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CONTRIBUTIONS TO THE THEORY OF INTERPOLATION OF OPERATIONS II

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1. Introduction

In this paper we shall discuss the interpolation of operations on intermediate spaces. Our method is the so-called real method. Our purpose is to treat the critical case which appears in singular integral operators. If we consider for example the Hilbert transform $\tilde{f}$ of function f of the class $L\log^+ L(\pm\infty,\infty)$, $\tilde{f}$ exist a.e. but the only local integrability holds. Then we shall discuss their integral estimation on the whole space.

The intermediate space between two Banach spaces was introduced by W.A.J. Luxemburg [6, 7]. This is defined as follows. Given a topological vector space V and two Banach spaces A_1 and A_2 which are contained and continuously embedded in V. If f is an element of A_i $(i=1, 2)$, we denote its norm by $||f||[A_i]$ $(i=1, 2)$. We shall consider the space A_1+A_2 and introduce in it the norm

$$||f||[A_1+A_2] = inf\,(||g||[A_1]+||h||[A_2])$$

where the infimum is taken over all pairs $g\in A_1$ and $h\in A_2$ such that $f=g+h$, then A_1+A_2 also becomes a Banach space. Since A_1 and A_2 are continuously embedded in V, it is evident that A_1+A_2 is also continuously embedded in V.

In what follows we shall consider totally σ-finite measure space (R, μ) and the space V of equivalent classes of real valued measurable functions on R. The equivalent relation here is that of coincidence almost everywhere. If in V we introduce a topology of convergence in measure on sets of finite measure, V becomes a topological vector space. If we take as the interpolation pair $A_1=L^1_\mu$, $A_2=L^p_\mu(1<p<\infty)$ then these are continuously embedded in V. We shall also consider another measure space (S, ν).

Let us consider operation T which transforms measurable functions on R to those on R. The operation T is called quasi-linear if

(i) $T(f_1+f_2)$ is uniquely defined whenever Tf_1 and Tf_2 are defined and

$$|T(f_1+f_2)| \leqq \kappa(|Tf_1|+|Tf_2|)$$

where κ is a constant independent of f_1 and f_2,

(ii) $T(cf)$ is uniquely defined whenever Tf is defined and

$$|T(cf)| = |c| \ |Tf|$$

for all scalars c.

We say that the operation T is of type (a, b), $1 \leqq a \leqq b \leqq \infty$, if Tf is defined for each $f \in L^a_\mu(R)$ and belongs to $L^b_\mu(S)$ such that

$$(1.1) \qquad ||Tf||[L^b_\nu] \leqq M||f||[L^a_\mu]$$

where M is a constant independent of f. The least admissible value of M in (1.1) is called the (a, b)-norm of operation T. Next we shall define the weak type (a, b) of operations. Suppose first that $1 \leqq b < \infty$. Given any $r > 0$, denote by $E_r = E_r[Tf]$ the set of points of the space S where $|Tf| > r$, and write $\nu(E_r)$ for the ν-measure of the set E_r. We say that the operation T is of weak type (a, b) if

$$(1.2) \qquad \nu(E_r[Tf]) \leqq \left(\frac{M}{r}||f||[L^a_\mu]\right)^b$$

where M is a constant independent of f. The least admissible value of M in (1.2) is called the weak (a, b)-norm of operation T. It is clear that being of type (a, b) implies being of weak type (a, b). We shall define weak type (a, ∞) as identical with type (a, ∞). Hence T is of weak type (a, ∞) if

$$(1.3) \qquad ess.\, sup||Tf|| \leqq M||f|| \ [L^a_\mu]$$

Beside the space L^a_μ we shall need the space $L^a_\mu(\log^+ L_\mu)^c$ which consist of functions to be μ-measurable and such that

$$||f||[L^a_\mu(\log^+ L_\mu)^c] = \left(\int_R |f|^a(1+(\log^+|f|)^c\, d\mu\right)^{1/a} < \infty \, .$$

This is not a Banach space but we shall use conveniently the same notations $L^{a_1}_\mu(\log^+ L_\mu)^{a_1/b_1} + L^{a_2}_\mu$ as in the preceding case, which consist of functions to be μ-measurable and such that

$$||f||[L^{a_1}_\mu(\log^+ L_\mu)^{a_1/b_1} + L^{a_2}_\mu] = inf(||h||[L^{a_1}_\mu(\log^+ L_\mu)^{a_1/b_1}] + ||g||[L^{a_1}_\mu]) < \infty$$

where the infimum is taken over all pairs $h \in L^{a_1}_\mu(log^+ L_\mu)^{a_1/b_1}$ and $g \in L^{a_2}_\mu$ such that $f = g + h$

In our preceding paper [4] we proved that

Theorem A. *Suppose that a quasi-linear operation T is of weak type* $(1, 1)$ *and of type* (p, p) *for some* $1 < p < \infty$. *Then we have*

$$\int_{|Tf|\leqq 1} |Tf|^p d\nu + \int_{|Tf|>1} |Tf| d\nu$$
$$\leqq K\Big(\int_{|f|\leqq 1} |f|^p d\mu + \int_{|f|>1} |f| (1+\log^+|f|) d\mu\Big)$$

where K is a constant independent of f.

We can grow it up to the following form

Theorem 1. *Under the hypotheses of theorem A, we have $Tf \in L_\nu + L_\nu^p$ for any $f \in L_\mu \log^+ L_\mu + L_\mu^p$ and*

$$||Tf|| \leqq KM^p ||f|| \{1+\log^+ ||f||^{-1}\}$$

where K is a constant depending on p, κ but not on M_1, M_2 and f; $M=$ $\max(M_1, M_2, 1)$

Furthermore we shall prove the following theorem which gives an answer to the conjecture of [4: p. 148～9].

