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STRUCTURE PRESERVING GROUP ACTIONS ON STABLY ALMOST COMPLEX MANIFOLDS

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1. Introduction

Conner and Floyd in [1, 2] introduced the notion of periodic maps preserving a complex structure, applying bordism methods quite successfully. In a discussion with Gary Hamrick it became apparent that a somewhat weaker notion was also quite plausible, and the object of this note is to analyze this weaker structure.

Being given a manifold with boundary V and a differentiable action $\phi\colon G\times V\to V$, with G a finite group, the differential $d\phi\colon G\times\tau(V)\to\tau(V)$ induces a G action on the tangent bundle to V. Being given a real representation $\theta\colon G\times W\to W$ of G on a vector space W, one may form a G-bundle $W\times V\xrightarrow{\pi}V$, where G acts by $\theta\times\phi$ on $W\times V$. Then the Whitney sum of $\tau(V)$ and the bundle π has a G-action given by $d\phi$ and θ. Thinking of $E(\tau(V)\oplus\pi)$ as identified with $E(\tau(V))\times W$, the action is $d\phi\times\theta$.

A bundle map $J\colon\tau(V)\oplus\pi\to\tau(V)\oplus\pi$ which covers the identity map on V and such that $J^2=-1$ in the fibers gives $\tau(V)\oplus\pi$ a complex structure and if J commutes with the G action $d\phi\times\theta$, $\tau(V)\oplus\pi$ becomes a complex G-bundle over V.

If $\psi\colon G\times T\to T$ is a complex representation of G one may form the bundle $\bar\pi\colon T\times V\to V$ with G action given by $\psi\times\phi$, and if $i\colon T\to T$ is the function with $i^2=-1$giving the complex structure, $\tau(V)\oplus\pi\oplus\bar\pi$ is a complex G bundle if G acts by $d\phi\times\theta\times\psi$ and the complex structure is $J\times i$.

A stably almost complex structure on (V, ϕ) preserved by G would then be an equivalence class of systems (W,θ,J), where two such (W,θ,J) and (W',θ',J') are equivalent if there are complex representations (T,ψ,i) and (T',ψ',i') so that $\tau(V)\oplus\pi\oplus\bar\pi$ and $\tau(V)\oplus\pi'\oplus\bar\pi'$ are equivalent complex G-bundles.

The boundary of V inherits a stably almost complex structure preserved by G for $\tau(\partial V)\cong\tau(V)|_{\partial V}\oplus 1$ as G-bundles, where 1 is the trivial line bundle coming from the trivial representation of G.

It is clear that this differs from the Conner-Floyd approach in which (W,θ) and (T,ψ) are restricted to be trivial representations.

One may form bordism groups using the new structure preserving actions, which will be denoted $\omega_*^U(G, \mathcal{F}, \mathcal{F}')$ given by G actions preserving a complex structure which are $\mathcal{F}$-free and such that the boundary action is $\mathcal{F}'$-free, where, $\mathcal{F}, \mathcal{F}'$ are families in G as in Conner-Floyd [3]. The corresponding groups using the Conner-Floyd definition of "structure preserving" will be denoted $\Omega_*^U(G, \mathcal{F}, \mathcal{F}')$, and the forgetful homomorphism will be denoted by

$$\rho\colon \Omega_*^U(G, \mathcal{F}, \mathcal{F}') \to \omega_*^U(G, \mathcal{F}, \mathcal{F}').$$

The remainder of this paper will be devoted to analyzing $\omega_*^U(G, \mathcal{F}, \mathcal{F}')$ and ρ in the case when G is cyclic of prime order. Surprisingly, the cases $G=Z_2$ and $G=Z_p$ with p odd are considerably different, which is not the case for the Conner-Floyd groups.

2. Structure preserving involutions

Now consider the special case $G=Z_2$, writing (V, ϕ) as (V, t) where t is the involution generating the Z_2 action. There are three families for Z_2, the empty family ϕ, the family Free$=\{\{1\}\}$, and the family All of all subgroups. Letting $\omega_*^U(Z_2, \mathcal{F})=\omega_*^U(Z_2, \mathcal{F}, \phi)$, the groups of interest are related by an exact sequence

$$\begin{tikzcd} \omega_*^U(Z_2, \text{Free}) \arrow[rr, "i"] & & \omega_*^U(Z_2, \text{All}) \arrow[dl, "j"] \\ & \omega_*^U(Z_2, \text{All}, \text{Free}) \arrow[ul, "\partial"] & \end{tikzcd}$$

where i, j are induced by inclusion of families and ∂ by taking the boundary.

