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CHARACTERIZATION OF SLICES AND RIBBONS

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When the definition of ribbon knot was made [2, p. 172], it was with the expectation that it would subsequently be proved that every slice knot is a ribbon knot (the converse being obvious), thereby establishing a simple characterization in 3-dimensional space R^3 of slice knots. Unfortunately this has turned out to be a very difficult thing either to prove or disprove[1]. Although such a 3-dimensional characterization may easily be obtained by suitably modifying the definition of ribbon knot, unless an example of a slice knot that is not a ribbon knot is found this would be somewhat unsatisfactory, because the striking simplicity of the original definition would be lost.

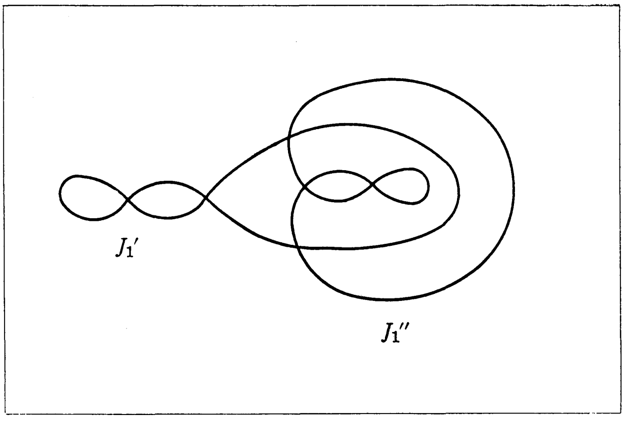
At any rate what I am going to do here is to give new 3-dimensional characterizations of ribbon knots and slice knots. It is hoped that these new characterizations may throw some light on the relationship between ribbon knots and slice knots. In this direction they do lead to an extremely simple derivation of a condition satisfied by the Seifert matrix of a ribbon knot, which condition yields at once all of the known restrictions on the algebraic invariants of a ribbon knot. It also shows that no knot invariants derivable from a Seifert matrix can ever be used to show that a slice knot is not a ribbon knot.

Extending slightly a terminology introduced by Papakyriakopoulos [8, p.5], let me call a normal singular surface f: $S \rightarrow M$ *canonical* if there are no branch points and the boundary ∂S of S is mapped topologically into M by f. (For simplicity it is assumed that the 3-dimensional manifold M is orientable and that the surface S is compact and orientable.) The singularity of a canonical surface consists of a finite number of triple points and a finite number of double lines which cross themselves and each other at the triple points. Each double line J is one or another of the following three types:

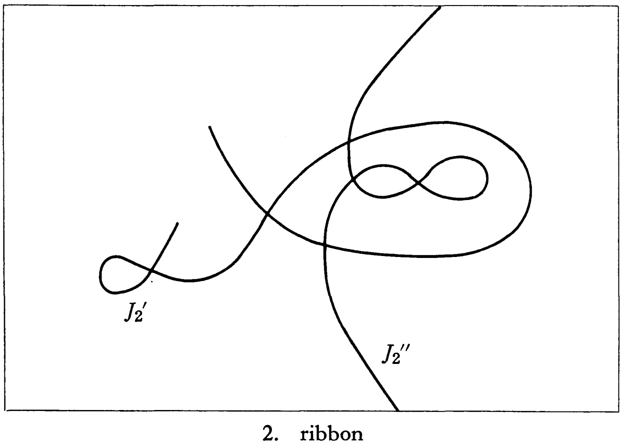
(1) a closed curve whose antecedents are closed curves J' and J'' that lie in the interior $\mathcal{J}S$ of S;