Theorem 2. *Let us write $\alpha_i = 1/a_i$, $\beta_i = 1/b_i$ $(i=1, 2)$. Let (α_1, β_1) and (α_2, β_2) be any two point of the triangle*

$$\Delta;\ 0<\beta\leqq\alpha\leqq 1$$

such that $\beta_1 \neq \beta_2$. Let us suppose that a quasi-linear operation $\tilde{f} = Tf$ is of weak type $(1/\alpha_1, 1/\beta_1)$ and of type $(1/\alpha_2, 1/\beta_2)$ with norms M_1 and M_2 respectively. Then if $\alpha_1 > \alpha_2$ and $\beta_1 > \beta_2$ we have $Tf \in L_\nu^{b_1} + L_\nu^{b_2}$ for any $f \in L_\mu^{a_1}(\log^+ L_\mu)^{a_1/b_1} + L_\mu^{a_2}$ and

$$||Tf|| \leqq KM^{b_2/b_1} ||f|| (1+(\log^+ ||f||^{-1})^{a_1/b_1})^{1/a_1}$$

where K is a constant depending on α_1, α_2, β_1, β_2, κ and not on M_1, M_2 and f; M $=\max(M_1, M_2\ 1)$.

Corollary 1. *Let us write*

$$\begin{cases} a_\theta = (1-\theta)a_1 + \theta a_2 \\ b_\theta = (1-\theta)b_1 + \theta b_2 \quad (0<\theta\leqq 1)\,. \end{cases}$$

Then under the hypotheses of theorem 2, we have $Tf \in L_\nu^{b_1} + L_\nu^{b_\theta}$ for any $f \in L_\mu^{a_1}$ $(\log^+ L_\mu)^{a_1/b_1} + L_\mu^{a_\theta}$ and

$$||Tf|| \leqq K_\theta M^{b_2/b_1} ||f|| \{1+(\log^+ ||f||^{-1})^{a_1/b_1}\}^{1/a_1}\,.$$

where K_θ is a constant depending on θ, α_1, α_2, β_1, β_2, but not on M_1, M_2, f.

Since by the Marcinkiewicz-Zygmund theorem it follows that the operation

T is of type (a_θ, b_θ) by the hypotheses of theorem 2 and so we can derive corollary 1 immediately.

Next T. Kawata [3] proved the following theorem

Theorem B. *Let $f(x)$ be a measurable function of the class $L(-\infty, \infty)$. Then its Hillbert transform*

$$\tilde{f}(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy$$

exists and satisfies the following inequality

$$\int_{-\infty}^{\infty} \frac{|\tilde{f}(x)|}{1+|\log|\tilde{f}(x)||^{1+\varepsilon}} dx \leqq M\int_{-\infty}^{\infty} |f(x)|\, dx$$

where ε is any positive number and M is a constant independent of f.

Furthermore the $\varepsilon>0$ can not be omitted as a counter example shows.

Here we introduce the space $L_\nu^{b_1}/(\log^+ L_\nu)^{1+\varepsilon}$ which consist of functions such that

$$||f||[L_\nu^{b_1}/(\log^+ L_\nu)^{1+\varepsilon}] = \left(\int_S \frac{|f(x)|^{b_1}}{1+(\log^+|f(x)|)^{1+\varepsilon}} d\nu\right)^{1/b_1} < \infty$$

and also the space $L_\nu^{b_1}/(\log^+ L_\nu)^{1+\varepsilon} + L_\nu^{b_2}$ which consist of functions

$$||f||[L_\nu^{b_1}/(\log^+ L_\nu)^{1+\varepsilon} + L_\nu^{b_2}] = \inf\{||h||[L_\nu^{b_1}/(\log^+ L_\nu)^{1+\varepsilon}] + ||g||[L_\nu^{b_2}]\} < \infty$$

where the infimum is taken over all pairs $h \in L_\nu^{b_1}/(\log^+ L_\nu)^{1+\varepsilon}$ and $g \in L_\nu^{b_2}$ such that $f=g+h$.

Then if we discuss his result from the stand point of views of the theory of interpolation of operation, we shall prove

Theorem 3. *Under the hypotheses of theorem 2, we have $Tf \in L_\nu^{b_1}/(\log^+ L_\nu)^{1+\varepsilon} + L_\nu^{b_2}$ for any $f \in L_\mu^{a_1} + L_\mu^{a_2}$ and*

$$||Tf|| \leqq KM^{b_2/b_1}||f||\{1+(\log^+||f||^{-1})^{1+\varepsilon/b_1}\}$$

where ε is any positive real number, K is a constant independent of f and $M=\max(M_1, M_2, 1)$.

Corollary 2. *Under the hypotheses of theorem 3, we have $Tf \in L_\nu^{b_1}/(\log^+ L_\nu)^{1+\varepsilon} + L_\nu^{b_\theta}$ for any $f \in L_\mu^{a_1} + L_\mu^{a_\theta}$ and*

$$||Tf|| \leqq K_\theta M^{b_2/b_1}||f||\{1+(\log^+||f||^{-1})^{1+\varepsilon/b_1}\}.$$

Furthermore we shall study the case $\theta=0$. Let us denote by $L_\nu^{b_1}/(\log^+ L_\nu^{-1})^{1+\varepsilon}$ the set of functions such that f is ν-measurable and

$$||f||[L_\nu^{b_1}/(\log^+ L_\nu^{-1})^{1+\varepsilon}]=\Big(\int_S \frac{|f|^{b_1}}{1+(\log^+|f|^{-1})^{1+\varepsilon}}d\nu\Big)^{1/b_1}<\infty .$$

Then we have

Theorem 4. *Let us suppose that the operation T is of weak type (a_1, b_1), $1\leqq a_1, b_1<\infty$ then we have $Tf\in L_\nu^{b_1}/(\log^+ L_\nu)^{1+\varepsilon}+L_\nu^{b_1}/(\log^+ L_\nu^{-1})^{1+\varepsilon}$ for any $f\in L_\mu^{a_1}$ and*

$$||Tf||[L_\nu^{b_1}/(\log^+ L_\nu)^{1+\varepsilon}+L_\nu^{b_1}/(\log^+ L_\nu^{-1})^{1+\varepsilon}]\leqq KM_1||f||[L_\mu^{a_1}]$$

where K is a constant independent of f and ε is any positive number.

