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Osaka J. Math.
9 (1972), 421-426

ON QUASI-INJECTIVE MODULES WITH A CHAIN CONDITION OVER A COMMUTATIVE RING

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(Received November 25, 1971)

In the previous paper [4] the author and T. Ishii studied the endomorphism rings of noetherian quasi-injective modules. As an application of it, we consider, in this paper, quasi-injective modules over a commutative ring R. If R is noetherian, E. Matlis decided every indecomposable injective modules in [6].

Greatly making use of those results in [6], we shall decide all quasi-injective (resp. injective) modules which are either artinian or noetherian in §§2 and 3. Especially, we shall give necessary and sufficient conditions of R for existence of quasi-injective (resp. injective) modules which are either artinian or noetherian (cf. [7], Theorem 5).

In this paper, a ring R is always commutative unless otherwise stated and every R-module is unitary.

1. Preliminaries

Let K be any ring (not necessarily commutative) and M a right K-module. Put $S=\mathrm{Hom}_K(M, M)$, then we assume that M is a left S-module. Let N be a subset of M. Then we denote the annihilator ideal of N in S and in K by $l(N)$ and ann N, respectively. Similarly, by $r(A)$ we denote the annihilator submodule of M for a left ideal A in S.

We call M a *weakly distinguished* K-module if for any K-submodules $N_1 \supset N_2$ in M such that N_1/N_2 is K-irreducible, $\mathrm{Hom}_K(N_1/N_2, M) \neq 0$. If M is K-quasi-injective, then M is weakly distinguished if and only if $rl(N)=N$ for any K-submodule N in M, (see [1], Proposition 6).

Finally, we shall add here some direct consequences of [4]. From now on we shall assume that a ring R is commutative.

Proposition 1. *Let R be a commutative ring and M a quasi-injective module. If M is noetherian as an R-module, then $S=\mathrm{Hom}_R(M, M)$ is left and right artinian,* (see Theorem 1 below).

Proof. Since R is commutative, S is an R-submodule of a finite directsum of copies of M. Therefore, S is artinian by [4], Theorem 1.

Propostion 2, *Let R and M be as above. We assume further that M is weakly distinguished. Put $S=\mathrm{Hom}_R(M, M)$. Then M is R-noetherian if and only if S is left artinian. In this case, M is R-artinian, S-injective and R/A is artinian, where $A=\mathrm{ann}\ M$.*

Proof. If M is R-noetherian, S is artinian by Proposition 1. Hence, M is S-injective by [4], Theorem 2 and M is R-artinian from the above remark, since S is noetherian. Further, R/A is an R-submodule of finite directsum of copies of M. Hence, R/A is artinian. If S is (left) artinian, then M is R-noetherian as above.

2. Noetherian quasi-injective modules

We shall decide quasi-injective noetherian modules in this section.

Lemma 1. *Let K be any ring and M a quasi-injective and weakly distinguished right K-module. Put $S=\mathrm{Hom}_K(M, M)$ and $T=\mathrm{Hom}_S(M, M)$. Then every K-submodule of M is a T-submodule of M.*

Proof. Let N be a K-module of M. Then $rl(N)=N$ by the remark in §1. Hence, N is a T-module.

Let R be a commutative noetherian ring and P a prime ideal in R. Let $\mathrm{E}(R/P)=E$ be an injective hull of R/P. Then Matlis showed in [6] that $E=\bigcup_i A_i$ and $\mathrm{Hom}_R(E, E)$ is a complete local noetherian ring, where $A_i=\{x \mid \in E, xP^i=0\}$.

Lemma 2. *Let R be a commutative noetherian ring and $\{P_i\}$ a finite set of distinct maximal ideals in R. Then every R-submodule N of $\Sigma\oplus\mathrm{E}(R/P_i)$ is weakly distinguished and quasi-injective.*