(2) an arc whose antecedents are an arc J' that spans the boundary ∂S of S and

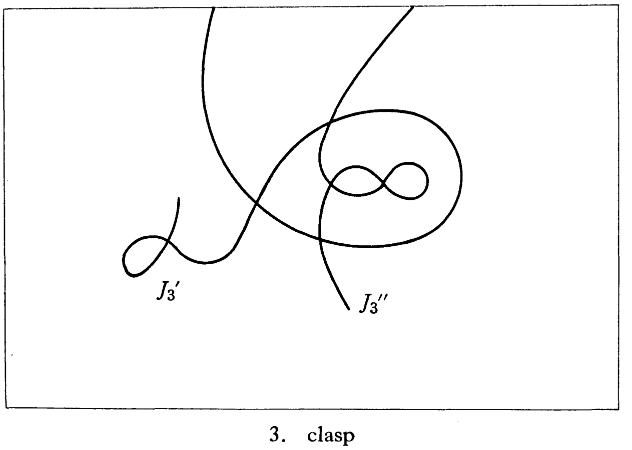
[1] The proof presented in [4] has an error in the second paragraph of p. 380. This fact was communicated to me by the authors, who cited diagram 2 to illustrate the difficulty. Diagram 1, from which diagram 2 may be generated was communicated to me by I. Johansson.



1. Dehn

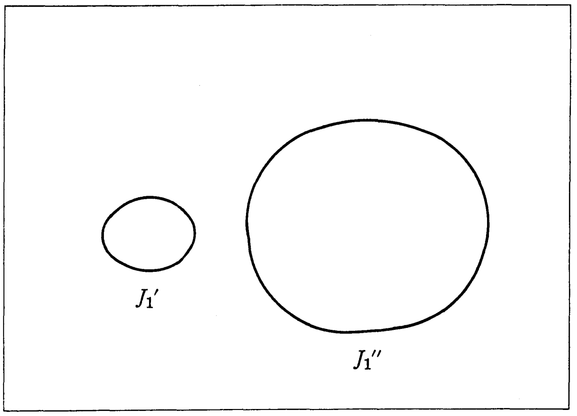


2. ribbon

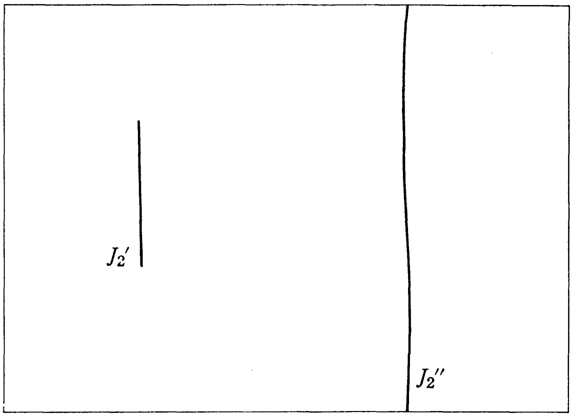


3. clasp

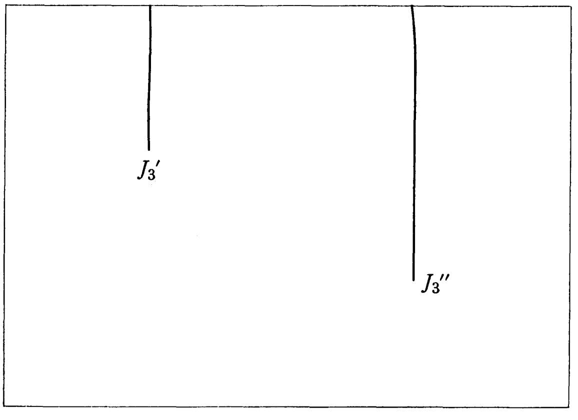
Type of pre-singularities on a disk



1°. simple Dehn



2°. simple ribbon



3°. simple clasp

an arc J'' that lies entirely in $\mathcal{I}S$;

(3) an arc whose antecedents are arcs J' and J'', each of which has an endpoint on ∂S and otherwise lies in $\mathcal{I}S$.

In the Dehn lemma only singularities of the first kind occur, so I call these *Dehn singularities*; singularities of the second kind may be called *ribbon singularities*; those of the third kind may perhaps be called *clasping singularities*[2]. In each case I call the singularity *simple* when J' and J'' are disjoint from one another and have no self-intersections.