First, to analyze $\omega_*^U(Z_2, \text{All}, \text{Free})$, consider an involution (V, t) on an n-dimensional manifold, with t acting freely on ∂V with J the complex operator on $\tau(V)\oplus k\oplus l$ with involution $dt\oplus 1\oplus(-1)$, where k, l denote trivial bundles of dimensions k and l respectively.

The fixed point set of t in V is a disjoint union of closed submanifolds F^{n-q} of dimension $n-q$, with normal bundles ν_q. A neighborhood of the fixed set of t may be identified with the disjoint union of the disc bundles $D(\nu_q)$, and since t acts freely on the complement of this neighborhood, one may cut the remainder away up to cobordism.

Along F^{n-q}, the bundle $\tau(V)\oplus k\oplus l$ decomposes into the eigen-bundles of $dt\oplus 1\oplus(-1)$ which are preserved by J, so that $\tau(F^{n-q})\oplus k$, the $+1$ eigen-bundle, and $\nu^q\oplus l$, the (-1) eigen-bundle are complex bundles. Thus F^{n-q} is a stably almost complex manifold and ν_q is a q-plane bundle with a stable complex structure.

Letting B_q be the bundle over BO_q induced from the fibration $BU\to BO$, the bundle ν_q is induced by a map into B_q. Thus one has:

Proposition 2.1. $\omega_n^U(Z_2, \text{All}, \text{Free}) \cong \bigoplus_{q=0}^{n} \Omega_{n-q}^U(B_q)$.

The group $\Omega_n^U(Z_2, \text{All}, \text{Free}) \cong \bigoplus_{q=0}^{[n/2]} \Omega_{n-2q}^U(BU_q)$ and the restriction homomorphism ρ is induced by the obvious maps $BU_j \to B_{2j}$.

The homology of the space B_q was computed in [4], and is torsion free, so $\Omega_*^U(B_q)$ is computable explicitly. Since the homomorphism $\Omega_*^U(BU_j) \to \Omega_*^U(BU)$ is a monomorphism onto a direct summand, and factors through ρ, one has:

Proposition 2.2. *$\omega_*^U(Z_2$, All, Free) is a free Ω_*^U module and the restriction*

$$\rho\colon \Omega_*^U(Z_2, \textit{All}, \textit{Free}) \to \omega_*^U(Z_2, \textit{All}, \textit{Free})$$

is a monomorphism onto a direct summand.

Turning to $\omega_*^U(Z_2, \text{Free})$, consider an involution (V, t) on an n-dimensional manifold, with t acting freely and with J the complex operator on $\tau(V) \oplus k \oplus l$ with involution $dt \oplus 1 \oplus (-1)$. By identifying x and $t(x)$ in V, one obtains the orbit space V/t and a quotient map $\pi\colon V \to V/t$, with V/t also being an n-dimen -sional manifold. Since $dt \oplus 1 \oplus (-1)$ covers t which is free, $dt \oplus 1 \oplus (-1)$ is free and the orbit space $E(\tau(V) \oplus k \oplus l)/(dt \oplus 1 \oplus (-1))$ may be indentified with the total space of the bundle $\tau(V/t) \oplus k \oplus l\xi$ where ξ is the line bundle associated with the double cover $\pi\colon V \to V/t$. Since J commutes with $dt \oplus 1 \oplus (-1)$, one has induced a complex structure on $\tau(V/t) \oplus k \oplus l\xi$ and a complex structure on $\tau(V) \oplus k \oplus l\xi$ induces a complex structure on $\tau(V) \oplus k \oplus l$, which is the bundle induced by π, commuting with the action.

Now $2\xi \cong \xi \otimes_R C$ has a complex structure, so a complex structure on $\tau(V/t) \oplus k \oplus l\xi$ is equivalent to a stable complex structure on $\tau(V/t)$ if l is even, or to a stable complex structure on $\tau(V/t) \oplus \xi$ if l is odd. Since the parity of l for V and ∂V is the same, $\omega_*^U(Z_2, \text{Free})$ decomposes into two direct summands, $\omega_*^U(Z_2, \text{Free})^+$ and $\omega_*^U(Z_2, \text{Free})^-$ for l even and odd respectively.