The case $a_1=b_1=1$ and $\theta=0$ corresponds to the theorem of T. Kawata. Finally we shall prove another type of theorems which correspond to the theorem of N. Kolmogoroff [5] about conjugatae functions and the theorem of A.P. Calderon-A. Zygmund [1] about singular integral operators.

Theorem 5. *Under the hypothese of theorem 2, we have $Tf\in L_\nu^{b_1-\varepsilon}+L_\nu^{b_2}$ for any $f\in L_\mu^{a_1}+L_\mu^{a_2}$ and*

$$||Tf||[L_\nu^{b_1-\varepsilon}+L_\mu^{b_2}]\leqq KM^{b_2/b_1-\varepsilon}||f||[L_\mu^{a_1}+L_\mu^{a_2}]$$

where ε is any positive number, K is a constant independent of f and $M=\max(M_1, M_2, 1)$.

Corollary 3. *Under the hypothese of theorem 5, we have $Tf\in L_\nu^{b_1-\varepsilon}+L_\nu^{b_\theta}$ for any $f\in L_\mu^{a_1}+L_\mu^{a_\theta}$ and*

$$||Tf||[L_\nu^{b_1-\varepsilon}+L_\nu^{b_\theta}]\leqq K_\theta M^{b_2/b_1-\varepsilon}||f||[L_\mu^{a_1}+L_\mu^{a_\theta}] \quad (0<\theta\leqq 1)$$

Corresponding to the case $\theta=0$, we have

Theorem 6. *Under the hypothese of theorem 4, we have $Tf\in L_\nu^{b_1-\varepsilon}+L_\nu^{b_1+\varepsilon}$ for any $f\in L_\mu^{a_1}$ and*

$$||Tf||[L_\nu^{b_1-\varepsilon}+L_\nu^{b_1+\varepsilon}]\leqq KM_1^{b_1/b_1-\varepsilon}||f||[L_\mu^{a_1}]$$

where K is a constant independent of f.

Furthermore we shall prove

Theorem 7. *Under the hypotheses of theorem 2 except that $\alpha_1>\alpha_2$ and $\beta_1<\beta_2$, we have $Tf\in L_\nu^{b_1+\varepsilon}+L_\nu^{b_2}$ for any $f\in L_\mu^{a_1}+L_\mu^{a_2}$ and*

$$||Tf||[L_\nu^{b_1+\varepsilon}+L_\nu^{b_2}]\leqq KM||f||[L_\mu^{a_1}+L_\mu^{a_2}]$$

where ε is any positive number, K is a constant independent of f and $M=\max(M_1, M_2, 1)$.

Corollary 4. *Let us write*

$$\begin{cases} a_\theta = (1-\theta)a_1+\theta a_2 \\ b_\theta = (1-\theta)b_1+\theta b_2 \quad (0<\theta\leqq 1) \end{cases}$$

Then under the hypotheses of theorem 7, we have $Tf \in L_{\nu}^{b_1+\varepsilon}+L_{\nu}^{b_\theta}$ *for any* $f\in L_{\mu}^{a_1}+L_{\mu}^{a_\theta}$ *and*

$$||Tf||[L_{\nu}^{b_1+\varepsilon}+L_{\nu}^{b_\theta}]\leqq K_\theta M||f||[L_{\mu}^{a_1}+L_{\mu}^{a_\theta}].$$

As an application we shall consider some singular integral operators. One of them is that of the Hilbert-Calderon-Zygmund which is defined as follows.

$$\tilde{f}(x) = P.V.\int_{R^n} K(x-y)f(y)\,dy$$

where the kernel $K(x)$ has the form

$$K(x) = |x|^{-n}\Omega(x'),\ x' = x/|x|\ .$$

Let us denote by Σ the unit sphere on which the $\Omega(x')$ is defined. Let us also denote by $\omega(\delta)$ the modulus of continuity of $\Omega(x')$,

$$|\Omega(x')-\Omega(y')|\leqq\omega(x'-y')\,.$$

Let us suppose that

(i) $\int_\Sigma \Omega(x')\,dx = 0$

(ii) $\Omega(x')\in L(\Sigma)$ and its modulus of continuity $\omega(\delta)$ satisfy the Dini condition:

$$\int_0^1 \frac{\omega(\delta)}{\delta}\,d\delta<\infty\ .$$

Then they proved that the operation $Tf=\tilde{f}$ is linear and type (p, p) for every $p>1$, weak type $(1, 1)$ respectively.

Another one is that of Hardy-Littlewood-Sobolev and they considered the singular integral operator of potential type

$$\tilde{f}_\lambda(x) = \int_{R^n} \frac{f(y)}{|x-y|^{n\lambda}}\,dy, \quad (0<\lambda<1)\,.$$

If we write $1<r<s<\infty$, $1/r-1/s=1-\lambda$, then it is proved that this operator is of type (r, s) in the one-dimensional case by G.H. Hardy-J.E. Littlewood [2] in the n-dimensional case by S.L. Sobolev [8] respectively. A. Zygmund [10] also proved that it is of werk type $(1, 1/\lambda)$ in the n-dimensional case.

From now on, if no confusion arises we shall omit the symbols of spaces and measures μ and ν for norms. The author thanks to Prof. H. Tanabe for his sincere advices as a referee.

