fields under consideration on the following subbundle F_0 of U_0

$$F_0 = \{u \in U_0;\ \phi_a = \sum_b H_{ab}\theta^b,\ \phi_r = 0 \text{ at } u\}^{3)}.$$

Lemma 4.1 *If $m < n-1$, then $f_r = 0$ for all r.*

Proof. Put $i=r$ in (3.9) and compare the coefficients of $\theta^s \wedge \theta^t$ using $\phi_r = 0$ to get

$$f_t J^r_s - f_s J^r_t - 2 f_r J^s_t = 0. \tag{4.1}$$

Put $t=r$ in (4.1) to get $f_r J^s_r = 0$. Therefore multiplying (4.1) by f_r we get $f_r J^s_t = 0$. If $f_r \neq 0$ for some r, then $J^s_t = 0$ for all s,t, which contradicts the fact that the rank of J is equal to $2n-2$ and $m < n-1$. Q.E.D.

By Lemma 4.1 we have $m \geqq 1$ and may assume that $f_1 = 1$ and $f_i = 0$ for $2 \leqq i \leqq m$. Then from (3.12) we have

$$\theta^1_a = \sum_a J^b_a \phi_b,\quad \theta^1_r = \sum_b J^a_r \phi_a. \tag{4.2}$$

Put $i=r$ in (3.9) to get

$$\sum_a \phi_a \wedge \theta^a_r = c \sum_i J^r_i \theta^1 \wedge \theta^i. \tag{4.3}$$

Now assume that $m=1$. Then $\theta^1_r = \sum_a J^a_r \phi_a = J^1_r \phi_1 = 0$ since $0 = f_1 J^1_r = J^1_r$ by (3.11). Hence $J^r_s = 0$ for all r, s since $0 = \phi_1 \wedge \theta^1_r = c \sum_s J^r_s \theta^1 \wedge \theta^s$ by (4.2) and (4.3). If $n \geqq 3$, this contradicts the fact that the ran of J is equal to $2n-2 (>2)$. Thus we proved

Theorem 4.2. *Let ι be an isometric immersion of M into $P_n(C)$ ($n \geqq 3$). Then in any nonempty open set of $F(M)$ there exists a point u where $t(u) \geqq 2$.*

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3) In the following the indices a, b, c run from 1 to m and the indices r,s,t run from $m+1$ to $2n-1$.

From Theorem 4.2 we have the following theorem

Theorem 4.3. *Let ι, $\hat{\iota}$ be two isometric immersions of M into $P_n(C)$ $(n\geqq 3)$ such that $J=\hat{J}$ and $f=\hat{f}$, or $J=\hat{J}$ and $f=-\hat{f}$. If the type number of (M, ι) or of $(M, \hat{\iota})$ is not equal to 2 at any point of $F(M)$, then ι, $\hat{\iota}$ are rigid.*

Proof. Let u be any point of $F(M)$. Then by Theorem 4.2 any neighbourhood of u contains a point v where $t(v)\geqq 3$. Hence $H=\pm\hat{H}$ at v. Since we have a sequence $\{u_w\}$ of points of $F(M)$ such that u_w tends to u as $w\to\infty$ and $H=\pm\hat{H}$ at u_w, we have $H=\pm\hat{H}$ at u. We define two closed subsets F_+ and F_- of $F(M)$ by

$$F_+ = \{u\in F(M);\, H=\hat{H} \text{ at } u\}\ ,$$
$$F_- = \{u\in F(M);\, H=-\hat{H} \text{ at } u\}\ .$$

Then $F(M)=F_+\cup F_-$. Moreover F_- can not contain any nonempty open set of $F(M)$. In fact, suppose that U' is a nonempty open set of $F(M)$ contained in F_-. Then we have on U'

$$d\phi_i+\sum_j \phi_j\wedge\theta^j_i = -(d\hat{\phi}_i+\sum_j\hat{\phi}_j\wedge\theta^j_i)\ .$$

On the other hand, from the assumption we have

$$\sum_{j,k}(f_jJ^i_k+f_iJ^j_k)\theta^j\wedge\theta^k = \sum_{j,k}(\hat{f}_j\hat{J}^i_k+\hat{f}_i\hat{J}^j_k)\theta^j\wedge\theta^k\ .$$

Theses equations and (3.9) imply

$$\sum_{j,k}(f_jJ^i_k+f_iJ^j_k)\theta^j\wedge\theta^k = 0 \quad \text{for all } i\,,$$

from which we have a contradiction $f_kJ^i_j=0$ as in the proof of Lemma 4.1. Thus we showed that the boundary of F_+ contains F_-, that is, $F_+=F(M)$ since F_+ is closed. Now Theorem 4.3 was reduced to Theorem 3.2. Q.E.D.

Corollary 4.4. *Let ι be an isometric immersion of M into $P_n(C)$ $(n\geqq 3)$. Assume that the type number of (M, ι) is not equal to 2 at any point of M. If a group of isometries of M leaving the almost Grayan structure (J, f) of (M, ι) invariant acts on M transitively, then M is congruent to a model space.*

The proof is similar to that of Theorem 3.3.

REMARK. Theorems 3.2, 4.2 and 4.3 are valid for a complex space form of negative constant holomorphic sectional curvature instead of $P_n(C)$.

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