A *ribbon knot* is a tame knot that bounds in R^3 a singular disk whose only singularities are mutually disjoint simple ribbon singularities. A *slice knot* is, of course, a tame knot in $R^3 = R^3 \times [0]$ that bounds in $R^3 \times [0, \infty)$ a locally flat disk.

Extension of these concepts to links of more than one component is not unique. In [2, p. 172] three different generalizations of the concept of slice knot were given; only one of them is of interest here: a *weak slice link*[3] is a tame link of, say, μ components in $R^3 \times [0]$ that bounds in $R^3 \times [0, \infty)$ a locally flat surface of genus 0 (that may or may not be connected). It is more or less obvious how to make analogous generalizations of the ribbon-knot concept: for example[3], a *weak ribbon link* should be a tame link of, say, μ components that bounds a singular surface of genus 0 whose only singularities are mutually disjoint simple ribbon singularities.

Now what is presently known about the relationship between slices and ribbons may be summed up in the following theorem (cf. Murasugi [7, lemma 8.1, p. 414] or Hosokawa andYanagawa [4, lemma 2, p. 377]), where 0 denotes the trivial knot type, and ϕ & ψ denotes the link type that splits into ϕ and ψ.

Theorem A. *A knot type κ is a slice knot type if and only if the link type κ & 0 &$\cdots$& 0 of μ components is, for sufficiently large μ, a weak ribbon link type.*

I come now to the basic result of this paper. A system of annuli $K_1 \cup \cdots \cup K_h$ will be called *trivial* if they are mutually disjoint and such that the link $\partial K_1 \cup \cdots \cup \partial K_h$ of $2h$ components is of trivial type in R^3. (In other words the

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[2] A knot that bounds a singular disk whose only singularity is a single simple clasp is a doubled knot (Schlingknoten); this kind of a singularity was studied by H. Seifert [10].

[3] A *strong slice link* is a tame link of, say, μ components in $R^3 \times [0, \infty)$ the union of μ mutually disjoint locally flat disks; a *strong ribbon link* would be a tame link of, say, μ components that bounds (in R^3) the union of μ singular disks the only singularities of which are mutually disjoint simple ribbon singularities. With proofs similar to those of theorem 1′ and corollary 1′ one can prove the following: THEOREM 1″. *A link of μ components is a strong ribbon link if and only if it bounds a non-singular surface F of, say, genus h on which there is a trivial system of annuli $K_1, \cdots, K_{k+\mu-1}$ which is such that each component of $F-(K_1 \cup \cdots \cup K_{h+\mu-1})$ has one of the components of the link as its boundary. Corollary 1″. A strong ribbon link of μ components has a standard Seifert matrix V of size, say $(2h+\mu-1)\times(2h+\mu-1)$ whose leading principal $h \times h$ minor is 0, as are its last $\mu-1$ rows and columns.*

annuli are untwisted, unknotted and unlinked.) Let me call a connected non-singular oriented surface F in R^3 *semi-unknotted* if on it there is a trivial system of annuli $K_1 \cup \cdots \cup K_h$ which is such that $F-(K_1 \cup \cdots \cup K_h)$ is a (connected) surface of genus 0. Note that in such a case the genus of F is h, and the number of boundaries of $F-(K_1 \cup \cdots \cup K_h)$ is $2h$ more than the number of boundaries of F.

Theorem 1. *A knot is a ribbon knot if and only if it bounds a semi-unknotted surface.*

Proof.[4] Suppose first that k is a ribbon knot, and consider a singular disk f: $D \rightarrow R^3$ bounded by k, whose only singularities are mutually disjoint ribbon singularities $(J_i', J_i'') \rightarrow J_i$, $i=1, 2, \cdots, h$. For each i let u_i be a simple closed curve on $\mathcal{J}D$ that bounds a disk d_i in $\mathcal{J}D$ that contains J_i'' and is otherwise disjoint from the pre-singularity and from the other disks d_j $(j \neq i)$. If, for each i, one makes the orientation-preserving cut [8, p. 12] along J_i one gets a non-singular orientable surface F of genus h. On this surface a suitably thin neighborhood K_i of $f(u_i)$ is an annulus, and $F-(K_1 \cup \cdots \cup K_h)$ is connected, hence a surface of genus 0. Since the disks $D_i = \mathrm{f}(d_i)$, $i=1, \cdots, h$, are pairwise disjoint the link $\partial K_1 \cup \cdots \cup \partial K_h$(of $2h$ components) is of trivial type.