2. Proof of Theorem 2. We shall begin to prove auxiliary result.

Proposition 1. *Under the hypotheses of theorem* 2, *if* $h\in L_{\mu}^{a_1}(\log^+ L_\mu)^{a_1/b_1}$ *we have*

$$(1)\quad \Big(\int_{|Th|\leqq 1} |Th|^{b_2}d\nu\Big)^{1/b_2}\leqq KM^{b_1/b_2}(||h||[L_{\mu}^{a_1}])^{b_1/b_2}$$

and

$$(2)\quad \Big(\int_{|Th|>1} |Th|^{b_1}d\nu\Big)^{1/b_1}\leqq KM^{b_2/b_1}(||h||[L_{\mu}^{a_1}(\log^+ L_\mu)^{a_1/b_1}])^{\gamma}$$

where K is a constant independent of M_1, M_2 and f, $M=\max(M_1, M_2, 1)$ and γ equals to min $(1, a_1b_2/a_2b_1)$ or max $(1, a_1b_2/a_2b_1)$ according to $||h||\leqq 1$ or $||h||>1$ respectively.

Proof of Proposition 1. Let us denote by $n(y)$ the distribution function of Th, then we have

$$\begin{aligned}\int_{|Th|\leqq 1} |Th|^{b_2}d\nu &= -\int_0^1 y^{b_2}dn(y)\\ &= -y^{b_2}n(y)\Big|_{y=0}^{y=1}+b_2\int_0^1 y^{b_2-1}n(y)\,dy\\ &\leqq b_2\int_0^1 y^{b_2-1}\Big(\frac{M_1}{y}||h||[L_{\mu}^{a_1}]\Big)^{b_1}dy = \frac{b_2}{b_2-b_1}M_1^{b_1}(||h||[L_{\mu}^{a_1}])^{b_1}.\end{aligned}$$

Next we have

$$\begin{aligned}\int_{|Th|>1} |Th|^{b_1}d\nu &= -\int_1^\infty y^{b_1}dn(y)\\ &= -y^{b_1}n(y)\Big|_{y=1}^{y=\infty}+b_1\int_1^\infty y^{b_1-1}n(y)\,dy.\end{aligned}$$

and

$$n(1)\leqq M_1^{b_1}(||h||[L_{\mu}^{a_1}])^{b_1}.$$

Now we shall run on lines of A. Zygmund[10]. Let us decompose h into h_1+h_2 such that

$$h_2=\begin{cases} h, & \text{if}\quad |h|\leqq z\\ e^{i\arg h}z, & \text{if}\quad |h|>z\end{cases}\qquad h_1=h-h_2.$$

Here z is a positive number greater than 1 and will be determined later. Since $h_2\in L_{\mu}^{a_2}$ and $h_1\in L_{\mu}^{a_1}(\log^+ L_\mu)^{a_1/b_1}\subset L_{\mu}^{a_1}$, if we denote by $n_i(y)$ $(i=1, 2)$ the distribution function of Th_i $(i=1, 2)$, then we have

$$n(y) \leqq n_1(y/2\kappa)+n_2(y/2\kappa)$$
$$\leqq M_1{}^{b_1}(y/2\kappa)^{-b_1}(||h_1||[L^{a_1}_{\mu}])^{b_1}+M_2{}^{b_2}(y/2\kappa)^{-b_2}(||h_3||[L^{a_2}_{\mu}])^{b_2}$$

therefore

$$b_1\int_1^\infty y^{b_1-1}n(y)\,dy$$
$$\leqq b_1(2\kappa)^{b_1}M_1{}^{b_1}\int_1^\infty y^{-1}||h_1||^{b_1}dy+b_1(2\kappa)^{b_2}M_2{}^{b_2}\int_1^\infty y^{b_1-b_2-1}||h_2||^{b_2}dy$$
$$= b_1(2\kappa)^{b_1}M_1{}^{b_1}I_1+b_1(2\kappa)^{b_2}M_2{}^{b_2}I_2,\ \text{say}.$$

Let us also denote by $m(y)$ and $m_i(y)$ the distribution function of h and h_i $(i=1, 2)$ respectively. Since $m_2(t)=m(t)$ if $0<t\leqq z$, $=m(z)$ if $t>z$, we have

$$I_2=\int_1^\infty y^{b_1-b_2-1}\Big(-\int_0^\infty t^{a_2}dm_2(t)\Big)^{k_2}dy,\ k_2=b_2/a_2$$
$$=\int_1^\infty y^{b_1-b_2-1}\Big(-t^{a_2}m(t)\Big|_{t=0}^{t=z}+a_2\int_0^z t^{a_2-1}m(t)\,dt\Big)^{k_2}dy$$
$$\leqq\int_1^\infty y^{b_1-b_2-1}\Big(a_2\int_0^z t^{a_2-1}m(t)\,dt\Big)^{k_2}dy$$

and

$$I_2^{1/k_2}\leqq\sup_{\chi}a_2\int_1^\infty y^{b_1-b_2-1}\Big(\int_0^z t^{a_2-1}m(t)\,dt\Big)\chi(y)dy$$

for $\int_1^\infty y^{b_1-b_2-1}\chi^{k_2'}(y)\,dy\leqq 1$, where $1/k_2+1/k_2'=1$. If we put $z=y^\xi$, $\xi>0$, we get

$$a_2\int_1^\infty y^{b_1-b_2-1}\Big(\int_0^z t^{a_2-1}m(t)\,dt\Big)\chi(y)\,dy$$
$$=a_2\int_1^\infty t^{a_2-1}m(t)\,dt\int_{t^{1/\xi}}^\infty y^{b_1-b_2-1}\chi(y)\,dy$$
$$+a_2\int_0^1 t^{a_2-1}m(t)\,dt\int_1^\infty y^{b_1-b_2-1}\chi(y)dy.$$

