ON THE PATHWISE UNIQUENESS OF SOLUTIONS OF ONE-DIMENSIONAL STOCHASTIC DIFFERENTIAL EQUATIONS

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Introduction

In this paper, we shall discuss a problem of the pathwise uniqueness for solutions of one-dimensional stochastic differential equations. Let \( a(x) \) and \( b(x) \) be bounded Borel measurable functions defined on \( \mathbb{R} \). We shall consider the following one-dimensional Itô's stochastic differential equation:

\[
\frac{dx_t}{dt} = a(x_t)dB_t + b(x_t)dt.
\]

K. Itô [1] proved that, if \( a(x) \) and \( b(x) \) are Lipschitz continuous, a solution is unique and it can be constructed on a given Brownian motion \( B_t \). On the other hand, if \( |a(x)| \) is bounded from below by a positive constant (i.e. uniformly positive), then a solution of (1) exists and it is unique in the law sense. This follows easily from a general result of one-dimensional diffusions (cf. [2]). However, though the distribution of \( \{x_s, B_s\} \) is unique, \( x_t \) is not always expressed as a measurable function of \( x_0 \) and \( \{B_s, s \leq t\} \). For example, if \( a(x) = \text{sgn} \ x, a(0) = 1 \) and \( x_0 = 0 \), it is not difficult to see that \( \sigma \{|x_s|; s \leq t\} = \sigma \{B_s; s \leq t\} \).

Here, we will show that, if \( a(x) \) is uniformly positive and of bounded variation on any compact interval, then the pathwise uniqueness holds for (1). This implies, in particular, that \( x_t \) is expressed as a measurable function of \( x_0 \) and \( \{B_s, s \leq t\} \) (cf. [5]). In this direction, M. Motoo (unpublished) already proved that the pathwise uniqueness holds for (1) if \( a(x) \) is uniformly positive and Lipschitz continuous and if \( b(x) \) is bounded measurable. Also, T. Yamada and S. Watanabe [5] proved the pathwise uniqueness of (1) if \( a(x) \) is Holder continuous of exponent \( \frac{1}{2} \) and \( b(x) \) is Lipschitz continuous. Our above mentioned result may be interesting in a point that it applies for many discontinuous \( a(x) \). It is still an open question whether only the uniform positivity of \( a(x) \) implies the pathwise uniqueness.

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A precise meaning of the equation (1) is as follows: \((\Omega, \mathcal{F}, P; \mathcal{F}_t)\) stands for a probability space \((\Omega, \mathcal{F}, P)\) with an increasing family \(\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}\) of sub-\(\sigma\)-algebras of \(\mathcal{F}\).

**Definition 1.** By a solution of (1), we mean a quadruplet \((\Omega, \mathcal{F}, P; \mathcal{F}_t)\) and a stochastic process \(\mathcal{X}_t = (x_t, B_t)\) defined on it such that
(i) with probability one, \(\mathcal{X}_t\) is continuous in \(t\) and \(B_0 = 0,\)
(ii) \(\mathcal{X}_t\) is an \(\{\mathcal{F}_t\}\)-adapted process and \(B_t\) is an \(\{\mathcal{F}_t\}\)-Brownian motion,
(iii) \(B_t\) satisfies
\[
x_t = x_0 + \int_0^t a(x_s) dB_s + \int_0^t b(x_s) ds \quad a.s.,
\]
where the integral by \(dB_s\) is understood in the sense of the stochastic integral of Itô.

**Definition 2.** We shall say that the pathwise uniqueness holds for (1), for any two solutions \(\mathcal{X}_t = (x_t, B_t), \mathcal{Y}_t = (y_t, B'_t)\) defined on a same quadruplet \((\Omega, \mathcal{F}, P; \mathcal{F}_t)\), \(x_0 = y_0\) and \(B_t \equiv B'_t\) implies \(x_t = y_t.\)