Suppose conversely that k bounds a semi-unknotted surface F of geuns h. Then there are mutually disjoint annuli $K_1, \cdots, K_h$ on F such that $F-(K_1 \cup \cdots \cup K_h)$ is a surface of genus 0, and pairwise disjoint disks $D_1^\sharp, \cdots, D_h^\sharp, D_1^\flat, \cdots, D_h^\flat$ in R^3 such that $\partial D_j^\sharp \cup \partial D_j^\flat = \partial K_j$. If one replaces $K_1 \cup \cdots \cup K_h$ by $D_1^\sharp \cup \cdots \cup D_h^\sharp \cup D_1^\flat \cup \cdots \cup D_h^\flat$ one obtains a singular disk bounded by k whose singularity consists of a certain number of mutually disjoint simple ribbon singularities $(J_i', J_i'') \rightarrow J_i$ $i=1, \cdots, l$, and perhaps also a number of simple Dehn singularities that are disjoint from each other and from $J_1 \cup \cdots \cup J_l$. Such Dehn singularities are easily eliminated in the usual way (there being no triple points) by making orientation preserving cuts along them.

With virtually identical proof one may obtain a generalization to links:

Theorem 1′. *A link is a weak ribbon link if and only if it bounds a semi-unknotted surface.*

Of course a characterization of slice knots results from theorem A and 1′:

Theorem 1*. *A knot type κ is a slice knot type if and only if, for some $\mu \geqq 1$, there is a semi-unknotted surface whose boundary is of the link type κ & 0 &$\cdots$ & 0 of μ components.*

These conditions on spanning surfaces lead to conditions on Seifert matrices

[4] I am indebted to F. Hosokawa for pointing out an error in the original proof.

obtained from them. First, recall the definition of a Seifert matrix for a knot or link [9, p. 586; 11, p. 64]. Let F be a non-singular oriented connected surface in R^3 of genus h, whose boundary ∂F is a link of μ components. Since F is 2-sided in R^3, any 1-dimensional cycle b on F can be deformed into a cycle $b^\sharp$ of R^3 lying slightly above F, and it can also be deformed into a cycle $b^\flat$ of R^3 that lies slightly below F. To any family of cycles $b_1, \cdots, b_{2h+\mu-1}$ representing a basis for the first homology group $H_1(F)$ of F one can associate a matrix $V=||v_{ij}||$ with $2h+\mu-1$ rows and columns by defining the entry v_{ij} to be the linking number $L(b_i^\sharp, b_j^\flat)$ of $b_i^\sharp$ and $b_j^\flat$ in R^3. Any such matrix is called a *Seifert matrix* of the link ∂F (or of the surface F). It should be noted that $V-V'$ is the matrix of intersection numbers on F of these basic cycles [9, p. 585].

Suppose that F is put in the canonical form of a disk with h pairs of bands $B_i, B_{h+i}, i=1, \cdots, h$, and $\mu-1$ unpaired bands $C_1, \cdots, C_{\mu-1}$. Cf [9, p. 584; 11, p. 63]. A basis $b_1, \cdots, b_{2h+\mu-1}$ for $H_1(F)$ represented by the median lines of $B_1, \cdots, B_h, B_{h+1}, \cdots, B_{2h}, C_1, \cdots, C_{\mu-1}$, in that order, oriented so that the intersection number on F of b_i with b_{h+1} is $+1$ for $i=1, \cdots, h$, is called a *standard basis*. In such a case the matrix $V-V'$ of intersection number will have the form

$$V-V'=\begin{pmatrix} 0 & -E & 0 \\ E & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where E denotes the $h \times h$ identity matrix. I shall call a matrix V *standard* if $V-V'$ has this form.