If $t\geqq 1$, we have

$$\int_{t^{1/\xi}}^\infty y^{b_1-b_2-1}\chi(y)\,dy\leqq\Big(\int_{t^{1/\xi}}^\infty y^{b_1-b_2-1}\,dy\Big)^{1/k_2}\Big(\int_{t^{1/\xi}}^\infty y^{b_1-b_2-1}\chi^{k_2'}(y)\,dy\Big)^{1/k_2'}$$
$$\leqq(b_2-b_1)^{-1/k_2}t^{-(b_2-b_1)/\xi k_2}.$$

If we write $a_2-1-(b_2-b_1)/\xi k_2=a_1-1$ and solve with respect to ξ,

$$\xi=\frac{b_2-b_1}{k_2(a_2-a_1)}=\frac{a_2(b_2-b_1)}{b_2(a_2-a_1)}>0$$

thus we get

$$a_2\int_1^\infty y^{b_1-b_2-1}\Big(\int_0^z t^{a_2-1}m(t)\,dt\Big)\chi(y)\,dy$$
$$\leqq a_2(b_2-b_1)^{-1/k_2}\int_0^\infty t^{a_1-1}m(t)\,dt=\frac{a_2}{a_1}(b_2-b_1)^{-1/k_2}(\,||h||[L_\mu^{a_1}])^{a_1}$$

and so

$$I_2\leqq\Big(\frac{a_2}{a_1}\Big)^{k_2}(b_2-b_1)^{-1}(\,||h||[L_\mu^{a_1}])^{a_1k_2}$$

Next, since $m_1(t)=m(t+z)$ for all $t\geqq0$, we have

$$I_1=\int_1^\infty y^{-1}\Big(-\int_0^\infty t^{a_1}dm_1(t)\Big)^{k_1}dy,\ k_1=b_1/a_1$$
$$=\int_1^\infty y^{-1}\Big(-t^{a_1}m_1(t)\Big|_{t=0}^{t=\infty}+a_1\int_0^\infty t^{a_1-1}m_1(t)\,dt\Big)^{k_1}dy$$
$$=\int_1^\infty y^{-1}\Big(a_1\int_z^\infty (t-z)^{a_1-1}m(t)\,dt\Big)^{k_1}dy$$
$$\leqq\int_1^\infty y^{-1}\Big(a_1\int_z^\infty t^{a_1-1}m(t)\,dt\Big)^{k_1}dy$$

and

$$I_1^{1/k_1}\leqq\sup_\omega a_1\int_1^\infty y^{-1}\Big(\int_z^\infty t^{a_1-1}m(t)\,dt\Big)\omega(y)\,dy$$

for $\int_1^\infty y^{-1}\omega^{k_1'}(y)\,dy\leqq1$, where $1/k_1+1/k_1'=1$. Since $z=y^\xi$, $\xi>0$ we get

$$a_1\int_1^\infty y^{-1}\Big(\int_z^\infty t^{a_1-1}m(t)\,dt\Big)\omega(y)dy=a_1\int_1^\infty t^{a_1-1}m(t)\,dt\int_1^{t^{1/\xi}}y^{-1}\omega(y)\,dy$$
$$\leqq a_1\int_1^\infty t^{a_1-1}m(t)\,dt\Big(\int_1^{t^{1/\xi}}y^{-1}dy\Big)^{1/k_1}\Big(\int_1^{t^{1/\xi}}y^{-1}\omega^{k_1'}(y)\,dy\Big)^{1/k_1'}$$
$$=a_1\int_1^\infty t^{a_1-1}(\log t^{1/\xi})^{1/k_1}m(t)\,dt=a_1\xi^{-1/k_1}\int_1^\infty t^{a_1-1}(\log t)^{1/k_1}m(t)\,dt$$

and since

$$a_1t^{a_1-1}(\log t)^{1/k_1}=(t^{a_1}(\log t)^{1/k_1})'-(1/k_1)t^{a_1-1}(\log t)^{1/k_1-1}$$
$$<(t^{a_1}(\log t)^{1/k_1})',\quad \text{if}\quad t>1$$

The last formula does not exceed the following,

$$\xi^{-1/k_1}\int_1^\infty (t^{a_1}(\log t)^{1/k_1})'\,m(t)\,dt$$
$$=\xi^{-1/k_1}\Big(t^{a_1}(\log t)^{1/k_1}\,m(t)\Big|_{t=1}^{t=\infty}-\int_1^\infty t^{a_1}(\log t)^{1/k_1}dm(t)\Big)$$
$$=\xi^{-1/k_1}\int_{|h|>1}|h|^{a_1}(\log|h|)^{1/k_1}d\mu\,.$$

Therefore we get

$$I_1 \leqq \xi^{-1}(||h||[L_{\mu}^{a_1}(\log^+ L_\mu)^{a_1/b_1}])^{b_1}$$

where $\xi = a_2(b_2-b_1)/b_2(a_2-a_1)>0$. Summing up these estimations we get

$$\int_{|Th|>1} |Th|^{b_1} d\nu \leqq M_1^{b_1}(||h||[L_{\mu}^{a_1}])^{b_1} + b_1(2\kappa)^{b_1} M_1^{b_1} \xi^{-1} (||h||[L_{\mu}^{a_1}(\log^+ L_\mu)^{a_1/b_1}])^{b_1} + b_1(2\kappa)^{b_2} M_2^{b_2}\left(\frac{a_2}{a_1}\right)^{k_2}(b_2-b_1)^{-1}(||h||[L_{\mu}^{a_1}])^{a_1k_2}.$$

We shall need the following lemma

Lemma 1. *Let us suppose the inequality $A \leqq \kappa(B+C)$ between three non-negative number A, B and C. Then we have the following inequalities:*

(i) *if* $0 \leqq A \leqq 1$,

$$A \leqq \begin{cases} \kappa(B+C), & \text{if } 0 \leqq C \leqq 1 \\ \kappa(B+C^p), & \text{if } C>1 \end{cases}$$

(ii) *if* $A>1$,

$$A \leqq \begin{cases} B^q + C^q, & \text{if } 0 \leqq C \leqq 1 \\ B^q + C, & \text{if } C>1 \end{cases}$$

where $p = b_1/b_2$ and $q = b_2/b_1$ respectively.