Proof. We may assume that N is an essential submodule of $E=\Sigma \oplus E_i$, where $E_i=\mathrm{E}(R/P_i)$. Then $\mathrm{ann}\ x\supset\Pi P_i^n$ for any x in N. Let N_1, N_2 be R-submodules of N such that N_1/N_2 is R-irreducible, then $N_1/N_2\approx R/P_i$ for some P_i. Since $N\cap R/P_i\neq(0)$, $\mathrm{Hom}_R(N_1/N_2, N)\neq(0)$, which means that N is weakly distinguished. Hence, E is an R-weakly distinguished injective module. Moreover, if we put $S=\mathrm{Hom}_R(E, E)$, $S=\mathrm{Hom}_S(E, E)$. Hence, every R-submodule M is an S-submodule by Lemma 1. Let E' be an injective hull of M contained in E. Then $E=E'\oplus E''$ and $E'\supset M$. $S'=\mathrm{Hom}_R(E', E')$ may be regarded as a subring of S. Hence, M is also an S'-module. Therefore, M is R-quasi-injective by [5], Theorem 1. 1.

We are interested in a noetherian or artinian quasi-injective module M and hence, we may assume that M is directly indecomposable.

Theorem 1. *Let M be a directly indecomposable module over a commutative ring R. Then M is quasi-injective and noetherian if and only if there exist an ideal*

I such that R/I is noetherian and a maximal ideal P containing I and M is contained in a submodule A_n of $\mathrm{E}_{R/I}(R/P)$. In this case, M is R-artinian, and hence R/I is artinian[1].

Proof. We assume that M is R-noetherian and quasi-injective. Put $I=\mathrm{ann}\, M$. Then $\bar{R}=R/I$ is noetherian as the proof of Proposition 2. Hence, we may assume that R is noetherian. Let E be an injective hull of M. Then $E=\mathrm{E}_R(R/P)$ with P prime by [6], Proposition 3. 1. Put $S=\mathrm{Hom}_R(E, E)$. We know from [6], Theorem 3. 4 and its proof that $A_1=S(R/P)\approx Sa\approx K$ for any non-zero element a in A_1, where K is the quotient field of R/P. Since $M\cap A_1\neq(0)$ and M is quasi-injective, M contains a submodule which is isomorphic to K by [5], Theorem 1. 1. Hence, P is a maximal ideal in R, and M is contained in some A_n, since M is R-finitely generated and each A_n has a composition length by [6], Theorem 3. 9. The remaining part is clear from the above and Lemma 2.

Corollary. *Let R be a commutative ring. Then there exists a noetherian injective module if and only if R contains a maximal ideal P such that R_P is artinian,* (cf. [6], Theorem 3. 11).

Proof. It is an immediate consequence of Theorem 1 and [7], Theorem 5,

3. Artinian, quasi-injective modules

We shall decide quasi-injective, (resp. injective) artinian modules in this section.

Theorem 2. *Let R be a commutative ring and M a directly indecomposable R-module and $S=\mathrm{Hom}_R(M, M)$. If M is quasi-injective and artinian, then*

i. *There exists a maximal ideal P in R such that $M=\bigcup A_i$, where $A_i=\{x\mid \in M, xP^i=0\}$, and M may be regarded as an R_P-module and R_P-quasi-injective.*

ii. *M is S-injective and S is a commutative $\mathfrak{P}$-adic complete local noetherian ring, where $\mathfrak{P}$ is a unique maximal ideal of S. Furthermore, the set of the S-submodules of M coincides with that of R-submodules of M.*

iii. *R is dense in S with repect to $\mathfrak{P}$-adic topology and hence, for any finite elements m_i in M and an element s in S, there exists an element r in R such that $m_is=m_ir$ for all i.*

Conversely, if S satisfies the first parts of ii *and* iii, *then M is a quasi-injective and artinian R-module.*

Proof. We assume that M is a quasi-injective and artinian R-module. Let $m\neq 0$ be an element in M, then $mR\approx R/\mathrm{ann}\, m$ is an artinian ring. Hence, there

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Added in proof: 1) In this case M is R/Ann M-injective by Theorem 1 of C. Faith *Modules finite over endomorphism ring,* Lecture Notes in Math., Springer, Heidelberg, 246.

exists a unique maximal ideal P such that $P\supset \operatorname{ann} m$ and $P^n\subset \operatorname{ann} m$, since M is indecomposable and quasi-injective. Therefore, M contains a unique minimal R-module R/P and P does not depend on a choice of m. Let s be in $R-P$ and $x\in l(s)\cap R/P$. Since P is maximal, there exist $p\in P$, $r\in R$ such that $1=p+rs$. Hence, $x=xp+xrs=0$. Therefore, $l(s)=(0)$. Since M is artinian, s gives an automorphism of M. Hence, M may be regarded as an R_P-module. It is clear that M is R_P-quasi-injective.