**Remark 1.** In Definition 2, it is sufficient to assume that \(x_0 = y_0 = x\) for some constant \(x \in \mathbb{R}.\)

**Remark 2.** A definition of the pathwise uniqueness may be defined in a stronger way as follows; the pathwise uniqueness holds if \(\mathcal{X}_t = (x_t, B_t)\) is a solution on \((\Omega, \mathcal{F}, P; \mathcal{F}_t)\) and \(\mathcal{Y}_t = (y_t, B'_t)\) is a solution on \((\Omega, \mathcal{F}, P; \mathcal{F}'_t)\) \((\mathcal{F}_t\) and \(\mathcal{F}'_t\) may be different) such that \(x_0 = y_0\) and \(B_t \equiv B'_t\) implies \(x_t = y_t.\) It is not difficult to show, using a result in [5], that this definition of the pathwise uniqueness is equivalent to Definition 2.

**Lemma.** Let \((M_t, V_t)_{t \in \mathbb{R}_+}^\mathbb{T}\) be a pair of continuous real process defined on a probability space \((\Omega, \mathcal{F}, P).\) Suppose that the total variation \(|||V(\omega)|||_T\) of \(V_t(\omega)\) on \([0, T]\) has a finite expectation. Further, suppose \(M_t\) is a martingale satisfying the following conditions;
(i) \(M_0 = 0\) a.s.,
(ii) there exist positive constants \(m_1\) and \(m_2\) such that
\[
M_t(\omega) \leq V_t(\omega) \leq m_2 M_t(\omega) \quad a.s.,
\]
for \((t, \omega) \in \{t; t \in [0, T]\}\) and \(M_t(\omega) \geq 0\).

Then, \(M_t = 0\) a.s. for \(0 \leq t \leq T.\)

Proof. For \(y \in \mathbb{R},\) let \(N_t(y, \omega)\) be the number of \(t \in [0, T]\) such that \(V_t(\omega) = y.\) By a theorem of Banach (cf. [4] pp. 280), we have
\[
|||V(\omega)|||_T = \int_{-\infty}^{\infty} N_t(y, \omega) dy.
\]
Obviously we may assume that $m_1 < m_2$. For $y > 0$, let $N_z(y, \omega)$ be the number of $\left[ \frac{1}{m_2}, \frac{1}{m_1} y \right]$-downcrossings of $M_t(\omega)$ on $[0, T]$. The condition (2) implies that

$$N_t(y, \omega) \geq N_z(y, \omega) \quad \text{for } y > 0.$$  

(4)

For $y > 0$, we define a sequence of stopping times $\{T_n\}$ in the following way;

$$T_0 = 0,$$

$$T_{2n+1} = \inf \left\{ t \geq T_{2n}; M_t > \frac{1}{m_1} y \right\} \wedge T \quad n = 0, 1, 2, \ldots,$$

$$T_{2n+2} = \inf \left\{ t \geq T_{2n+1}; M_t < \frac{1}{m_2} y \right\} \wedge T \quad n = 0, 1, 2, \ldots.$$  

Then, for $n = 1, 2, \ldots$, we can obtain the following inequality;

$$\left( \frac{1}{m_1} y - \frac{1}{m_2} y \right) (N_z(y, \omega) \wedge n + 1) \geq \sum_{k=1}^{n} \{ M_{T_{2k-1}}(\omega) - M_{T_{2k}}(\omega) \} + \{ (M_T(\omega) - \frac{1}{m_1} y) \vee y_0 \} \chi_{\{N_z(y, \omega) < n\}}.$$  

Taking the expectation, we have

$$E[N_z(y) \wedge n] \geq \frac{E[\{ (M_T - \frac{1}{m_1} y) \vee y_0 \} \chi_{\{N_z(y) < n\}}]}{\left( \frac{1}{m_1} - \frac{1}{m_2} \right) y} - 1.$$  

Letting $n \to \infty$, we have

$$E[N_z(y)] \geq \frac{E[\{ (M_T - \frac{1}{m_1} y) \vee y_0 \}]}{\left( \frac{1}{m_1} - \frac{1}{m_2} \right) y} - 1.$$  

(5)

Now, we assume that $P(M_T = 0) > 0$. Then, there exist positive constants $\varepsilon$ and $\delta$ such that

$$E[\{ (M_T - \frac{1}{m_1} y) \vee y_0 \}] > \varepsilon \quad \text{for } 0 < y < \delta.$$  

(6)

The inequalities (5) and (6) provide us with the equality

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1) Let $x$ and $y$ be real numbers. $x \wedge y$ means $\min(x, y)$.
2) $x \vee y$ means $\max(x, y)$.
3) $\chi_A$ denotes the indicator function of a set $A$. 