End of proof of Theorem 2. We shall use a constant K depending only on α_1, α_2, β_1, β_2 and κ and use the same letter at each occurence. We shall also denote by M the maximam value of M_1, M_2 and 1. If $f \in L_{\mu}^{a_1}(\log^+ L_\mu)^{a_1/b_1} + L_{\mu}^{a_2}$, for any positive number η there exists a decomposition of $f=g+h$ with $g \in L_{\mu}^{a_2}$, $h \in L_{\mu}^{a_1}(\log^+ L_\mu)^{a_1/b_1}$ such that $||g||+||h|| \leqq ||f||+\eta$. Here if $||f||<1$ then we should take the η so small that $||f||+\eta \leqq 1$.
From now on we shall denote simply $||f||$, $||g||$ and $||h||$ respectively. Let us denote by S_1 and S_2 the set of points $|Tf|>1$ and $|Tf| \leqq 1$ respectively. Let us also denote by S_{11} and S_{12} the set of points $|Th|>1$ and $|Th| \leqq 1$ respectively. Then applying the first part of lemma 1 with $A=|Tf|$, B$=|Tg|$, $C=|Th|$ and integrating over S_2, we get

$$\left(\int_{S_2} |Tf|^{b_2} d\nu\right)^{1/b_2}$$

$$\leqq \kappa\left(\int_{S_2 \cap S_{12}} (|Tg|+|Th|)^{b_2} d\nu\right)^{1/b_2} + \kappa\left(\int_{S_2 \cap S_{11}} (|Tg|+|Th|^p)^{b_2} d\nu\right)^{1/b_2}$$

$$\leqq 2\kappa\left(\int_S |Tg|^{b_2} d\nu\right)^{1/b_2} + \kappa\left(\int_{S_{12}} |Th|^{b_2} d\nu\right)^{1/b_2} + \kappa\left(\int_{S_{11}} |Th|^{b_1} d\nu\right)^{1/b_2}$$

by the Minkowsky inequality. Since $g \in L_{\mu}^{a_2}$ we have by hypotheses

(3) $||Tg||[L_{\nu}^{b_2}] \leqq M_2||g||[L_{\mu}^{a_2}]$.

If we substitute estimations (1), (2) and (3), we get

$$\Big(\int_{S_2} |Tg|^{b_2} d\nu\Big)^{1/b_2} \leqq KM(||f||+\eta)^{\gamma_1}$$

where the index γ_1 equals to $\min(b_1/b_3, a_1/a_2)$ or 1 according to $||f||<1$ or $||f|| \geqq 1$ respectively. Next if we apply the second part of lemma 1, we get by repetitions of the same discussion

$$\begin{aligned}&\Big(\int_{S_1} |Tf|^{b_1} d\nu\Big)^{1/b_1}\\ &\leqq\Big(\int_{S_1\cap S_{12}} (|Tg|^q+|Th|^q)^{b_1} d\nu\Big)^{1/b_1}+\Big(\int_{S_1\cap S_{11}} (|Tg|^q+|Th|^{b_1} d\nu\Big)^{1/b_1}\\ &\leqq 2\Big(\int_{S} |Tg|^{b_2} d\nu\Big)^{1/b_1}+\Big(\int_{S_{12}} |Th|^{b_2} d\nu\Big)^{1/b_1}+\Big(\int_{S_{11}} |Th|^{b_1} d\nu\Big)^{1/b_1}\\ &\leqq KM^{b_2/b_1}(||f||+\eta)^{\gamma_2}\end{aligned}$$

where the index γ_3 equals to $\min(1, a_1b_2/a_2b_1)$ or b_2/b_1 according to $||f||<1$ or $||f|| \geqq 1$ respectively.

Let us first suppose that $||f||=1$, then by the above estimations we have if η tends to zero,

$$\Big(\int_{|Tf|\leqq 1} |Tf|^{b_2} d\nu\Big)^{1/b_2}+\Big(\int_{|Tf|>1} |Tf|^{b_1} d\nu\Big)^{1/b_1} \leqq KM^{b_2/b_1}$$

thus we have proved

(4) $||Tf|| \leqq KM^{b_2/b_1}||f||, \quad M=\max(M_1, M_2, 1)$.

Now we shall exclude the assumption $||f||=1$. We use the properties as for $||f||$ which is neither norm nor quasi-norm and so we shall prove these for the sake of completeness.

Lemma 2. *The pseudo-norm* $||f||[L_{\mu}^{a_1}(\log^+ L_{\mu})^{a_1/b_1}+L_{\mu}^{a_2}]$ *has the following properties*:

(i) *if* λ *is any positive real number, it is satisfied*

$$2^{-1/b_1}\lambda\{1+(\log^+\lambda^{-1})^{a_1/b_1}\}^{-1/a_1}||f|| \leqq ||\lambda f|| \leqq 2^{1/b_1}\lambda\{1+(\log^+\lambda)^{a_1/b_1}\}^{1/a_1}||f||$$

(ii) *The pesudo-norm* $||\lambda f||$ *is a continuous function of* λ.