First considering $\omega_*^U(Z_2, \text{Free})^+$, the class of V, if ∂V is empty, is completely determined by the stably almost complex manifold V/t with its double cover V. Hence $\omega_*^U(Z_2, \text{Free})^+ \cong \Omega_*^U(RP(\infty))$, by assigning to the class of V the class of the map $V/t \to RP(\infty)$ classifying the double cover. The homomorphism ρ sends $\Omega_*^U(Z_2, \text{Free})$ into $\omega_*^U(Z_2, \text{Free})^+$ and composing to $\Omega_*^U(RP(\infty))$ is the usual isomorphism for computing $\Omega_*^U(Z_2, \text{Free})$.

For $\omega_*^U(Z_2, \text{Free})^-$, one has a classifying map $V/t \xrightarrow{f} RP(\infty)$ with ξ induced from the canonical bundle λ over $RP(\infty)$. The tangent bundle of $D(\xi)$, the disc bundle, is the pullback of $\tau(V/t) \oplus \xi$, so that $D(\xi)$ is a stably almost complex manifold. One then has the map $(D\xi, S\xi) \to (D\lambda, S\lambda) \to (T\lambda, *) \simeq (RP(\infty), *)$ where S is the sphere bundle and T is the Thom space, which defines a homomorphism from $\omega_*^U(Z_2, \text{Free})^-$ into the reduced bordism group $\tilde{\Omega}_{*+1}^U(RP(\infty))$. By applying transverse regularity arguments with $RP(\infty)$ considered as the Thom space of λ, one may reverse this process to recover V, so $\omega_*^U(Z_2, \text{Free})^- \cong \tilde{\Omega}_{*+1}^U(RP(\infty))$.

Combining these results gives:

Proposition 2.3. *$\omega_*^U(Z_2. \text{Free}) \cong \Omega_*^U(RP(\infty)) \oplus \tilde{\Omega}_{*+1}^U(RP(\infty))$ and ρ sends $\Omega_*^U(Z_2, \text{Free})$ isomorphically onto the first summand.*

Note. The Smith homomorphism is much more reasonably defined in $\omega_*^U(Z, \text{Free})$ than in Conner-Floyd's groups. Specifically, if (M, t) is a structure preserving involution, then splitting M gives a submanifold M' invariant under t whose normal bundle in M is the trivial line bundle of the non-trivial representation. Thus the Smith homomorphism maps the summands $\omega_*^U(Z_2, \text{Free})^+$ and $\omega_*^U(Z_2, \text{Free})^-$ into each other. In particular

$$\Delta\colon \omega_n^U(Z_2, \text{Free})^+ = \Omega_n^U(RP(\infty)) \to \omega_{n-1}^U(Z_2, \text{Free})^- = \tilde{\Omega}_n^U(RP(\infty))$$

is the reduction homomorphism, and

$$\Delta\colon \omega_n^U(Z_2, \text{Free})^- = \tilde{\Omega}_{n+1}^U(RP(\infty)) \to \omega_{n-1}^U(Z_2, \text{Free})^+ = \Omega_{n-1}^U(RP(\infty))$$

is obtained by dualizing $\xi \oplus \xi$.

To compute $\omega_*^U(Z_2, \text{All})$, one makes use of the exact sequence of the families. Being given a map $F^{n-q} \to B_q$ representing an element of $\omega_n^U(Z_2, \text{All}, \text{Free})$, the bundle $\nu_q \oplus l$ is complex over F^{n-q} and hence $q+l$ is even. Thus along the boundary of $D(\nu_q)$, $q+l$ must also be even, and the homomorphism

$$\partial\colon \omega_n^U(Z_2, \text{All}, \text{Free}) \to \omega_{n-1}^U(Z_2, \text{Free})$$

sends $\bigoplus_{q \text{ odd}} \Omega_{n-q}^U(B_q)$ into $\omega_{n-1}^U(Z_2, \text{Free})^-$ and $\bigoplus_{q \text{ even}} \Omega_{n-q}^U(B_q)$ into $\omega_{n-1}^U(Z_2, \text{Free})^+$.