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\[
\int_0^\infty E[N_\tau(y)] \, dy = \infty.
\]

But this is a contradiction since, by (3) and (4),
\[
\int_0^\infty E[N_\tau(y)] \, dy \leq \int_{-\infty}^\infty E[N_\tau(y)] \, dy < \infty.
\]

Therefore we have

\[ P(M_t = 0) = 1 \quad \text{for} \quad 0 \leq t \leq T. \]

This completes the proof.

**Remark 3.** In the above lemma, we may suppose the following condition instead of (2); there exist positive constants \( m_1 \) and \( m_2 \) such that
\[
(7) \quad m_1 |M_t| \leq |V_t| \leq m_2 |M_t| \quad \text{a.s. for} \quad 0 \leq t \leq T.
\]

Now we will state our main result.

**Theorem.** Let \( a(x) \) and \( b(x) \) be bounded Borel measurable. Suppose \( a(x) \) is of bounded variation on any compact interval. Further, suppose there exists a constant \( c > 0 \) such that
\[
(8) \quad a(x) \geq c \quad \text{for} \quad x \in R.
\]

Then, the pathwise uniqueness holds for (1).

**Proof.** We assume that \( |a(x)| \leq M \) and \( |b(x)| \leq M \) for \( x \in R \). Let \( \xi_t = (x_t, B_t) \) and \( \eta_t = (y_t, B_t) \) be solutions of (1) such that \( x_0 = y_0 \) is a constant. For \( N > |x_0| \), we define that
\[
\tau_N = \left\{ \begin{array}{ll}
\inf \{ t \geq 0; \, |x_t| = N \} & \text{if} \quad \{ \} = \phi, \\
\infty & \text{if} \quad \{ \} = \phi,
\end{array} \right.
\]
\[
\eta_N = \left\{ \begin{array}{ll}
\inf \{ t \geq 0; \, |y_t| = N \} & \text{if} \quad \{ \} = \phi, \\
\infty & \text{if} \quad \{ \} = \phi,
\end{array} \right.
\]
\[
\gamma_N = \tau_N \wedge \eta_N.
\]

Let, for \( x \in R \),
\[
f(x) = -2 \int_0^x \frac{b(y)}{a(y)} \, dy, \quad \varphi(x) = \int_0^x \exp [f(y)] \, dy.
\]

By the time substitution and Cameron-Martin's formula (cf. [3]), there exists a constant \( K_1 > 0 \) depending only on \( c, M, N \) and \( t \) such that
\[
(9) \quad E\left[ \int_0^{\tau_N \wedge \eta_N} g(x_s) \, ds \right] \leq K_1 \|g\|_{L^1([-N, N])}, \quad \text{for} \quad g \in L^1([-N, N]).
\]

Since \( \varphi'(x) \) is absolutely continuous and \( \varphi''(x) \) is locally integrable, the inequality (9) assures us that Itô's formula applies to \( \varphi \) and we have
\[ \varphi(x_{t,N}) = \varphi(x_0) + \int_0^{t/N} \varphi'(a(x_s)) dB_s \quad a.s. \]

Since \( \varphi \) is a homeomorphism \( R \) onto \( I = (\varphi(-\infty), \varphi(\infty)) \), we can define that

\[
\sigma(x) = \varphi' a \circ \varphi^{-1}(x), \quad h(x) = \int_0^x \frac{1}{\sigma(y)} dy \quad \text{for} \quad x \in I.
\]

Obviously \( \sigma \) is of bounded variation on any compact interval of \( I \). Let \( |||\sigma|||_N \) be the total variation of \( \sigma \) on \( [\varphi(-N), \varphi(N)] \).