Proof of Lemma 2. (i) Since for any positive number λ

$$1+(\log^+|\lambda h|)^{a_1/b_1}\leqq 1+(\log^+\lambda+\log^+|h|)^{a_1/b_1}$$
$$\leqq 2^{a_1/b_1}\{1+(\log^+\lambda)^{a_1/b_1}\}\{1+(\log^+|h|)^{a_1/b_1}\}$$

we have by the decomposition of $f=g+h$,

$$||\lambda f||\leqq||\lambda g||+||\lambda h||\leqq\lambda||g||+2^{1/b_1}\lambda\{1+(\log^+\lambda)^{a_1/b_1}\}^{1/a_1}||h||$$
$$\leqq 2^{1/b_1}\lambda\{1+(\log^+\lambda)^{a_1/b_1}\}^{1/a_1}(||f||+\eta)\,.$$

If we let η tend to 0, the second inequality of (i) has proved. Next in this inequality, let us put $\lambda^{-1}f$ instead of f.

$$||f||\leqq 2^{1/b_1}\lambda\{1+(\log^+\lambda)^{a_1/b_1}\}^{1/a_1}||\lambda^{-1}f||$$

and write $\lambda^{-1}=\mu$, we have

$$2^{-a_1/b_1}\mu\{1+(\log^+\mu^{-1})^{a_1/b_1}\}^{-1/a_1}||f||\leqq||\mu f||$$

This is just the first inequality of (i).

(ii) Let us suppose that $\lambda_n\to\lambda$. The case $\lambda=0$ is trivial from the inequality (i). We suppose that $\lambda>0$. For any given positive number, there exists a decomposition $\lambda f=G+H$, $G\in L_{\mu}^{a_2}$, $H\in L_{\mu}^{a_1}(\log^+ L_{\mu})^{a_1b_1}$ such that $||G||+||H||\leqq||\lambda f||+\eta$. If we put $G=\lambda g$, $H=\lambda h$, then $g+h=f$ and os $||\lambda_n f||\leqq||\lambda_n g||+||\lambda_n h||$. Both $||\lambda g||$ and $||\lambda h||$ are continuous with respect to λ, therefore

$$\overline{\lim_{n\to\infty}}||\lambda_n f||\leqq\overline{\lim_{n\to\infty}}\{||\lambda_n g||+||\lambda_n h||\}$$
$$=||\lambda g||+||\lambda h||=||G||+||H||\leqq||\lambda f||+\eta\,.$$

Next we shall prove that $||\lambda f||\leqq\underline{\lim}_{n\to\infty}||\lambda_n f||$. Since $||\lambda f||$ is a monotone non-decreasing function with respect to $\lambda>0$, we can assume that $\lambda_n<\lambda$. We have a similar decomposition $\lambda_n f=G_n+H_n$ such that $G_n\in L_{\mu}^{a_2}$, $H_n\in L_{\mu}^{a_1}(\log^+ L_{\mu})^{a_1/b_1}$ and $||G_n||+||H_n||\leqq||\lambda_n f||+\eta$. If we write $G_n=\lambda_n g_n$, $H_n=\lambda_n h_n$, then $g_n+h_n=f$ for all n.

Therefore we get

$$||\lambda f||\leqq||\lambda g_n||+||\lambda h_n||$$
$$\leqq||G_n||+||H_n||+(||\lambda g_n||-||\lambda_n g_n||)+(||\lambda h_n||-||\lambda_n h_n||)$$
$$\leqq||\lambda_n f||+(||\lambda g_n||-||\lambda_n g_n||)+(||\lambda h_n||-||\lambda_n h_n||)+\eta$$

where we can prove easily

$$||\lambda g_n||-||\lambda_n g_n||\to 0\quad(n\to\infty)$$
$$||\lambda h_n||-||\lambda_n h_n||\to 0\quad(n\to\infty)$$

by the use of $||G_n|| \leqq ||\lambda f|| + \eta$ and $||H_n|| \leqq ||\lambda f|| + \eta$ for sufficiently large n. Thus we obtain $||\lambda f|| \leqq \varliminf_{n\to\infty} ||\lambda_n f||$.

From lemma 2, we can conclude that if $||f|| \neq 1$, there exist a positive real number λ such that $||\lambda f|| = 1$. If $||f|| > 1$, then the λ to make $||f|| = 1$ is less than 1 and on the contrary if $||f|| < 1$ then the λ is greater than 1. Since $\tilde{f} = Tf$ is a quasi-linear operation and $L_\nu^{b_1} + L_\nu^{b_2}$ is a Banach space, we have

$$(5) \quad ||T(\lambda f)||[L_\nu^{b_2} + L_\nu^{b_1}] = \lambda ||Tf||[L_\nu^{b_1} + L_\nu^{b_2}].$$

If $||f|| > 1$, then we use the inequality (4) with respect to λf such that $||\lambda f|| = 1$ and applying the second part of lemma 2 (i) and (5), we can derive that the inequality (4) is also true. If $||f|| < 1$ then we use the inequality (4) with respect to λf such that $||\lambda f|| = 1$ again and applying the second part of lemma 2, (i) and (5) we have

$$||Tf|| \leqq 2^{1/b_1} M^{b_2/b_1} \{1 + (\log^+ \lambda)^{a_1/b_1}\}^{1/a_1} ||f||.$$

Since $2^{-1/b_1}\lambda ||f|| \leqq ||\lambda f|| = 1$ by the first part of lemma 2, (i) we can derive that

$$(6) \quad ||Tf|| \leqq K M^{b_2/b_1} ||f|| \{1 + (\log^+ ||f||^{-1})^{a_1/b_1}\}^{1/a_1}.$$

Thus theorem 2 has proved completely.

3. Proofs of Theorems 3 and 4. Since the method of proofs of theorems 3 and 4 is just the same as the preceding section, we only sketch the outlines.