ii. Put $S=\mathrm{Hom}_R(M, M)$. Then S is left noetherian by [3], Proposition 1. Furthermore, we know from [3], Theorem 2 that M is S-injective, since M is R-weakly distinguished (cf. the proof of Lemma 2 and i). On the other hand, we put $S'=\mathrm{Hom}_S(M, M)$. Then $S'\subset S$ and hence, S' is the center of S. Moreover, since M is an artinian S-injective, S' is noetherian as above. Let N be the radical of S then S/N is a division ring by [2], Theorem 1 in p. 44 and Theorem 6 in p. 48, and $R/P\approx S/N$ as S-modules. Hence, M is S-weakly distinguished. Thus, M is also S'-injective as above. Since $S=\mathrm{Hom}_{S'}(M, M)$, $S=S'$ is a complete local ring with respect to a $\mathfrak{P}$-adic topology by [6], Theorem 3. 7, where $\mathfrak{P}$ is a unique maximal ideal in S' and $\mathfrak{P}\cap R=P$. The last part of ii is clear from the above and Lemma 1.

iii. The following argument is analogous to [6], Theorem 3. 7. Put $\bar{A}_i=\{x\mid \in M, x\mathfrak{P}^i=0\}$. We shall show for s in S that there exists r_i in R for each $\bar{A}_i$ such that $l(s-r_i)\supset \bar{A}_i$, Since $\bar{A}_1=R/P=S/\mathfrak{P}$, we have r_1. We assume that there exists r_i in R such that $l(s-r_i)\supset \bar{A}_i$. Let $\{m_1, m_2, \cdots, m_t\}$ be a system of minimal generators of $\bar{A}_{i+1}$ as an S-module (see Theorem 1), then we obtain elements b_i in R such that $m_ib_i\neq 0$, $m_jb_i=0$ if $i\neq j$ by [5], Theorem 2. 3. Put $g=s-r_i$, $g(\bar{\mathrm{A}}_i)=0$ and hence, $g(m_i)\mathfrak{P}=g(m_i\,\mathfrak{P})=0$, which means $g(m_i)\subset \bar{A}_1$. Since $\bar{A}_1$ is essential in M as an R-module and R/P is irreducible, there exists c_i in R such that $m_ib_ic_i=g(m_i)$ for each i. Put $r'_{i+1}=\Sigma\, b_jc_j$, then $g(m_j)=m_jb_jc_j=m_jr'_{i+1}$ for all j. Hence, $(s-(r_i+r'_{i+1}))\,\bar{\mathrm{A}}_{i+1}=(0)$. Since $r(\bar{\mathrm{A}}_{j+1})=\mathfrak{P}^{j+1}$ by [6], Theorem 3. 4, $s=\lim r_j$, $r_j\in R$. Let $\{m_i\}$ be a finite elements in M, then there exists an $\bar{\mathrm{A}}_n$ containing all m_i. Hence, if we take an element r in R such that $s-r\in\mathfrak{P}^n$, $m_ir=m_is$ for all i.

Conversely, we assume that S satisfies the first parts of ii and iii. Then every R-submodule N of M is an S-module and every R-homomorphism of N to M is an S-homomorphism. Hence, M is a quasi-injective and artinian by Lemma 2, since M is S-artinan.

Corollary. *Let M, R and S be as above. If M is a quasi-injective, artinian R-module, then for any intermediate ring T between R and S, M is T-quasi-injective.*

REMARK. In Theorem 2 we have shown that S is noetherian, however R/A is not noetherian in general, where $A=\operatorname{ann} M$. For example, let Z be the ring of integers and P a prime. Z_{P^∞} is Z_P-artinian, injective and indecomposable.

We can obtain a non-noetherian intermediate local ring T between Z_P and $\hat{Z}_P = \mathrm{Hom}_{Z_P}(Z_{P^\infty}, Z_{P^\infty})$ (see [3], Lemma 1) and M is T-quasi-injective and T-artinian.

Next, we shall consider a case of injective modules.