The diagram

$$\begin{tikzcd}
\Omega_n^U(Z_2, \text{All}, \text{Free}) \arrow[r, "\partial"] \arrow[d] & \Omega_{n-1}^U(Z_2, \text{Free}) \arrow[d, "\rho", "\cong"'] & \\
\bigoplus_{q \text{ even}} \Omega_{n-q}^U(BU_{q/2}) \arrow[d] & \omega_{n-1}^U(Z_2, \text{Free})^+ = \Omega_{n-1}^U(RP(\infty)) & \\
\bigoplus_{q \text{ even}} \Omega_{n-q}^U(B_q) \arrow[ur, "\partial"'] & &
\end{tikzcd}$$

commutes, and $\rho\partial$ is known to map onto $\tilde{\Omega}_{n-1}^U(RP(\infty))$. The summand Ω_{n-1}^U complementary to $\tilde{\Omega}_{n-1}^U(RP(\infty))$ is realized as the manifolds $M \times Z_2$ with M stably almost complex and t interchanging the two copies of M. Applying i and the augmentation $\varepsilon\colon \omega_{n-1}^U(Z_2, \text{All}) \to \Omega_{n-1}^U$ which takes the cobordism class of the underlying manifold, one obtains $2[M]$. Thus i is monic on this summand and the image of ∂ in $\omega^U_{n-1}(Z_2, \text{Free})^+$ is precisely $\tilde{\Omega}^U_{n-1}(RP(\infty))$.

Now considering $\omega^U_*(Z_2, \text{Free})^- \cong \tilde{\Omega}^U_{*+1}(RP(\infty))$, one notes that $\tilde{\Omega}^U_*(RP(\infty))$ is generated as Ω^U_* module by the inclusion maps $RP(2i+1) \to RP(\infty)$ which are obtained by Thomifying the inclusion $RP(2i) \to RP(\infty)$, for which the induced double cover is the antipodal involution on S^{2i}. The complex structure imparted may be considered as that given by considering $S^{2i} \subset C^{i+1}$, where C^{i+1} has the involution given by multiplication by -1, and the complex stucture given by multiplication by $\sqrt{-1}$, imparting the appropriate structure to $\tau(S^{2i}) \oplus 1 \oplus 1$. The same construction gives an involution on $D^{2i+1} \subset C^{i+1}$ with appropriate structure on $\tau(D^{2i+1}) \oplus 0 \oplus 1$. Thus these classes are in the image of ∂, and since ∂ is a Ω^U_* module homomorphism, $\omega^U_*(Z_2, \text{Free})^-$ is contained in the image of ∂.

Thus one has compatible splittings for the sequences to obtain a commutative diagram

$$\begin{array}{ccccccccc}
0 \to & \Omega^U_n & \to & \Omega^U_n(Z_2, \text{All}) & \to & \bigoplus\limits_{q \text{ even}} \Omega^U_{n-q}(BU_{q/2}) & \to & \tilde{\Omega}^U_{n-1}(RP(\infty)) & \to 0 \\
 & {\scriptstyle 1}\downarrow & & {\scriptstyle \rho}\downarrow & & {\scriptstyle \rho'}\downarrow & & {\scriptstyle \rho''}\downarrow & \\
0 \to & \Omega^U_n & \to & \omega^U_n(Z_2, \text{All}) & \to & \bigoplus\limits_{q} \Omega^U_{n-q}(B_q) & \to & \tilde{\Omega}^U_{n-1}(RP(\infty)) \oplus \tilde{\Omega}^U_n(RP(\infty)) & \to 0
\end{array}$$

in which both ρ' and ρ'' are monomorphisms onto direct summands, and 1 is the identity.

Rather than belabor the point further, one has:

Proposition 2.4. ρ: *$\Omega^U_*(Z_2, All) \to \omega^U_*(Z_2, All)$ is a monomorphism.*

3. Maps of odd prime period

Now consider the case $G = Z_p$ with p an odd prime, again writing (V, ϕ) as (V, t) where t is a diffeomorphism of period p. Again there are three families: ϕ, Free, and All and one has an exact sequence

$$\begin{tikzcd}
\omega^U_*(Z_p, \text{Free}) \arrow[rr, "i"] & & \omega^U_*(Z_p, \text{All}) \arrow[dl] \\
& \omega^U_*(Z_p, \text{All}, \text{Free}) . \arrow[ul] &
\end{tikzcd}$$

To begin, consider $\omega^U_*(Z_p, \text{Free})$. If (V, t) is a free action of Z_p on an n-manifold with $dt \times s$ acting on $\tau(V) \oplus \pi$, where π is given by the representation (W, θ), then one may form the orbit space V/Z_p which is an n-manifold with pr: $V \to V/Z_p$ the projection. Since $dt \times s$ acts freely on $E(\tau(V) \oplus \pi)$, $E(\tau(V) \oplus \pi)/Z_p \to V/Z_p$ is a vector bundle and complex structures preserved by $dt \times s$ are given by complex structures on the quotient bundle.