Proposition 2. *Let us suppose that the quasi-linear operatin T is of weak type (a_1, b_1), $1 \leqq a_1, b_1 < \infty$. Then if $h \in L_\mu^{a_1}$ we have*

$$(7) \quad \left(\int_{|Th|>1} \frac{|Th|^{b_1}}{1 + (\log |Th|)^{1+\varepsilon}} d\nu \right)^{1/b_1} \leqq K M_1 ||h||[L_\nu^{a_1}],$$

and

$$(8) \quad \left(\int_{|Th|\leqq 1} \frac{|Th|^{b_1}}{1 + (\log |Th|^{-1})^{1+\varepsilon}} d\nu \right)^{1/b_1} \leqq K M_1 ||h||[L_\mu^{a_1}],$$

where ε is any positive number.

Proof of Proposition 2. We have by the same way as the proof of proposition 1,

$$\int_{|Th|>1} \frac{|Th|^{b_1}}{1 + (\log |Th|)^{1+\varepsilon}} d\nu = -\int_1^\infty \frac{y^{b_1}}{1 + (\log y)^{1+\varepsilon}} dn(y)$$

$$= -\frac{y^{b_1}}{1 + (\log y)^{1+\varepsilon}} n(y) \Big|_{y=1}^{y=\infty} + \int_1^\infty \left(\frac{y^{b_1}}{1 + (\log y)^{1+\varepsilon}} \right)' n(y)\, dy$$

If $y>1$, we have

$$\left(\frac{y^{b_1}}{1+(\log y)^{1+\varepsilon}}\right)' = \frac{b_1 y^{b_1-1}\{1+(\log y)^{1+\varepsilon}\}-(1+\varepsilon)y^{b_1-1}(\log y)^{\varepsilon}}{\{1+(\log y)^{1+\varepsilon}\}^2}$$
$$\leqq b_1 y^{b_1-1}/\{1+(\log y)^{1+\varepsilon}\}$$

and therefore

$$\int_{|Th|>1}\frac{|Th|^{b_1}}{1+(\log|Th|)^{1+\varepsilon}}d\nu \leqq (M_1||h||)^{b_1}+b_1\int_1^\infty \frac{y^{b_1-1}}{1+(\log y)^{1+\varepsilon}}n(y)\,dy$$
$$\leqq (M_1||h||)^{b_1}+b_1 M_1^{b_1}\int_1^\infty \frac{dy}{y\{1+(\log y)^{1+\varepsilon}\}}||h||^{b_1} \leqq KM_1^{b_1}||h||^{b_1}$$

The remaining part is proved by the same way.

For the proof of theorem 3, we need the following lemmas.

Lemma 3. *From an inequality $A\leqq\kappa(B+C)$ between three non-negative numbers A, B and C, we can derive the following*

(i) *if* $0\leqq A\leqq 1$,

$$A\leqq\begin{cases}\kappa(B+C), & \text{if } 0\leqq C\leqq 1\\ \text{const.}\left(B+\left(\dfrac{C^{b_1}}{1+(\log C)^{1+\varepsilon}}\right)^{1/b_2}\right), & \text{if } C>1\end{cases}$$

(ii) *if* $A>1$,

$$\frac{A^{b_1}}{1+(\log A)^{1+\varepsilon}}\leqq\begin{cases}(2\kappa)^{b_2}(B^{b_2}+C^{b_2}), & \text{if } 0\leqq C\leqq 1\,.\\ \text{const.}\left(B^{b_2}+\dfrac{C^{b_1}}{1+(\log C)^{1+\varepsilon}}\right), & \text{if } C<1\end{cases}$$

Lemma 4. *The pseudo-norm $||f||[L_\nu^{b_1}](\log^+ L_\nu)^{1+\varepsilon}+L_\nu^{b_2}]$ satisfy the following inequality: there exists a constant C such that*

$$\frac{\lambda}{C\{1+(\log^+\lambda)^{(1+\varepsilon)/b_1}\}}||f||\leqq||\lambda f||\leqq C\lambda\{1+(\log^+\lambda)^{(1+\varepsilon)/b_1}\}||f||.$$

If we repeat the discussion of the proof of theorem 2, we can prove theorem 3. The theorem 4 is an immediate consequence of proposition 2.

4. Proofs of Theorem 5, 6 and 7. We shall need the following proposition and lemma.

Proposition 3. *Let us suppose that the quasi-linear operation T is of weak type (a_1, b_1), $1\leqq a_1, b_1<\infty$. Then if $h\in L_\mu^{a_1}$ we have*

$$(9)\quad \left(\int_{|Th|>1}|Th|^{b_1-\varepsilon}d\nu\right)^{1/b_1-\varepsilon}\leqq KM_1^{b_1/b_1-\varepsilon}(||h||[L_\mu^{a_1}])^{b_1/b_1-\varepsilon}$$

and

$$(10) \quad \left(\int_{|Th|\leqq 1} |Th|^{b_1+\varepsilon} dv \right)^{1/b_1+\varepsilon} \leqq KM_1^{b_1/b_1+\varepsilon}(\| h \| [L_{\mu}^{a_1}])^{b_1/b_1+\varepsilon}$$

where ε *is any positive number.*

Lemma 5. *From an inequality* $A \leqq \kappa(B+C)$ *between three non-negative numbers* A, B *and* C, *we can derive the following*

(i) *if* $0 \leqq A \leqq 1$,

$$A \leqq \begin{cases} \kappa(B+C), & \text{if } 0 \leqq C \leqq 1 \\ B+C^{b_1-\varepsilon/b_2}, & \text{if } C>1 \end{cases}$$

(ii) *if* $A>1$,

$$A \leqq \begin{cases} (2\kappa)^{b_2/b_1-\varepsilon}(B^{b_2/b_1-\varepsilon}+C^{b_2/b_1-\varepsilon}), & \text{if } 0 \leqq C \leqq 1 \\ (2\kappa)^{b_2/b_1-\varepsilon}(B^{b_2/b_1-\varepsilon}+C), & \text{if } C>1\,. \end{cases}$$

Now proofs of these theorems are repetitions of those of preceding section and need not be gone into the details.

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