Theorem 3. *Let R be a commutative ring and M an R-artinian, injective module. Then there exists a finite set of maximal ideals $P_1, P_2, \cdots, P_n$ such that R_T is noetherian, where $T = R-(P_1 \cup P_2 \cup \cdots \cup P_n)$ and n is the number of non-isomorphic indecomposable direct summands of M. Conversely, if R_T is noetherian, there exists an R-artinian, injective module which is a directsum of n non isomorphic indecomposable modules.*

Proof. Let $M = \sum_1^n \oplus M_i$ and the M_i be directly indecomposable. We may assume $M_i \not\approx M_j$ if $i \neq j$. Each M_i corresponds to a maximal ideal P_i and M_i may be regraded as R_{P_i}-module by Theorem 2. Further, M_i is an injective hull of R/P_i as an R-module. Put $T = R-(P_1 \cup \cdots \cup P_n)$, then $R_T/P_iR_T \approx R/P$. Hence, M is an R_T-cogenerator. Therefore, R_T is noetherian by [8], Lemma 2. Conversely, we assume R_T is noetherian and put $M_i = \mathrm{E}_R(R/P_i)$. Since R/P_i is a unique minimal sub-module of M_i, $M_i = \mathrm{E}_{R_{P_i}}(R/P_i)$. Let $\varphi_i; R \to R_{P_i}$ be the canonical homomorphism. Then the operation of elements r in R on $M_i = \mathrm{E}_{R_{P_i}}(R/P_i)$ is given via φ_i. Hence, $M_i = \mathrm{E}_{R_T}(R_T/P_iR_T)$ and $\mathrm{Hom}_{R_{P_i}}(M_i, M_i) = \mathrm{Hom}_R(M_i, M_i)$ by the standard argument. Furthermore, since R_{P_i} is noetherian, for any element x in M_i ann$_{R_{P_i}} x \supseteq P_i^{n_i}R_{P_i} \supseteq \varphi_i(P_i^{n_i})$ for some n_i and hence, $xP_i^{n_i} = (0)$. Put $M = \Sigma \oplus M_i$, then M is an R-weakly distinguished module from the above, (cf. the proof of Lemma 2). Since R_T is noetherian, $\mathrm{Hom}_{R_T}(M, M) = \Sigma \oplus \mathrm{Hom}_{R_{P_i}}(M_i, M_i) = \mathrm{Hom}_R(M, M)$ is noetherian by [6], Theorem 3.9. Therefore, M is R-artinian, since M is R-weakly distinguished.

Lemma 3. *Let R be a local noetherian ring with maximal ideal P and $M = \mathrm{E}_R(R/P)$. Let $S = \mathrm{Hom}_R(M, M)$ and T be an intermediate ring between R and S. If for any element x in $\mathrm{E}_T(M)$, $xP^n = (0)$ for some n, then M is T-injective.*

Proof. $\mathrm{E}_T(M) = M \oplus K$ as R-modules. If $K \neq (0)$, for any $k \neq 0$ in K, $kP^n = (0)$ by the assumption. Hence, ann $k' = P$ for some $k' \in K$. Since $\mathrm{E}_T(M)$ is indecomposable, it contains a unique minimal T-module R/P. Which is a contradiction.

Proposition 3. *Let R, M and S be as in Lemma 3. Then for any intermediate local ring T between R and S, M is T-injective if and only if T is noetherian and $\mathfrak{P} \cap T = P'$, where $\mathfrak{P}$ and P' are maximal ideals in S and T, respectively.*

Proof. "Only if part" is an immediate consequence of Theorem 3. We assume that T is noetherian as in the proposition. Since $M = \mathrm{E}_R(R/P)$ and

$(R/P)S=R/P$, $R/P\approx T/P'$ and $P'\cap R=P$. Let $M'=\mathrm{E}_T(T/P')$, then for any x in M' $xP^n\subset xP'^n=(0)$ for some n. Hence, $M=M'$ by Lemma 3.

REMARK. Let Z, P be as in the previous remark. Then there exists a tower of noetherian local rings $Z_P\subset R_1\subset R_2\subset\cdots$ such that R_i dominates R_{i-1} and $T=\cup R_i$ is not noetherian. Then M is R_i-injective for each i, but not T-injective.

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