Now (W, θ) may be decomposed by means of the irreducible representations into a direct sum of subrepresentations W_0, which is trivial, and W_k for $1\leq k\leq (p-1)/2$ where W_k is a complex vector space in which s acts as multiplication by $\exp\left(\frac{2\pi ik}{p}\right)$. In particular, $E(\pi)/Z_p\to V/Z_p$ is then the Whitney sum of a trivial bundle ξ_0 with fiber W_0 and the complex vector bundles ξ_k with fiber W_k associated with the p-fold cover $V\to V/Z_p$. Thus $E(\tau(V)\oplus\pi)/Z_p$ is the total space of the bundle $\tau(V/Z_p)\oplus\xi_0\oplus(\oplus\xi_k)$. Since $(\oplus\xi_k)$ has been given a complex structure, the complex structures on $\tau(V)$ preserved under the action are given precisely by stably almost complex structures on V/Z_p. Thus a structure preserving Z_p action is just a principal Z_p bundle over a stably almost complex manifold. Assigning to (V, t) the map $V/Z_p\to BZ_p$ classifying the cover then defines an isomorphism of $\omega_*^U(Z_p, \text{Free})$ with $\Omega_*^U(BZ_p)$. When applied to structure preserving actions of Z_p in the sense of Conner and Floyd, one also obtains an isomorphism and so one obtains:

Proposition 3.1 *The restriction homomorphism* $\rho\colon \Omega_*^U(Z_p, \mathit{Free})\to \omega_*^U(Z_p, \mathit{Free})$ *is an isomorphism.*

In the commutative diagram

$$\begin{array}{ccc}
\Omega_*^U(Z_p, \text{All, Free}) & \xrightarrow{\partial'} & \Omega_*^U(Z_p, \text{Free}) \\
\rho\downarrow & & \cong\downarrow\rho \\
\omega_*^U(Z_p, \text{All, Free}) & \xrightarrow{\partial} & \omega_*^U(Z_p, \text{Free})
\end{array}$$

it is known that the image of ∂' is $\tilde{\Omega}_*^U(BZ_p)$, and the composite

$$\Omega_*^U \to \omega_*^U(Z_p, \text{Free}) \xrightarrow{i} \omega_*^U(Z_p, \text{All}) \xrightarrow{\varepsilon} \Omega_*^U$$

is multiplication by p on the complementary summand, so the image of ∂ is precisely $\tilde{\Omega}_*^U(BZ_p)$.

Thus one has a splitting, giving the diagram

$$\begin{array}{ccccccccc}
0 \to & \Omega_*^U & \to & \Omega_*^U(Z_p, \text{All}) & \to & \Omega_*^U(Z_p, \text{All, Free}) & \to & \tilde{\Omega}_*^U(BZ_p) & \to 0 \\
& \downarrow 1 & & \rho\downarrow & & \rho\downarrow & & \downarrow 1 & \\
0 \to & \Omega_*^U & \to & \omega_*^U(Z_p, \text{All}) & \to & \omega_*^U(Z_p, \text{All, Free}) & \to & \tilde{\Omega}_*^U(BZ_p) & \to 0\,.
\end{array}$$

Now consider the group $\omega_*^U(Z_p, \text{All, Free})$. Letting (V, t) be an action which is free on ∂V, the fixed point set of V is a disjoint union of closed submanifolds F^{n-q} with normal bundles ν_q and V may be replaced by the disc bundles of the ν_q. At points of F^{n-q}, the bundle $\tau\oplus\pi$ decomposes into $\tau(F^{n-q})\oplus\xi_0$, where ξ_0 is the trivial bundle of W_0, which is the trivial eigen-bundle, and bundles $(\nu_q)_k\oplus\xi_k$, where ξ_k is the trivial bundle with fiber W_k and $(\nu_q)_k$ is a sub-bundle of $\nu_q|F^{n-q}$, giving the eigen-bundle corresponding to multiplication by $\exp\left(\frac{2\pi ik}{p}\right)$ for $1\leq k\leq(p-1)/2$. Considered as a complex Z_p bundle, the bundle $\tau\oplus\pi$ decomposes into complex sub-bundles η_0, the trivial eigen-bundle, and η_j, $1\leq j\leq p-1$ on which $dt\times s$ acts as multpilication by $\exp\left(\frac{2\pi ij}{p}\right)$. Taking the parts of the complex decomposition which give the real decomposition, one has $\eta_0\cong\tau(F^{n-q})\oplus\xi_0$. so F^{n+q} is stably almost complex, and $(\nu_q)_k\oplus\xi_k\cong\eta_k\oplus\eta_j$ where $(p-1)/2\leq j\leq p-1$ and $\exp\left(\frac{2\pi ij}{p}\right)$ is the complex conjugate of $\exp\left(\frac{2\pi ik}{p}\right)$, or $j=p-k$.