We can take an approximate sequence \( \{\sigma_n(x)\}_{n=1}^\infty \) such that

1. \( \sigma_n(x) \in C^1(R) \) and \( c \exp \left[ \frac{-2MN}{c^2} \right] \leq \sigma_n(x) \leq M \exp \left[ \frac{2MN}{c^2} \right] \)
   for \( x \in R \),

2. \( ||\sigma - \sigma_n||_{L^1([\varphi(-N), \varphi(N)])} \leq \frac{1}{n!} \) and \( ||\sigma_n||_{L^1([\varphi(-N), \varphi(N)])} \leq |||\sigma|||_N \).

Let

\[
h_n(x) = \int_0^x \frac{1}{\sigma_n(y)} dy \quad \text{for} \quad x \in I.
\]

Since \( h_n(x) \in C^2(R) \), we can apply Itô's formula to \( h_n \) and have

\[
h_n(\varphi(x_{t,N})) = h_n(\varphi(x_0)) + \int_0^{t/N} \sigma(\varphi(x_s)) \frac{\partial}{\partial \varphi(x_s)} \varphi(\varphi'(a(x_s))) dB_s - \frac{1}{2} \int_0^{t/N} \sigma_n(\varphi(x_s))^2 (\varphi'(a(x_s)))^2 ds.
\]

\[
= h_n(\varphi(x_0)) + L_t^n + W_t^n.
\]

It follows from (ii) that there exists a constant \( K_2 > 0 \) depending only on \( c, M \) and \( N \) such that

\[
|h(x) - h_n(x)| \leq K_2 \frac{1}{n!} \quad \text{for} \quad x \in [\varphi(-N), \varphi(N)].
\]

From this, we see that \( h_n(\varphi(x_{t,N})) \) converges almost surely to \( h(\varphi(x_{t,N})) \).

There exists a constant \( K_2 > 0 \) depending only on \( c, M, N, \) and \( t \) such that

\[
E[(L_t^n - B_{t/N})^2] \leq K_2 \frac{1}{n!}.
\]

Therefore \( L_t^n \) converges almost surely to \( B_{t/N} \). Let

\[
W_t = h(\varphi(x_{t,N})) - h(\varphi(x_0)) - B_{t/N}.
\]

From the above results, \( W_t^n \) converges almost surely to \( W_t \).
It is easy to see that there exists a constant $K > 0$ depending only on $c, M,$ and $N$ such that

$$E[|||W^n|||] \leq K E[^{t/\gamma_N} \int_0^t |\sigma'_n(x_s)| ds],$$

where $|||W^n|||$ is the total variation of $W^n_\tau$ on $[0, t]$. Using the time substitution, we easily see that there exists a constant $K > 0$ depending only on $c, M, N,$ and $t$ such that

$$E[^{t/\gamma_N} \int_0^t |\sigma'_n(x_s)| ds] \leq K |||W^n|||_{L^1([-N, N])}.$$

Hence it holds that

$$E[|||W|||] \leq K |||W|||_{L^1([-N, N])},$$

where $|||W|||$ is the total variation of $W_s$ on $[0, t]$.

From the definition of $h(x)$, there exists positive constants $m_1$ and $m_2$ such that

$$m_1 (x-y) \leq h(x) - h(y) \leq m_2 (x-y) \quad \text{for } y \leq x \quad \text{and} \quad x, y \in [\phi(-N), \phi(N)].$$

Let

$$M_t = \int_0^{t/\gamma_N} (\sigma(\phi(x_s)) - \sigma(\phi(y_s))) dB_s,$$

$$V_t = h(\phi(x_{t/\gamma_N})) - h(\phi(y_{t/\gamma_N})).$$

We can apply Lemma to $(M_t, V_t)$ and it follows that

$$P(\phi(x_{t/\gamma_N}) = \phi(y_{t/\gamma_N})) = 1.$$

Therefore we have

$$P(x_{t/\gamma_N} = y_{t/\gamma_N}) = 1.$$

Since $\lim_{\gamma_N \to \infty} \gamma_N = \infty \text{ a.s.}$, we obtain that $P(x_t = y_t) = 1$ and the proof is complete.

Remark 4. In Theorem, if $a(x)$ is continuous, we may assume that $a(x)$ is positive instead of (8).

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References