If the surface F is semi-unknotted it can be put into canonical form in such a way that the annuli $K_1, \cdots, K_h$ follow the median lines of $B_1, \cdots, B_h$. When this is done it is obvious that $L(b_i^\sharp, b_j^\flat)=0$ for all $i, j \leq h$. Thus Theorems 1 and 1′ have the following consequence.

Corollary 1. *A ribbon knot has a standard Seifert matrix V of size, say, $2h \times 2h$, whose leading principal $h \times h$ minor* $\begin{Vmatrix} v_{11} \cdots v_{1h} \\ v_{h1} \cdots v_{hh} \end{Vmatrix}$ *is* 0.

Corollary 1′. *A weak ribbon link of μ components has a standard Seifert matrix V of size, say, $(2h+\mu-1) \times (2h+\mu-1)$, whose leading principal $h \times h$ minor is* 0.

It is known that any $(2h+\mu-1) \times (2h+\mu-1)$ integral matrix $V=||v_{ij}||$ that satisfies the condtion $V-V'=\begin{pmatrix} 0 & -E & 0 \\ E & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is a Seifert matrix for some link of μ components [9, pp. 586–587]. If the leading principal $h \times h$ minor is 0 then, among the various surfaces F for which V is a Seifert matrix, there will be at least one that is semi-unknotted.

One can also obtain a corollary from Theorem 1*. Suppose that F is a semi-unknotted surface whose boundary is a link of μ components of type κ & 0 &···& 0. Then it can be put into canonical form in such a way that the annuli $K_1, \cdots, K_h$ follow the median lines of $B_1, \cdots, B_h$, and the curves $C_1, \cdots, C_{\mu-1}$ represent the trivial constituents of κ & 0 &···& 0. Then $v_{ij}=0$ not only when $i, j \leq h$ but also whenever either $i>2h$ and $j>2h$. Thus

Corollary 1*. *If κ is the type of a slice knot then, for some $\mu \leq 1$, the link type κ & 0 &···& 0 of μ components has a standard Seifert matrix V of size, say, $(2h+\mu-1)\times(2h—\mu-1)$ of the form*

$$\begin{pmatrix} & & v_{1h+1} & \cdots v_{12h} & v_{12h+1} & \cdots v_{12h+\mu-1} \\ & 0 & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ & & v_{hh+1} & \cdots v_{h2h} & v_{h2h+1} & \cdots v_{h2h+\mu-1} \\ v_{h+11} & \cdots v_{h+1h} & v_{h+1h+1} & \cdots v_{h+12h} & v_{h+12h+1} & \cdots v_{h+12h+\mu-1} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ v_{2h1} & \cdots v_{2hh} & v_{2hh+1} & \cdots v_{2h2h} & v_{2h2h+1} & \cdots v_{2h2h+\mu-1} \\ v_{2h+11} & \cdots v_{2h+1h} & v_{2h+1h+1} & \cdots v_{2h+12h} & & \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & 0 & \\ v_{2h+\mu-11} & \cdots v_{2h+\mu-1h} & v_{2h+\mu-1h+1} & \cdots v_{2h+\mu-12h} & & \end{pmatrix}.$$

From Corollary 1 it is easy to deduce all the known algebraic conditions for ribbon knots, i.e. that [3. pp. 262–263] the Alexander polynomial must be of the form $\Delta(t)=f(t)\,f(1/t)$, that [1] the Minkowski units must all be$=+1$, and that [7, theorem 8.3, p. 414] the signature must be$=0$. These are, of course, also known conditions for slice knots, cf [6]. The analogue of corollary 1 for slice knots has been proved by J. Levine [5]. It follows that, in order to show that some slice knot is not a ribbon knot, if indeed that is true, deeper invariants than those derived from the Seifert matrix will be required.

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