After stabilization, the bundles η_k and η_{p-k} are stable complex bundles subject only to the condition that $\eta_k\oplus\eta_{p-k}$ should be stably isomorphic as complex bundle with $(\nu_q)_k$. Thus, the class of (V, t) is completerly determined by the bordism classes $F^{n-q}_{(r)}\rightarrow BU_{r_1}\times BU\times\cdots\times BU_{r_{(p-1/2)}}\times BU$ where $r_1+.\cdots+r_{(p-1/2)}=q/2$, where $F^{n-q}_{(r)}$ are the portions of F^{n-q} over which $(\nu_q)_k$ has real dimension $2r_k$, the map into BU_{r_k} classifying $(\nu_q)_k$, and that into the k-th BU factor classifying η_k. Thus, one has

Proposition 3.2 *$\omega^U_n(Z_p, All, Free)$ is isomorphic to*

$$\bigoplus_{(r)}\Omega^U_{n-2r}(BU_{r_1}\times BU\times\cdots\times BUr_{(p-1/2)}\times BU)\,,$$

the sum being over all sequences $(r)=(r_1,\cdots,r_{(p-1/2)})$ of non-negative integers, and with $r=r_1+\cdots+r_{(p-1/2)}$.

In order to analyze $\rho\colon\Omega^U_n(Z_p$, All Free$)\rightarrow\omega^U_*(Z_p$, All, Free),one may simply note that analogously $\Omega^U_n(Z_p$, All, Free) is isomorphic to

$$\bigoplus_{(s,t)}\Omega^U_{n-2r}(BU_{s_1}\times BU_{t_1}\times\cdots\times BU_{s_{(p-1/2)}}\times BU_{t_{(p-1/2)}})$$

where $\frac{q}{2}=r=s_1+\cdots+s_{(p-1/2)}+t_1+\cdots+t_{(p-1/2)}$ and the map of $F^{n-q}_{(s,t)}$ into BU_{sk} classifies η_k and into BU_{tk} classifies η_{p-k}, with $(\nu_q)_k\cong\eta_k\oplus\eta_{p-k}$ in this case. The map ρ is then induced by the maps $\bigcup_{s_k+t_k=r_k} BU_{sk}\times BU_{tk}\rightarrow BU_{rk}\times BU$ given by the Whitney sum map $BU_{sk}\times BU_{tk}\rightarrow BU_{rk}$ and by $BU_{sk}\times BU_{tk}\xrightarrow{pr}BU_{sk}\xrightarrow{\sigma}BU$ where pr is the projection and σ is stabilization.

One may then observe that ρ is anything but monic, for many summands in $\Omega^U_n(Z_p$, All, Free) map to the same summand in $\omega^U_n(Z_p$, All, Free). (One need only look at the terms with $n=2r$ in which many copies of Z map to a single copy of Z). Since, by the commutative diagram, the kernels of the homomorphisms $\rho\colon\Omega^U_n(Z_p$, All, Free$)\rightarrow\omega^U_n(Z_p$, All, Free) and $\rho\colon\Omega^U_n(Z_p$, All$)\rightarrow\omega^U_n(Z_p$, All) are isomorphic, one sees that $\rho\colon\Omega^U_n(Z_p$, All$)\rightarrow\omega^U_n(Z_p$, All) is also not monic.

The homomorphism ρ is also not epic, for the map

$\bigcup_{s_k+t_k=r_k} BU_{s_k}\times BU_{t_k}\to BU_{r_k}\times BU$ factors through $BU_{r_k}\times BU_{r_k}$. One can, of course, compute ρ: $\Omega_n^U(Z_p$, All, Free$)\to\omega_n^U(Z_p$, All, Free) explicitly since the groups and map are completely known, but it hardly seems worthwhile,.

As a final note, one should consider the reason why the Z_2 and Z_p cases, p odd, are so different. Clearly the problem is the dissimilarity between the nature of real representations in the two cases. In studying $\Omega_*^U(G, *, *)$ only the complex representations really play a role, while in $\omega_*^U(G, *, *)$ both types enter.

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